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A NEW DIAGNOSTIC TEST FOR CROSS-SECTION UNCORRELATEDNESS IN NONPARAMETRIC PANEL DATA MODELS

JIA CHEN
Monash University
and
University of Queensland

JITI GAO
Monash University
and
University of Adelaide

DEGUI LI
Monash University

In this paper, we propose a new diagnostic test for residual cross-section uncorrelatedness (CU) in a nonparametric panel data model. The proposed nonparametric CU test is a nonparametric counterpart of an existing parametric cross-section dependence test proposed in Pesaran (2004, Cambridge Working paper in Economics 0435). Without assuming cross-section independence, we establish asymptotic distribution for the proposed test statistic for the case where both the cross-section dimension and the time dimension go to infinity simultaneously, and then analyze the power function of the proposed test under a sequence of local alternatives that involve a nonlinear multifactor model. The simulation results and real data analysis show that the nonparametric CU test associated with an asymptotic critical value works well.

1. INTRODUCTION

Existing studies in nonparametric and semiparametric estimation and model specification testing mainly assume cross-section independence. Such an assumption is far from realistic, because cross-section dependence is common in practice and

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it may arise from the presence of common shocks and unobserved components that become a part of the error term. If observations are cross-section correlated, parametric and nonparametric estimators based on the assumption of cross-section uncorrelatedness (CU) may be inconsistent. Hence, it is appealing to test for CU before one attempts to make statistical inference for a panel data model.

There is substantial literature on diagnostic tests for CU in parametric panel data models. Breusch and Pagan (1980) proposed a Lagrange multiplier (LM) test statistic, which is based on the average of the squared pairwise correlation coefficients of the residuals. As pointed out by Pesaran (2004), the LM test is only applicable to the case where $T \to \infty$ while N is fixed, where T and N are the time dimension and the cross-section dimension, respectively. Frees (1995) thus proposed a test statistic that is based on the squared Spearman rank correlation coefficients and allows N to be larger than T. Recently, Pesaran (2004) introduced a cross-section dependence (CD) test. The main idea of the CD test is to use a simple average of all pairwise correlation coefficients of the residuals from the individual parametric linear regressions in the panel. The advantage of the CD test is that it is correctly centered when both N and T are fixed. However, the tests proposed by Frees (1995) and Pesaran (2004) can only be used to test for uncorrelatedness rather than independence unless the residuals are further assumed to be normally distributed. For other recent contributions to diagnostic tests of cross-section independence, we refer to Ng (2006), Huang, Kab, and Urga (2008), Pesaran, Ullah, and Yamagata (2008), Sarafidis, Yamagata, and Robertson (2009), and Su and Ullah (2009).

This paper proposes a new diagnostic test for CU in a nonparametric panel data model. We construct a local linear estimator for each individual regression function and then propose a nonparametric CU test statistic in a similar fashion to that proposed in Pesaran (2004) for the parametric case. Without assuming cross-section independence, we show that the proposed test has an asymptotically normal distribution under the null hypothesis and is also consistent under a sequence of local alternatives that involve a nonlinear multifactor model.

The rest of this paper is organized as follows. A nonparametric test for CU in a panel data model is proposed in Section 2. An asymptotic distribution of the proposed nonparametric CU test statistic is established in Section 3. One simulation example is given in Section 4. An empirical analysis of a set of Consumer Price Index (CPI) data in Australian capital cities is given in Section 5. An outline of the proofs of the main results is given in Appendix A. Appendix B provides some useful lemmas. Several other simulation examples are available from Section 5 of the working paper by Chen, Gao, and Li (2009). Detailed proofs of some useful lemmas are also available from the working paper.

2. NONPARAMETRIC CU TEST STATISTIC

Consider a nonparametric panel data model of the form

$$Y_{it} = g_i(X_{it}) + \sigma_i(X_{it})\, u_{it}, \qquad i = 1, \ldots, N; \qquad t = 1, \ldots, T, \tag{2.1}$$

where $\{g_i(\cdot)\}$ is a sequence of individual regression functions, $\{X_{it}\}$ is a sequence of random regressors, $\{\sigma_i(\cdot)\}$ is a sequence of unknown positive functions, and $\{u_{it}\}$ is independent of $\{X_{it}\}$ with $\mathbb{E}[u_{it}] = 0$ and $\mathbb{E}[u_{it}^2] = 1$.

Note also that X_{it} are assumed to be one-dimensional. As shown in the construction of the test that follows, the same procedure remains applicable when X_{it} are multidimensional. In addition, each $g_i(\cdot)$ may be approximated by an additive function of one-dimensional functions when X_{it} are high-dimensional (see, e.g., Gao, 2007, Ch. 2). To avoid tedious technicalities, this paper focuses on the scalar case where X_{it} are only one-dimensional.

The aim of this paper is to test the null hypothesis

$$H_0: \mathbb{E}\left[u_{it}u_{jt}\right] = 0 \quad \text{for all } t \geq 1 \quad \text{and} \quad \text{all } i \neq j, \tag{2.2}$$

without assuming the cross-section independence between u_{it} and u_{jt} for all $i \neq j$. This kind of independence has been assumed in some existing studies (see, e.g., Frees, 1995, Assump. 1; Pesaran, 2004, Assump. 2).

Following Pesaran (2004) in the parametric linear panel data case, this paper proposes to use the simple average of all pairwise correlation coefficients of the residuals from the individual nonparametric regression in the panel. In this paper, we propose using a local linear kernel estimation method to construct our nonparametric CU test. The local linear kernel estimator of $g_i(x)$ can be expressed as (see, e.g., Fan and Gijbels, 1996)

$$\widehat{g}_i(x) = \sum_{t=1}^{T} w_{it}(x)Y_{it}, \qquad 1 \leq i \leq N, \tag{2.3}$$

where $w_{it}(x) = \left(\widetilde{K}_{x,h}(X_{it})\right) / \left(\sum_{t=1}^{T} \widetilde{K}_{x,h}(X_{it})\right)$, in which

$$\widetilde{K}_{x,h}(X_{it}) = \frac{1}{h} K\left(\frac{X_{it}-x}{h}\right)\left[S_{i2}(x) - \left(\frac{X_{it}-x}{h}\right) S_{i1}(x)\right]$$

with $S_{ij}(x) = \frac{1}{Th}\sum_{t=1}^{T}\left((X_{it}-x)/h\right)^j K\left((X_{it}-x)/h\right)$ for $j = 0, 1, 2$.

Let $v_{it} = \sigma_i(X_{it})u_{it}$. Note that $\mathbb{E}\left[v_{it}v_{jt}\right] = \mathbb{E}\left[\sigma_i(X_{it})\sigma_j(X_{jt})\right]\mathbb{E}\left[u_{it}u_{jt}\right] = 0$ when H_0 of (2.2) holds. Thus, to test for the uncorrelatedness of $\{u_{it}\}$, it suffices to test for the uncorrelatedness of $\{v_{it}\}$. With the help of the local linear smoother defined previously, we estimate v_{it} by $\widetilde{v}_{it} = Y_{it} - \widehat{g}_i(X_{it})$. We are now ready to propose a nonparametric CU test statistic of the form

$$NCU = \sqrt{\frac{T}{N(N-1)}} \sum_{i=1}^{N}\sum_{j\neq i}^{N} \widetilde{\rho}_{ij}, \tag{2.4}$$

where

$$\widetilde{\rho}_{ij} = \frac{\sum_{t=1}^{T} \overline{v}_{it}\overline{v}_{jt}}{\sqrt{\sum_{t=1}^{T} \overline{v}_{it}^2}\sqrt{\sum_{t=1}^{T} \overline{v}_{jt}^2}} = \frac{\frac{1}{T}\sum_{t=1}^{T} \overline{v}_{it}\overline{v}_{jt}}{\sqrt{\frac{1}{T}\sum_{t=1}^{T} \overline{v}_{it}^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T} \overline{v}_{jt}^2}},$$

in which $\overline{v}_{it} = \widetilde{v}_{it}\widehat{f}_i(X_{it})$ and $\widehat{f}_i(x) = \frac{1}{T}\sum_{s=1}^{T} \widetilde{K}_{x,h}(X_{is})$.

3. LARGE-SAMPLE THEORY

To establish asymptotic theory of the test statistic, we need the following assumptions.

A1.

(a) The kernel function $K(\cdot)$ is a symmetric and continuous density function with a compact support.

(b) $g_i(\cdot)$, $1 \le i \le N$, are twice continuously differentiable. Let $g_i''(\cdot)$ be the second-order derivative of $g_i(\cdot)$. Then, $\max_{i\ge 1} \mathbb{E}\left[|g_i''(X_{i1})|^2\right] < \infty$.

(c) For fixed $i \ge 1$, both $\{X_{it},\ t \ge 1\}$ and $\{u_{it},\ t \ge 1\}$ are sequences of stationary random variables.

(d) Let $f_i(\cdot)$ be the density function of $\{X_{it}\}$; then $f_i(x)$ is continuous and bounded in $x \in R$. Define $\widetilde{X}_{ijst} = (X_{is_1}, \ldots, X_{is_l}, X_{jt_1}, \ldots, X_{jt_k})$ and let $f_{is_1,\ldots,is_l,jt_1,\ldots,jt_k}(\ldots,\ldots)$ be the joint density of $\widetilde{X}_{ijst}$. Then, $f_{is_1,\ldots,is_l,jt_1,\ldots,jt_k}(\ldots,\ldots)$ is also continuous and bounded for all $1 \le i,j \le N$, $1 \le l,k \le 4$.

A2.

(a) $0 < \min_{i,j\ge 1} \mathbb{E}\left[\sigma_i(X_{i1})\sigma_j(X_{j1})\right] \le \max_{i,j\ge 1} \mathbb{E}\left[\sigma_i(X_{i1})\sigma_j(X_{j1})\right] < \infty$ and $\max_{i\ge 1} \mathbb{E}\left[\sigma_i^8(X_{it})\right] < \infty$.

(b) Define $\tau_{i,j,*}^2 = \mu_2^4\mu_0^4\left(\sigma_{uij}^2\, \kappa_{i,j,*} + 2\sum_{t=2}^{\infty} \mathbb{E}[u_{i1}u_{it}u_{j1}u_{jt}]\, \kappa_{i,j,*}(t)\right)$, where $\mu_k = \int u^k K(u)du$,

$$\sigma_{uij}^2 = \mathbb{E}\left[u_{i1}^2 u_{j1}^2\right], \qquad \kappa_{i,j,*} = \iint f_i^4(x) f_j^4(y)\sigma_i^2(x)\sigma_j^2(y) \times f_{i1,j1}(x,y)dxdy \quad \text{and}$$

$$\kappa_{i,j,*}(t) = \iiiint f_i^2(x_1) f_i^2(x_2) f_j^2(y_1) f_j^2(y_2)\sigma_i(x_1)\sigma_i(x_2)\sigma_j(y_1)\sigma_j(y_2) \times f_{i1,it,j1,jt}(x_1,x_2,y_1,y_2)dx_1dx_2dy_1dy_2,$$

and $\sigma_{i,j*}^2 = \mu_2^4\mu_0^4 \iint f_i^5(x) f_j^5(y)\sigma_i^2(x)\sigma_j^2(y)dxdy$. Moreover, there exists some $\tau_0(*) > 0$ such that

$$\lim_{N\to\infty} \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j\neq i} \frac{\tau_{i,j,*}^2}{\sigma_{i,j*}^2} = \tau_0(*). \tag{3.1}$$

A3. The bandwidth h satisfies

$$\frac{T^\theta h}{\log T} \to \infty \quad \text{and} \quad N^2 T h^8 \to 0 \quad \text{as } T\to\infty \quad \text{and} \quad N\to\infty, \tag{3.2}$$

where $\theta = (\chi_0 - 3)/(\chi_0 + 2)$ and χ_0 is defined in A4(a) which follows. In addition, $N = O\left(T^{q_0-1}\right)$, where $q_0 > 1$ is a constant defined in A4(a).

A4.

(a) Let $\mathbf{u}_t = (u_{1t},\ldots,u_{Nt})^\tau$ and $\mathbf{X}_t = (X_{1t},\ldots,X_{Nt})^\tau$. Suppose that $\{(\mathbf{u}_t,\mathbf{X}_t)\}$ is a sequence of stationary α-mixing random variables with $\max_{i\geq 1}\mathbb{E}\left[|X_{i1}|^{2q_0}\right] < \infty$ for some $q_0 > 1$ and with mixing coefficient $\alpha(\cdot)$ satisfying $\alpha(k) = O\left(k^{-\chi_0}\right)$ for k large enough and some $\chi_0 > \max\left\{2 + 4/(q_0-1), 6\right\}$.

(b) Let $\max\limits_{i\geq 1}\mathbb{E}\left[u_{i1}^4\right] < \infty$ and for each $1\leq k\leq 4$, let $(j_1^*,\ldots,j_k^*)$ be any set of k positive integers satisfying $1\leq j_l^*\leq 2$, $1\leq l\leq k$, and $\sum\limits_{l=1}^{k} j_l^* = 4$. Suppose that

$$\sum_{k=3}^{4} \sum_{(i_1^*,\ldots,i_k^*)\in\Theta_k^*} \mathbb{E}\left[u_{i_1^*,1}^{j_1^*}\cdots u_{i_k^*,1}^{j_k^*}\right] = O\left(N^2\right), \tag{3.3}$$

where $\Theta_k^* = \left\{(i_1^*,\ldots,i_k^*):\ 1\leq i_l^*\leq N, \text{ and all } i_l^*,\ 1\leq l\leq k, \text{ are different}\right\}$.

(c) Let $\max\limits_{i\geq 1}\mathbb{E}\left[u_{i1}^8\right] < \infty$ and for each $1\leq k\leq 8$, let $(j_1,\ldots,j_k)$ be any set of k positive integers satisfying $1\leq j_l\leq 8$, $1\leq l\leq k$, and $\sum\limits_{l=1}^{k} j_l = 8$. Suppose that

$$\sum_{k=5}^{8} \sum_{(i_1,\ldots,i_k)\in\Theta_k} \mathbb{E}\left[u_{i_1,1}^{j_1}\cdots u_{i_k,1}^{j_k}\right] = O\left(N^4\right), \tag{3.4}$$

where $\Theta_k = \{(i_1,\ldots,i_k):\ 1\leq i_l\leq N, \text{ and all } i_l, 1\leq l\leq k, \text{are different}\}$.

Remark 3.1.

(a) The preceding assumptions are mild and can be satisfied in many cases. For example, A1(a) is a mild condition on the kernel function and is assumed by many authors in nonparametric inference of both stationary time series and panel data (see, e.g., Fan and Yao, 2003; Gao 2007; Cai and Li, 2008). A1(b) and A1(d) are some mild conditions on the individual regression functions.

(b) Note that when $\{u_{it}\}$ is independent in (i,t), and X_{it} and X_{jt} are independent for all $i \neq j$ and each given t, we have $\kappa_{i,j,*} = \int\int f_i^5(x) f_j^5(y) \sigma_i^2(x)\sigma_j^2(y)dxdy$ and $\tau_{i,j,*}^2 = \mu_2^4 \mu_0^4 \kappa_{i,j,*}\sigma_{uij}^2 = \sigma_{i,j,*}^2$. Thus, $\tau_0(*) \equiv 1$.

(c) Condition A3 is a set of conditions on the bandwidth and on T and N. The first bandwidth condition in A3 is proposed to apply the uniform consistency of the nonparametric kernel estimator in the proofs of Theorems 3.1 and 3.2 in Appendix A. The second bandwidth condition in A3 is needed in the proofs of Theorems 3.1 and 3.2. Note that the first two parts of A3 are satisfied in many cases. For example, when $g_i(X_{it}) = g(X_{it}) + \alpha_i$ (α_i is the individual effect) and $\sigma_i(\cdot) \equiv \sigma_0$, (2.3) may be replaced by a pooled local linear estimator and an optimal bandwidth will satisfy $h_{\text{optimal}} = O\left((NT)^{-1/5}\right)$. In this case, the first and the second parts of A3 require $\left(T^{3/2}/N\right) \to \infty$. Note also that $h_0 = O\left(T^{-1/5}\right)$ is not necessarily an optimal bandwidth in this kind of nonparametric panel data regression unless N is fixed.

Furthermore, the third condition of $N = O\left(T^{q_0 - 1}\right)$ allows for the following two cases. The first case is that the rate of $T \to \infty$ is faster than that of $N \to \infty$ when $1 < q_0 < 2$. The second case is that the rate of $T \to \infty$ is slower than that of $N \to \infty$ when $q_0 > 2$. In addition, the simulation studies in Section 4 support that the nonparametric CU test works well even when T is as small as $T = 20$, although it cannot be shown at this stage that the main theorems remain true when T is fixed.

(d) The α-mixing condition assumed in A4(a) is a commonly used condition in the time series case (see, e.g., Fan and Yao, 2003; Gao, 2007; Li and Racine, 2007). Assumptions A4(b) and (c) impose a kind of correlation assumptions as a weak alternative to the independence assumption that $\{u_{it}\}$ is a sequence of independent and identically distributed random errors for all (i,t) as used in Assumption 1 of Frees (1995) and Assumption 2 of Pesaran (2004). Obviously, equations (3.3) and (3.4) hold trivially when $\{u_{it}\}$ are cross-sectionally independent.

More detailed discussion about the plausibility and justifiability of conditions A1–A4 is given in Remarks 3.1 and 3.2 of the working paper by Chen et al. (2009). In Theorem 3.1, which follows, we show that the nonparametric CU test statistic has an asymptotically normal distribution. This has also been obtained by Pesaran (2004) and Hsiao, Pesaran, and Pick (2007) under the assumption that u_{it} and u_{jt} are cross-sectionally independent in the context of parametric linear and nonlinear panel data models. The proof of Theorem 3.1 is given in Appendix A.

THEOREM 3.1. *Assume that, for model (2.1), A1–A4 hold. Then under* H_0,

$$NCU = \sqrt{\frac{T}{N(N-1)}} \left(\sum_{i=1}^{N} \sum_{j \neq i}^{N} \widetilde{\rho}_{ij} \right) \xrightarrow{d} N(0, \tau_0(*)) \tag{3.5}$$

as $T \to \infty$ *and* $N \to \infty$ *simultaneously.*

We then analyze the power of the proposed test under a sequence of local alternatives. Naturally, the power of the proposed test for the cross-section correlation relies on the form of the alternative hypothesis. We consider a sequence of cross-section correlation alternatives via a nonlinear multifactor model of the form

$$H_1: \quad u_{it} = F_{NT}(z_t, \beta_i) + \varepsilon_{it} \quad \text{with } F_{NT}(z_t, \beta_i) = \frac{1}{N^{k/2} T^{k/4}} G(z_t, \beta_i) \tag{3.6}$$

for $k = 0, 1$, where $\{G(z_t, \beta_i)\}$ is a sequence of known functions indexed by $\{z_t\}$ and $\{\beta_i\}$, $\{z_t,\ t \geq 1\}$ is a sequence of stationary α-mixing random variables, $\{\beta_i,\ i \geq 1\}$ is a sequence of common factors, $\{\varepsilon_{it},\ t \geq 1\}$ is a sequence of stationary α-mixing random errors with $\mathbb{E}[\varepsilon_{i1}] = 0$ for each i and is independent of $\{z_t\}$, and $\mathbb{E}\left[\varepsilon_{it}\varepsilon_{jt}\right] = 0$ for all $i \neq j$. Note that (3.6) defines a global alternative when $k = 0$, whereas it gives a sequence of local alternatives when $k = 1$.

Before establishing the asymptotic distribution of the proposed test statistic under the alternative hypothesis H_1, we need the following set of conditions.

A5.

(a) $\{z_t\}$ is a sequence of stationary α-mixing random variables with mixing coefficient satisfying

$$\alpha_z(t) = O(t^{-\chi_1}) \quad \text{for } \chi_1 > \max\left\{ \frac{q_1(2q_1 + \delta_1)}{\delta_1}, 3 \right\},$$

where $\delta_1 > 0$ and $q_1 > 1$ are some constants such that equation (3.7), which follows, is satisfied.

(b) The nonlinear function $G(\cdot, \cdot)$ satisfies

$$\mathbb{E}[G(z_t, \beta_i)] = 0 \quad \text{and} \quad \max_{(i,j)} \mathbb{E}\left[\left| G(z_t, \beta_i) G(z_t, \beta_j) \right|^{4q_1 + 2\delta_1} \right] < \infty \tag{3.7}$$

for some $q_1 > 1$ and $\delta_1 > 0$. Let

$$\psi(*) = \lim_{N \to \infty} \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} \frac{\psi_{ij}(*)}{\sigma_{i,j*}}, \tag{3.8}$$

where $\psi_{ij}(*) = \mu_2^2 \mu_0^2 \mathbb{E}[\sigma_i(X_{it})\sigma_j(X_{jt}) f_i^2(X_{it}) f_j^2(X_{jt}) G(z_t, \beta_i) G(z_t, \beta_j)]$. The condition $\max_{i \geq 1} \mathbb{E}\left[|\sigma_i(X_{i1})|^8\right] < \infty$ in A2(a) is strengthened to

$$\max_{i \geq 1} \mathbb{E}\left[|\sigma_i(X_{i1})|^{8q_1 + 4\delta_1}\right] < \infty,$$

where $q_1 > 1$ and $\delta_1 > 0$ are the same as in (3.7).

(c) A4 is satisfied when $\{u_{it}\}$ is replaced by $\{\varepsilon_{it}\}$. Moreover, $\{\varepsilon_{it}\}$ is independent of $\{z_t\}$. Let $\tau_1(*)$ be defined in the same way as $\tau_0(*)$ with $\{u_{it}\}$ being replaced by $\{\varepsilon_{it}\}$.

Condition A5 allows for a general class of forms for $G(z_t, \beta_i)$. It obviously covers the linear multifactor case, $G(z_t, \beta_i) = z_t \beta_i$, which has been studied by Pesaran (2004). When the alternative hypothesis H_1 holds, we have the following asymptotic distribution for the test statistic *NCU*.

THEOREM 3.2. *For model (2.1), assume that conditions A1–A3 and A5 are satisfied. In addition,* $N = o\left(\sqrt{T^{q_1}}\right)$, *where* $q_1 > 1$ *was defined in A5(a).*

(i) Under H_1 with $k = 0$, we have, as $T \to \infty$ and $N \to \infty$ simultaneously,

$$\frac{NCU}{NT^{1/2}} \xrightarrow{P} \psi(*). \tag{3.9}$$

(ii) Under H_1 with $k = 1$, we have, as $T \to \infty$ and $N \to \infty$ simultaneously,

$$NCU \xrightarrow{d} N(\psi(*), \tau_1(*)), \tag{3.10}$$

where $\tau_1(*)$ was defined in A5(c).

The proof of Theorem 3.2 is given in Appendix A. To apply the asymptotic distribution of the test statistic in practice, we need to construct some consistent estimators for σ_{ij*}^2 and $\tau_{i,j,*}^2$. In practice, we propose using the following estimators of the form

$$\widehat{\sigma}_{i,j*}^2 = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \widehat{Z}_{is*}^2 \widehat{Z}_{jt*}^2 \quad \text{and} \quad \widehat{\tau}_{i,j,*}^2 = \frac{1}{T^2} \left(\sum_{t=1}^{T} \widehat{Z}_{it*} \widehat{Z}_{jt*} \right)^2,$$

where $\widehat{Z}_{it*} = \mu_2^2 \mu_0^2 \, \widetilde{v}_{it} \, \widehat{f}_i^{\,2}(X_{it})$, in which $\widetilde{v}_{it} = Y_{it} - \widehat{g}_i(X_{it})$ and $\widehat{f}_i(\cdot)$ was defined in Section 2. It may be shown that $\widehat{\sigma}_{i,j*}^2$ and $\widehat{\tau}_{i,j,*}^2$ are both consistent. The details are given in Appendix C of the working paper by Chen et al. (2009).

The simulation study in Section 4 shows that the power of the proposed test is satisfactory when $\psi(*) > 0$ (or $\psi(*) < 0$). However, when $\psi(*) = 0$, the asymptotic distribution in (3.10) is the same as that in Theorem 3.1, which implies that the test would not have a satisfactory power. In the context of parametric panel data models, Pesaran et al. (2008) proposed a bias-adjusted LM test to avoid the problem of poor power for the case of $\psi(*) = 0$. It is interesting to consider a nonparametric type of bias-adjusted LM test statistic. However, such an issue is left for our future study.

4. A SIMULATED EXAMPLE

In this section, we give a simulated example to demonstrate the finite-sample performance of the nonparametric CU test. We also compare its performance with those of two parametric CD tests in the working paper by Chen et al. (2009). Because both the sizes and power values of the proposed nonparametric CU test associated with an asymptotic critical value in each case are already comparable with those of the parametric CD test based on a critical value selected by a bootstrap resampling procedure, there is no need to adopt the bootstrap procedure to improve the finite-sample performance of the proposed NCU test.

In the following experiments, the uniform kernel $K(u) = \frac{1}{2} I\{|u| \le 1\}$ is used in the proposed nonparametric CU test. We now introduce a leave-one-out cross-validation method for the choice of h. Define the leave-one-out estimator of $g_i(x)$ as

$$\widehat{g}_{i,-t}(x) = \sum_{s=1,\neq t}^{T} w_{is,-t}(x) Y_{is}, \qquad i = 1,2,\ldots,N; \qquad t = 1,2,\ldots,T, \tag{4.1}$$

where $w_{is,-t}(x) = \widetilde{K}_{x,h}^{(-t)}(X_{is}) \Big/ \left(\sum_{s=1,\neq t}^{T} \widetilde{K}_{x,h}^{(-t)}(X_{is}) \right)$, in which

$$\widetilde{K}_{x,h}^{(-t)}(X_{is}) = \frac{1}{h} K\left(\frac{X_{is}-x}{h}\right) \left[S_{i2,-t}(x) - \left(\frac{X_{is}-x}{h}\right) S_{i1,-t}(x) \right]$$

with $S_{ij,-t}(x) = \frac{1}{Th} \sum_{s=1,\neq t}^{T} \left((X_{is}-x)/h\right)^j K\left((X_{is}-x)/h\right)$ for $j = 0,1,2$.

An optimal bandwidth, $\widehat{h}$, based on the leave-one-out cross-validation method is chosen such that

$$\widehat{h} = \arg \min_{\{\text{over all possible } h \text{ values}\}} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \widehat{g}_{i,-t}(X_{it}) \right]^2. \tag{4.2}$$

The bandwidth selection method is used in both the simulation study and empirical analysis.

Consider a nonparametric panel data model of the form

$$Y_{it} = \frac{X_{it}}{1+X_{it}^2} + u_{it}, \qquad i = 1,2,\ldots,N; \qquad t = 1,2,\ldots,T, \tag{4.3}$$

where $X_{it} \stackrel{i.i.d.}{\sim} N(0,1)$, $u_{it} = f(\beta_i, z_t) + e_{it}$, z_t are the time-variant common effects, $z_t \stackrel{i.i.d.}{\sim} N(0,1)$, $e_{it} \stackrel{i.i.d.}{\sim} N(0,1)$, $\{\beta_i\}$ is a sequence of nonrandom numbers indicating the degree of cross-section error correlations, and $f(\cdot,\cdot)$ takes one of the two forms: $f(\beta_i, z_t) = \beta_i z_t$ and $f(\beta_i, z_t) = \beta_i z_t \big/ \left(1 + \beta_i^2 z_t^2\right)$. Note that $\{u_{it}\}$ and $\{X_{it}\}$ are independently generated.

For $\beta_i = 0$, the sizes of the proposed nonparametric CU test are reported in Table 1, and for $\beta_i \sim U(0.1, 0.3)$, the power values are given in Table 2.

TABLE 1. Size of the nonparametric test for model (4.3) at the 5% level

$T\backslash N$	10	20	30	50	100
10	0.053	0.049	0.044	0.062	0.046
20	0.049	0.049	0.046	0.057	0.046
30	0.040	0.053	0.046	0.041	0.049
50	0.044	0.044	0.041	0.052	0.045
100	0.052	0.046	0.047	0.052	0.047

TABLE 2. Power of the nonparametric test for model (4.3) at the 5% level

	$f(\beta_i, z_t) = \beta_i z_t$					$f(\beta_i, z_t) = \frac{\beta_i z_t}{1+\beta_i^2 z_t^2}$				
$T\backslash N$	10	20	30	50	100	10	20	30	50	100
10	0.149	0.326	0.374	0.581	0.848	0.117	0.192	0.296	0.450	0.787
20	0.136	0.376	0.652	0.879	0.986	0.161	0.308	0.486	0.757	0.969
30	0.201	0.584	0.723	0.960	0.999	0.169	0.418	0.711	0.920	0.999
50	0.338	0.556	0.927	0.997	1.000	0.210	0.602	0.823	0.964	1.000
100	0.653	0.898	0.979	1.000	1.000	0.261	0.846	0.982	1.000	1.000

Table 1 shows that the nonparametric CU test has some reasonable sizes for the nonparametric panel data model (4.3), and Table 2 shows that the simulated power values of the nonparametric CU test are also satisfactory. This shows that the proposed nonparametric CU test is a generally applicable test in this kind of testing for CU, as the applicability does not require a model to be parametrically specified. In other words, it still works well without necessarily prespecifying the conditional mean function.

5. EMPIRICAL APPLICATION: AN ANALYSIS OF AUSTRALIAN CPI DATA

As an application of our testing method, we test for CU of CPI among the eight Australian capital cities during the period 1989–2008. The data, which were obtained from the web site of the Australian Bureau of Statistics, are recorded quarterly each year. Hence, they consist of the CPI numbers for eight cities ($N = 8$) at 80 different times ($T = 80$). We chose Y_{it} as the log of the food CPI for city i at time t and X_{it} as the log of all group CPI for city i at time t. For each city i, we computed the nonparametric regression function of Y_{it} on X_{it} ($t = 1, 2, \ldots, T$) using the local linear estimation method. Then, we used the estimation residuals $\tilde{v}_{it}$ to compute the nonparametric CU test statistic. In a similar way, we also computed the regression of the logarithm of the transportation CPI on the log of all group CPI for each city. The results are summarized in Table 3.

TABLE 3. Cross-section correlation of CPI in Australian capital cities

	Food	Transportation
Nonparametric CU test	47.2378	47.0227
Bootstrap 1% critical values	[−2.3130, 2.6100]	[−2.4895, 2.7300]
Bootstrap 5% critical values	[−1.8796, 1.8517]	[−1.8786, 1.8899]
Bootstrap 10% critical values	[−1.6086, 1.5584]	[−1.6532, 1.6203]

Note that the two-sided bootstrap critical values were calculated using 1,000 iterations. It follows from Table 3 that there is some evidence to suggest rejecting the null hypothesis that there is CU for both the food and transportation indexes. In addition, based on the bootstrap simulated critical value in each case, CU should be rejected at all the levels of 1%, 5%, and 10%. This suggests that the CU assumption in such empirical studies may not be appropriate. Further studies are needed to find ways of defining a suitable cross-section correlation structure to deal with panel data analysis when there is some cross-section correlation.

REFERENCES

Breusch, T.S. & A.R. Pagan (1980) The Lagrange multiplier test and its application to model specifications in econometrics. *Review of Economic Studies* 47, 239–253.

Cai, Z. & Q. Li (2008) Nonparametric estimation of varying coefficient dynamic panel data models. *Econometric Theory* 24, 1321–1342.

Chen, J., J. Gao, & D. Li (2009) Testing for Cross-Section Uncorrelatedness in Nonparametric Panel Data Models: Theory and Practice. Working paper, University of Adelaide. Available at http://www.adelaide.edu.au/directory/jiti.gao.

Fan, J. & J. Gijbels (1996) *Local Polynomial Modelling and Its Applications*. Chapman and Hall.

Fan, J. & Q. Yao (2003) *Nonlinear Time Series: Nonparametric and Parametric Methods*. Springer-Verlag.

Frees, E.W. (1995) Assessing cross sectional correlation in panel data. *Journal of Econometrics* 69, 393–414.

Gao, J. (2007) *Nonlinear Time Series: Semiparametric and Nonparametric Methods*. Chapman and Hall.

Hsiao, C., M.H. Pesaran, & A. Pick (2007) Diagnostic Tests of Cross Section Independence for Nonlinear Panel Data Models. IZA Discussion paper 2756.

Huang, H., C. Kab, & G. Urga (2008) Copula-based tests for cross-sectional independence in panel models. *Economics Letters* 100, 224–228.

Li, Q. & J. Racine (2007) *Nonparametric Econometrics: Theory and Practice*. Princeton University Press.

Ng, S. (2006) Testing cross section correlation in panel data using spacing. *Journal of Business & Economic Statistics* 24, 12–23.

Pesaran, M.H. (2004) General Diagnostic Tests for Cross Section Dependence in Panels. Cambridge Working paper in Economics 0435.

Pesaran, M.H., A. Ullah, & T. Yamagata (2008) A bias adjusted LM test of error cross section independence. *Econometrics Journal* 11, 105–127.

Phillips, P.C.B. & H. Moon (1999) Linear regression limit theory for nonstationary panel data. *Econometrica* 67, 1057–1111.

Sarafidis, V., T. Yamagata, & D. Robertson (2009) A test of cross section dependence for a linear dynamic panel model with regressors. *Journal of Econometrics* 148, 149–161.

Shao, Q. & H. Yu (1996) Weak convergence for weighted empirical processes of dependent sequences. *Annals of Probability* 24, 2098–2127.

Su, L. & A. Ullah (2009) Testing conditional uncorrelatednesss. *Journal of Business & Economic Statistics* 27, 18–29.

APPENDIX A: Proofs of the Main Results

Here and in what follows we use a double sum of the form $\sum_{t=1}^{T}\sum_{s\neq t}$ to replace $\sum_{t=1}^{T}\sum_{s=1,\neq t}^{T}$ for notational simplicity. Before proving the main theorems, we introduce some useful lemmas. Because the proof of Lemma A.1 is of general interest, the full details are given in this paper. As the proofs of Lemmas A.2–A.6 are quite technical, they are relegated to the working paper by Chen et al. (2009).

We now provide a new central limit theorem for quadratic forms of panel data series for the case where $T \to \infty$ and $N \to \infty$ simultaneously in Lemma A.1, which is of general interest in itself.

LEMMA A.1. *Assume that the following three sets of conditions hold:*

(i) $\mathbb{E}\left[U_{i1}\right] = 0$ *and* $\mathbb{E}\left[U_{i1}U_{j1}\right] = 0$ *for all* $i \neq j$*. For* $\mathbf{U}_t = (U_{1t}, \ldots, U_{Nt})^{\tau}$, $\{\mathbf{U}_t\}$ *is a vector of* α*-mixing random variables with mixing coefficient* $\alpha(\cdot)$ *satisfying* $\alpha(k) = O(k^{-\chi})$ *as* $k \to \infty$ *for some constant* $\chi > 6$.

(ii) $\max_{i\geq 1}\mathbb{E}\left[U_{i1}^{8}\right] < \infty$*. For each* $1 \leq k \leq 8$*, let* $(j_1, \ldots, j_k)$ *be any set of* k *positive integers satisfying* $1 \leq j_l \leq 8$, $1 \leq l \leq k$*, and* $\sum_{l=1}^{k} j_l = 8$*. Suppose that*

$$\sum_{k=5}^{8} \sum_{(i_1,\ldots,i_k)\in\Theta_k} \mathbb{E}\left[U_{i_1,1}^{j_1}\cdots U_{i_k,1}^{j_k}\right] = O\left(N^4\right), \tag{A.1}$$

where $\Theta_k = \{(i_1, \ldots, i_k) : 1 \leq i_l \leq N,$ *and* $i_l, 1 \leq l \leq k,$ *are all different*$\}$.

(iii) $0 < \Sigma(U) < \infty$ *with* $\Sigma(U) = \lim_{N\to\infty} \frac{1}{N(N-1)}\left\{\sum_{i=1}^{N}\sum_{j\neq i}\mathbb{E}\left[U_{i1}^2U_{j1}^2\right] + 2\sum_{t=2}^{\infty}\sum_{i=1}^{N}\sum_{j\neq i}\mathbb{E}[U_{i1}U_{it}U_{j1}U_{jt}]\right\}$.

Then, as $T \to \infty$ *and* $N \to \infty$ *simultaneously*

$$\frac{1}{\sqrt{N(N-1)T}}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T} U_{it}U_{jt} \xrightarrow{d} N\left(0, \Sigma(U)\right). \tag{A.2}$$

Proof. Let $\xi_0 = \Sigma(U)$ and

$$V_{T,N}(t) = \frac{1}{\sqrt{N(N-1)T}}\sum_{i=1}^{N}\sum_{j\neq i} U_{it}U_{jt} = \frac{1}{\sqrt{N(N-1)T}}\left[\left(\sum_{i=1}^{N} U_{it}\right)^2 - \sum_{i=1}^{N} U_{it}^2\right]. \tag{A.3}$$

Then,

$$\frac{1}{\sqrt{N(N-1)T}}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}U_{it}U_{jt}=\sum_{t=1}^{T}V_{T,N}(t).$$

Hence, to prove (A.2), it suffices for us to show that as $T\to\infty$ and $N\to\infty$ simultaneously,

$$\sum_{t=1}^{T}V_{T,N}(t)\xrightarrow{d}N(0,\xi_0). \tag{A.4}$$

Because $\{\mathbf{U}_t\}$ is assumed to be stationary with α-mixing and $\{V_{T,N}(t)\}$ is a sequence of simple functions of $\{U_{it}\}$, thus $\{V_{T,N}(t)\}$ is also stationary with α-mixing. We can then apply the large-block and small-block technique to prove (A.4). The large-block and small-block technique has been applied by many authors in the time series case (see, e.g., Gao, 2007, proof of Thm. A.1; Fan and Yao, 2003, proof of Thm. 2.21). The idea is to partition the set $\{1,\ldots,T\}$ into $2k_T+1$ subsets with a large block of size l_T and a small block of size s_T and the remaining set of size $T-k_T(l_T+s_T)$, where

$$l_T=\left[T^{(\lambda-1)/\lambda}\right],\qquad s_T=\left[T^{1/\lambda}\right],\quad\text{and}\quad k_T=[T/(l_T+s_T)]\quad\text{for any }\lambda>2. \tag{A.5}$$

Then define

$$\widetilde{V}_q=\sum_{t=(q-1)(l_T+s_T)+1}^{ql_T+(q-1)s_T}V_{T,N}(t),\qquad \overline{V}_q=\sum_{t=ql_T+(q-1)s_T+1}^{q(l_T+s_T)}V_{T,N}(t),\qquad q=1,\ldots,k_T,$$

and $\widehat{V}=\sum_{t=k_T(l_T+s_T)+1}^{T}V_{T,N}(t)$.

Note that

$$\mathrm{Var}\left(\sum_{q=1}^{k_T}\overline{V}_q\right)=\mathbb{E}\left[\sum_{q=1}^{k_T}\overline{V}_q\right]^2=\sum_{q=1}^{k_T}\mathbb{E}\left[\overline{V}_q^2\right]+2\sum_{q=2}^{k_T}(k_T-q+1)\mathbb{E}\left[\overline{V}_1\overline{V}_q\right].$$

By Lemma B.1 in Appendix B, (A.5), and the condition on the α-mixing coefficient, we have

$$\sum_{q=1}^{k_T}\mathbb{E}\left[\overline{V}_q^2\right]=O\left(\frac{k_Ts_T}{T}\right)\quad\text{and}\quad\sum_{q=2}^{k_T}(k_T-q+1)\mathbb{E}\left[\overline{V}_1\overline{V}_q\right]=o\left(\frac{k_Ts_T}{T}\right),$$

which implies that

$$\mathrm{Var}\left(\sum_{q=1}^{k_T}\overline{V}_q\right)=O\left(\frac{k_Ts_T}{T}\right)=o(1). \tag{A.6}$$

Analogously, we can also show that

$$\mathrm{Var}\left(\widehat{V}\right)=O\left(\frac{T-k_Tl_T}{T}\right)=o(1). \tag{A.7}$$

It follows from (A.6) and (A.7) that to prove (A.4), we need only to show

$$\sum_{q=1}^{k_T} \widetilde{V}_q \xrightarrow{d} N(0,\xi_0). \tag{A.8}$$

In Theorem 2 of Phillips and Moon (1999), the authors provided a central limit theorem for a partial sum of linear processes. The establishment of (A.8) will provide a central limit theorem for a quadratic form of stationary mixing processes.

We now turn to the proof of (A.8). By (A.5), Lemma B.1, and the condition on the α-mixing coefficient, we have

$$\left| \mathbb{E}\left[\exp\left(\sum_{q=1}^{k_T} \widetilde{V}_q\right)\right] - \prod_{q=1}^{k_T} \mathbb{E}\left[\exp\left(\widetilde{V}_q\right)\right]\right| \le C k_T \alpha(s_T) \to 0,$$

which implies that $\widetilde{V}_q$, $q = 1, \ldots, k_T$, are asymptotically independent. Moreover, as in the proof of Theorem 2.20(ii) in Fan and Yao (2003), we have

$$\mathbb{E}\left[\widetilde{V}_1^2\right] = \frac{l_T \xi_0}{T}[1+o(1)],$$

which implies that as $T \to \infty$ and $N \to \infty$ simultaneously,

$$\sum_{q=1}^{k_T} \mathbb{E}\left[\widetilde{V}_q^2\right] = k_T\, \mathbb{E}\left[\widetilde{V}_1^2\right] = \frac{k_T l_T}{T}\xi_0[1+o(1)] \to \xi_0. \tag{A.9}$$

Thus, the *Feller condition* is satisfied.

Furthermore, by Lemma B.3 in Appendix B, we have

$$\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j\neq i} U_{i1}U_{j1}\right]^4 = O(N^4),$$

which, in conjunction with Lemma B.2 (with $p = 3$ and $r = 4$) in Appendix B, implies

$$\mathbb{E}\left[\left|\widetilde{V}_q\right|^3\right] \le \left(\frac{l_T}{N^2 T}\right)^{3/2} \left\{\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j\neq i} U_{i1}U_{j1}\right]^4\right\}^{3/4} = O\left(\left(\frac{l_T}{T}\right)^{3/2}\right).$$

It follows from Cauchy–Schwarz inequality that, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{E}\left[\widetilde{V}_q^2 I\{|\widetilde{V}_q| \ge \epsilon\}\right] &\le \left\{\mathbb{E}\left[\left|\widetilde{V}_q\right|^3\right]\right\}^{2/3} \left[P(|\widetilde{V}_q| \ge \epsilon)\right]^{1/3} \\ &\le C\left\{\mathbb{E}\left[\left|\widetilde{V}_q\right|^3\right]\right\}^{2/3}\left\{\mathbb{E}\left[\left|\widetilde{V}_q\right|^2\right]\right\}^{1/3} = O\left(\left(\frac{l_T}{T}\right)^{4/3}\right) = o\left(\frac{l_T}{T}\right).\end{aligned} \tag{A.10}$$

By (A.10), we have, for any $\epsilon > 0$,

$$\sum_{q=1}^{k_T} \mathbb{E}\left[\widetilde{V}_q^2 I\{|\widetilde{V}_q| \ge \epsilon\}\right] = o(k_T l_T/T) = o(1). \tag{A.11}$$

Hence, the *Lindeberg condition* is satisfied. This, along with (A.9), proves (A.8). Thus, the proof of Lemma A.1 is completed. ■

LEMMA A.2. *Suppose that conditions A1(a) and (c) and A3 hold. If, in addition, $f_i(\cdot)$ is continuous and integrable uniformly in $i \geq 1$, then for $k = 0, 1, 2, \ldots,$*

$$\max_{1 \leq i \leq N} \sup_{x \in R} |S_{ik}(x) - f_i(x)\mu_k| = o_P(1),$$

where $\mu_k = \int u^k K(u) du$.

The proof of Lemma A.2, and also those of Lemmas A.3–A.6 in this Appendix, are available in the working paper by Chen et al. (2009).

Proof of Theorem 3.1. Observe that

$$\overline{v}_{it} = \left[Y_{it} - \widehat{g}_i(X_{it})\right] \widehat{f}_i(X_{it}) = v_{it} \widehat{f}_i(X_{it}) + \left[g_i(X_{it}) - \widehat{g}_i(X_{it})\right] \widehat{f}_i(X_{it}),$$

where $v_{it} = \sigma_i(X_{it}) u_{it}$. By a standard decomposition, we have

$$\begin{aligned}
\sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} \overline{v}_{it} \overline{v}_{jt} &= \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} v_{it} \widehat{f}_i(X_{it}) v_{jt} \widehat{f}_j(X_{jt}) \\
&\quad - \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} v_{it} \widehat{f}_i(X_{it}) \left(\frac{1}{T} \sum_{s=1}^{T} v_{js} \widetilde{K}_{st}^{j} \right) \\
&\quad + \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} v_{it} \widehat{f}_i(X_{it}) \left\{ \frac{1}{T} \sum_{s=1}^{T} \left[g_j(X_{jt}) - g_j(X_{js})\right] \widetilde{K}_{st}^{j} \right\} \\
&\quad - \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} v_{jt} \widehat{f}_j(X_{jt}) \left(\frac{1}{T} \sum_{s=1}^{T} v_{is} \widetilde{K}_{st}^{i} \right) \\
&\quad + \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} v_{jt} \widehat{f}_j(X_{jt}) \left\{ \frac{1}{T} \sum_{s=1}^{T} \left[g_i(X_{it}) - g_i(X_{is})\right] \widetilde{K}_{st}^{i} \right\} \\
&\quad + \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} \left[g_i(X_{it}) - \widehat{g}_i(X_{it})\right] \left[g_j(X_{jt}) - \widehat{g}_j(X_{jt})\right] \widehat{f}_i(X_{it}) \widehat{f}_j(X_{jt}) \\
&=: \sum_{i=1}^{N} \sum_{j \neq i} \sum_{k=1}^{6} \rho_T(i, j, k), \qquad \textbf{(A.12)}
\end{aligned}$$

where $\widetilde{K}_{st}^{i} = \widetilde{K}_{X_{it},h}(X_{is})$. Furthermore, by Lemma A.2 and $\mu_1 = 0$, we have

$$\sup_{x \in R} \left| h\widetilde{K}_{x,h}(X_{it}) - K\left(\frac{X_{it} - x}{h}\right) \left[f_i(x)\mu_2 - o_P(1) \left(\frac{X_{it} - x}{h}\right) \right] \right| = o_P(1)$$

uniformly in $i \geq 1$. This implies that

$$\widetilde{K}_{x,h}(X_{it}) = \frac{\mu_2 f_i(x)}{h} K\left(\frac{X_{it} - x}{h}\right) + o_P\left(h^{-1}\right). \qquad \textbf{(A.13)}$$

Define $K_{st}^{i} = K\left((X_{it} - X_{is})/h\right)$. Then, the conclusion of Lemma A.3 is still valid if we replace $\widetilde{K}_{st}^{i}$ with $h^{-1} K_{st}^{i}$.

It follows from (A.12) and Lemmas A.3–A.6 that

$$NCU = \sqrt{\frac{T}{N(N-1)}} \sum_{i=1}^{N} \sum_{j \neq i} \widetilde{\rho}_{ij}$$

$$= \sqrt{\frac{1}{N(N-1)T}} \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} \left[\frac{\sigma_i(X_{it})\sigma_j(X_{jt}) f_i^2(X_{it}) f_j^2(X_{jt}) \mu_2^2 \mu_0^2}{\sigma_{i*}\, \sigma_{j*}} \right] u_{it} u_{jt}$$

$$+ o_P(1), \tag{A.14}$$

where σ_{i*} is as defined in Lemma A.6.

In view of (A.14), we need only to show that the leading term of NCU has a joint limit distribution of normal. If we let $U_{it} = (\mu_2 \mu_0 / \sigma_{i*}) \sigma_i(X_{it}) f_i^2(X_{it}) u_{it}$ in Lemma A.1, then conditions A2 and A4, equation (A.14), and Lemma A.1 imply that Theorem 3.1 holds as $T \to \infty$ and $N \to \infty$ simultaneously. ■

LEMMA A.3. *Assume that the conditions of Theorem 3.1 are satisfied. Then under H_0, we have*

$$\sum_{i=1}^{N} \sum_{j \neq i} \rho_T(i, j, 2) = o_P\left(N\sqrt{T}\right) \quad \text{and} \quad \sum_{i=1}^{N} \sum_{j \neq i} \rho_T(i, j, 4) = o_P\left(N\sqrt{T}\right). \tag{A.15}$$

LEMMA A.4. *Assume that the conditions of Theorem 3.1 are satisfied. Then, under H_0, we have*

$$\sum_{i=1}^{N} \sum_{j \neq i} \rho_T(i, j, k) = o_P\left(N\sqrt{T}\right), \quad \text{for } k = 3, 5, 6. \tag{A.16}$$

LEMMA A.5. *Assume that the conditions of Theorem 3.1 are satisfied. Then, under H_0, we have*

$$\sum_{i=1}^{N} \sum_{j \neq i} \rho_T(i, j, 1) = \sum_{i=1}^{N} \sum_{j \neq i} \sum_{t=1}^{T} u_{it} u_{jt} \sigma_i(X_{it}) \sigma_j(X_{jt}) f_i^2(X_{it}) f_j^2(X_{jt}) \mu_2^2 \mu_0^2 + o_P\left(N\sqrt{T}\right). \tag{A.17}$$

LEMMA A.6. *Assume that the conditions of Theorem 3.1 are satisfied. Then, under H_0, we have*

$$\frac{1}{T} \sum_{t=1}^{T} \overline{v}_{it}^2 = \sigma_{i*}^2 + o_P(1) \quad \text{with } 0 < \sigma_{i*}^2 = \mu_2^2 \mu_0^2 \int \sigma_i^2(x) f_i^5(x) dx < \infty$$

uniformly in $i \geq 1$.

The main technique in the proofs of Lemmas A.3–A.6 is to evaluate the order of the following form:

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{N} u_{it}u_{js}\right]^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} \mathbb{E}\left[u_{it}^2 u_{js}^2\right] \\
&\quad + \sum_{(i_1,i_2,j_1,j_2;t_1,t_2,s_1,s_2)\in A_{1212}} \mathbb{E}\left[u_{i_1t_1}u_{i_2t_2}u_{j_1s_1}u_{j_2s_2}\right] \\
&= O\left(N^2T^2\right), \qquad \textbf{(A.18)}
\end{aligned}
$$

where $A_{1212} = \{(i_1, i_2, j_1, j_2; t_1, t_2, s_1, s_2)$: at least one of i_1, i_2, j_1, j_2 is different from the others and (or) at least one of t_1, t_2, s_1, s_2 is different from the others$\}$. Condition A4 and Lemma B.1 in Appendix B are used to show that the second term on the right-hand side of the first equality in (A.18) is at most of the same order as the first term. In the detailed evaluation, one will need to consider cases where i_1, i_2, j_1, j_2 are either the same or different and t_1, t_2, s_1, s_2 are the same or different.

Proof of Theorem 3.2. We start with the case of $k = 1$. To simplify the notations, let $\tilde{\sigma}_i(X_{it}) = \sigma_i(X_{it}) f_i^2(X_{it})/\sigma_{i*}$, where σ_{i*} was defined in Lemma A.6. Following the proof of Theorem 3.1, we need only to show that under H_1

$$
\frac{1}{N(N-1)T}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\tilde{\sigma}_i(X_{it})\tilde{\sigma}_j(X_{jt})G(z_t,\beta_i)G(z_t,\beta_j) \xrightarrow{P} \psi(*), \qquad \textbf{(A.19)}
$$

$$
\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\sigma_i(X_{it})F(z_t,\beta_i)\left[\frac{1}{T}\sum_{s=1}^{T}\varepsilon_{js}\sigma_j(X_{js})\widetilde{K}_{st}^{j}\right] = o_P(R_{NT}), \qquad \textbf{(A.20)}
$$

$$
\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\sigma_i(X_{it})F(z_t,\beta_i)\left[\frac{1}{T}\sum_{s=1}^{T}\sigma_j(X_{js})F(z_s,\beta_j)\widetilde{K}_{st}^{j}\right] = o_P(R_{NT}), \qquad \textbf{(A.21)}
$$

$$
\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\left[\frac{1}{T^2}\sum_{s_1=1}^{T}\sigma_i(X_{is_1})F(z_{s_1},\beta_i)\widetilde{K}_{s_1t}^{i}\right]\left[\sum_{s_2=1}^{T}\sigma_j(X_{js_2})F(z_{s_2},\beta_j)\widetilde{K}_{s_2t}^{j}\right] = o_P(R_{NT}), \qquad \textbf{(A.22)}
$$

$$
\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\left[\frac{1}{T^2}\sum_{s_1=1}^{T}\sigma_i(X_{is_1})F(z_{s_1},\beta_i)\widetilde{K}_{s_1t}^{i}\right]\left[\sum_{s_2=1}^{T}\varepsilon_{js_2}\sigma_j(X_{js_2})\widetilde{K}_{s_2t}^{j}\right] = o_P(R_{NT}), \qquad \textbf{(A.23)}
$$

$$
\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\varepsilon_{it}\sigma_i(X_{it})\left[\frac{1}{T}\sum_{s=1}^{T}\sigma_j(X_{js})F(z_s,\beta_j)\widetilde{K}_{st}^{j}\right] = o_P(R_{NT}), \qquad \textbf{(A.24)}
$$

$$
\frac{1}{N}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N}F^2(z_t,\beta_i) \xrightarrow{P} 0, \qquad \textbf{(A.25)}
$$

where $F(z_t, \beta_i) = F_{NT}(z_t, \beta_i)$ and $R_{NT} = N\sqrt{T}$.

We first prove (A.20). By conditions A3 and A5 and Lemma B.1, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\sum_{i=1}^{N}\sum_{j\neq i}\sum_{t=1}^{T}\sigma_i(X_{it})F(z_t,\beta_i)\left[\frac{1}{T}\sum_{s=1}^{T}\varepsilon_{js}\sigma_j(X_{js})\widetilde{K}_{st}^{j}\right]\right\}^2 \\
&\quad \leq \frac{C}{T^2h^2}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j\neq i_1,i_2}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\left|F(z_{t_1},\beta_{i_1})F(z_{t_2},\beta_{i_2})\right|\right]\mathbb{E}\left[\varepsilon_{js}^2\right]\mathbb{E}\left[K_{st_1}^{j}K_{st_2}^{j}\right]
\end{aligned}
$$

$$\leq \frac{C}{T^2h^2}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j\neq i_1,i_2}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\sum_{s=1}^{T} h^2\mathbb{E}\left[\left|F(z_{t_1},\beta_{i_1})F(z_{t_2},\beta_{i_2})\right|\right]\mathbb{E}\left[\varepsilon_{js}^2\right]$$

$$\leq \frac{C}{T^2h^2}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j\neq i_1,i_2}\frac{T^2h^2}{NT^{1/2}} = O\left(N^2T^{-1/2}\right),$$

which, by Markov inequality, implies that (A.20) holds. The proofs of (A.21) and (A.24) are similar to that of (A.20).

We then prove (A.22). By condition A5 and Lemma B.1, we have

$$\mathbb{E}\left\{\left[\frac{1}{T}\sum_{s_1=1}^{T}\sigma_i(X_{is_1})F(z_{s_1},\beta_i)\widetilde{K}^i_{s_1,t}\right]\left[\frac{1}{T}\sum_{s_2=1}^{T}\sigma_j(X_{js_2})F(z_{s_2},\beta_j)\widetilde{K}^j_{s_2,t}\right]\right\}^2$$

$$\leq \frac{C}{T^4h^4}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}\mathbb{E}\left[F(z_{s_1},\beta_i)F(z_{t_1},\beta_i)F(z_{s_2},\beta_j)F(z_{t_2},\beta_j)\right]\mathbb{E}\left[K^i_{s_1t}K^i_{t_1t}K^j_{s_2t}K^j_{t_2t}\right]$$

$$= O\left(\frac{1}{N^2T}\right),$$

which implies that (A.22) holds. By the same argument, we can show that (A.23) holds.

For the proof of (A.19), by (3.8) and Lemma 6(a) in Phillips and Moon (1999), it suffices for us to show that for any $\epsilon > 0$,

$$\lim_{N,T\to\infty} P\left\{\left|\frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j\neq i}\left[\frac{1}{T}\sum_{t=1}^{T}\sigma_i(X_{it})\sigma_j(X_{jt})G(z_t,\beta_i)G(z_t,\beta_j)-\psi_{i,j}(*)\right]\right| > \epsilon\right\} = 0. \tag{A.26}$$

Let $Q_T(i,j) = \frac{1}{T}\sum_{t=1}^{T}\sigma_i(X_{it})\sigma_j(X_{jt})G(z_t,\beta_i)G(z_t,\beta_j)-\psi_{ij}(*)$, $i\neq j$. Then, to show (A.26), we need only to prove that

$$\lim_{N,T\to\infty} P\left\{\max_{1\leq i\neq j\leq N}|Q_T(i,j)| > \epsilon\right\} = 0 \quad \text{for any } \epsilon > 0.$$

Note that

$$P\left\{\max_{1\leq i\neq j\leq N}|Q_T(i,j)| > \epsilon\right\}$$

$$\leq \frac{N(N-1)}{\epsilon^{2q_1}}\max_{i,j}\mathbb{E}\left[|Q_T(i,j)|^{2q_1}\right]$$

$$\leq CN^2T^{-2q_1}\max_{1\leq i,j\leq N}\mathbb{E}\left[\left|\sum_{t=1}^{T}\sigma_i(X_{it})\sigma_j(X_{jt})G(z_t,\beta_i)G(z_t,\beta_j)\right|^{2q_1}\right].$$

By Lemma B.2 and the conditions of Theorem 3.2, we have

$$
\mathbb{E}\left[\left|\sum_{t=1}^{T}\sigma_i(X_{it})\sigma_j(X_{jt})G(z_t,\beta_i)G(z_t,\beta_j)\right|^{2q_1}\right]
$$
$$
\leq CT^{q_1}\max_{1\leq i,j\leq N}\left\{\mathbb{E}\left[\left|G(z_t,\beta_i)G(z_t,\beta_j)\right|^{4q_1+2\delta_1}\right]\right\}^{2q_1/(4q_1+2\delta_1)}\left\{\mathbb{E}\left[|\sigma_i(X_{it})|^{8q_1+4\delta_1}\right]\right\}^{2q_1/(8q_1+4\delta_1)}
$$
$$
= O\left(T^{q_1}\right),
$$

which implies $P\left\{\max_{1\leq i\neq j\leq N}|Q_T(i,j)|>\epsilon\right\} = O\left(\frac{N^2}{T^{q_1}}\right) = o(1)$ by $N = o\left(T^{q_1/2}\right)$. Hence, (A.26) holds, and this completes the proof for the case of $k=1$.

Now we turn to the proof for the case of $k=0$. Define $\widetilde{G}(z_t,\beta_i) = \left(N^{1/2}T^{1/4}G(z_t,\beta_i)\right)$. Then,

$$
F_{NT}(z_t,\beta_i) = G(z_t,\beta_i) = \frac{1}{N^{1/2}T^{1/4}}\left[N^{1/2}T^{1/4}G(z_t,\beta_i)\right] = \frac{1}{N^{1/2}T^{1/4}}\widetilde{G}(z_t,\beta_i).
$$

The proof for the case of $k=0$ is similar and details are omitted here. ■

APPENDIX B: Technical Lemmas

This Appendix provides some technical lemmas on α-mixing sequences and U-statistics. Such lemmas are used in the proofs of the main results. Lemma B.1 provides a useful inequality for α-mixing processes. This lemma is taken from Lemma A.1 in Gao (2007). Lemma B.2 gives the moment inequality for the partial sum of α-mixing random variables, which follows from Theorem 4.1 of Shao and Yu (1996). Lemma B.3 provides the fourth-order moment for U-statistics under some mild conditions. For the detailed proofs of the first two lemmas, we refer to the aforementioned literature. The proof of Lemma B.3 is available from the working paper by Chen et al. (2009).

LEMMA B.1. *Suppose that M_m^n are the σ-fields generated by a stationary α-mixing process $\{\xi_i\}$ with mixing coefficient $\alpha(\cdot)$. For some positive integers m, let $\eta_i \in M_{s_i}^{t_i}$ where $s_1 < t_1 < s_2 < t_2 < \cdots < t_m$ and $t_i - s_i > \tau$ for all i. Assume further that $||\eta_i||_{p_i}^{p_i} = \mathbb{E}|\eta_i|^{p_i} < \infty$ for some $p_i > 1$ with $Q := \sum_{i=1}^{l} 1/p_i < 1$. Then,*

$$
\left|\mathbb{E}\left[\prod_{i=1}^{l}\eta_i\right] - \prod_{i=1}^{l}\mathbb{E}[\eta_i]\right| \leq 10(l-1)\alpha(\tau)^{(1-Q)}\prod_{i=1}^{l}||\eta_i||_{p_i}.
$$

LEMMA B.2. *Let $2 < p < r \leq \infty$ and $\{Z_t\}$ be α-mixing with $\mathbb{E}[Z_t] = 0$ and $\mathbb{E}[|Z_t|^r] < \infty$. Define $S_n = \sum_{t=1}^{n} Z_t$ and suppose that the α-mixing coefficient $\alpha(k)$ satisfies $\alpha(k) = O\left(k^{-\beta^*}\right)$ for k large enough and $\beta^* > (pr)/2(r-p)$. Then,*

$$
\mathbb{E}[|S_n|^p] \leq C_0 n^{p/2}\max_{1\leq t\leq n}\left(\mathbb{E}|Z_t|^r\right)^{p/r},
$$

where C_0 is a positive constant.

LEMMA B.3. *Let* $\{U_i,\ 1 \le i \le n\}$ *be a sequence of random variables with* $\mathbb{E}\left[U_i\right] = 0$, $\mathbb{E}\left[U_i U_j\right] = 0$ *for all* $i \neq j$, *and* $\max_{i \ge 1} \mathbb{E}\left[U_i^8\right] < \infty$. *For each* $1 \le k \le 8$, *let* $(j_1, \ldots, j_k)$ *be any set of k positive integers satisfying* $1 \le j_l \le 8$, $1 \le l \le k$, *and* $\sum_{l=1}^{k} j_l = 8$. *Suppose that*

$$\sum_{k=5}^{8} \sum_{(i_1,\ldots,i_k)\in\Theta_k} \mathbb{E}\left[U_{i_1,1}^{j_1} \cdots U_{i_k,1}^{j_k}\right] = O\left(N^4\right), \tag{B.1}$$

where $\Theta_k = \{(i_1, \ldots, i_k) :\ 1 \le i_l \le N,\ \text{and } i_l, 1 \le l \le k, \text{are all different}\}$. *Then, for large enough* N,

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} \sum_{j \neq i} U_i U_j\right)^4\right] = O\left(N^4\right).$$