THE UNIVERSITY OF ADELAIDE AUSTRALIA

School of Mathematical Sciences
Discipline of Applied Mathematics

Financial Risk Measures

— The Theory and Applications of

Backward Stochastic Difference/Differential Equations with respect to the Single Jump Process

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Abstract

This thesis studies financial risk measures which dynamically assign a value to a risk at a future date which can be interpreted as the present value of a future monetary value. In particular, the theories of backward stochastic difference equations in discrete time and differential equations in continuous time (BSDE) with respect to a single jump process are developed. Based on these theories, some associated dynamic risk measures are defined.

Chapter 1 is an introduction to the background of BSDEs, risk measures, and the single jump process, and also outlines the structure of this thesis.

Part I considers backward stochastic difference equations related to a discrete finite time single jump process (Chapter 2) and backward stochastic differential equations related to a finite continuous time single jump process (Chapter 3). We prove the existence and uniqueness of solutions of these BSDEs under some assumptions. Comparison Theorems for these solutions are also given. Applications to the theory of nonlinear expectations are then investigated.

Part II considers some applications of the theories established in Part I. In Chapter 4, risk measures related to the solutions of backward stochastic difference equations with respect to a discrete time single jump process are defined and some simple numerical examples are given. In Chapter 5, we consider the question of an optimal transaction between two investors to minimize their risks. We define a dynamic entropic risk measure using backward stochastic differential equations related to a continuous time single jump process. The inf-convolution of dynamic entropic risk measures is a key transformation in solving the optimization problem.

Signed Statement

I, Bin Shen (Leo Shen) certify that this work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Dedication

To Amy

Chapter 1

Introduction

1.1 Overview

Risk is a facet of life faced everyday by both organizations and individuals. To mitigate risk it must be modelled and quantified. Mathematical finance derives and extends the mathematical models concerned with the financial markets. One of the important aims of mathematical finance is to determine the financial risk associated with any form of financing based upon various models. Risk measures were introduced in the literature to evaluate future losses, to give some criteria on the acceptability of risk exposures and also for pricing purposes.

The purpose of this thesis is to consider backward stochastic differential/difference equations (BSDE) with respect to the single jump process. One the one hand, jump processes play now a key role for modelling financial market fluctutations, such as risk management and option pricing [7]. The simplest jump process is a process with just one jump, i.e. a single jump process. Consequently it is also an important model. On the other hand, backward stochastic differential equations appear in numerous problems in finance: the theory of contingent claim valuation in a complete market, the pricing theory in an incomplete market, and recursive utility, especially in risk measures [11, 12, 13]. Risk measures can be defined using nonlinear expectations based on backward stochastic differential equations [22, 23, 24].

1.2 Risk measures in mathematical finance

1.2.1 Static risk measures

A financial position is a mapping X from some set Ω of possible future scenarios to the real numbers, where $X(\omega)$ is the future discounted net worth of a position at the end of the trading period if the scenario ω is realized. Write $\mathcal{X}$ for the set of financial positions. Following [15], a *static risk measure* $\rho(\cdot)$ is a mapping $\rho : \mathcal{X} \to \mathbb{R}$, which satisfies some of the following properties:

- monotonicity: $\rho(X) \geq \rho(Y), \forall X, Y \in \mathcal{X}$ and $X \leq Y$ a.s.;
- convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$,$\forall \alpha \in (0,1), \forall X, Y \in \mathcal{X}$
- positivity: $X \geq 0 \Rightarrow \rho(X) \leq \rho(0)$;
- constancy: $\rho(\alpha) = -\alpha, \forall \alpha \in \mathbb{R}$;
- translatability: $\rho(X + \beta) = \rho(X) - \beta, \forall \beta \in \mathbb{R}, \forall X \in \mathcal{X}$;
- positive homogeneity: $\rho(\alpha X) = \alpha\rho(X), \forall \alpha \geq 0, \forall X \in \mathcal{X}$;
- subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y), \forall X, Y \in \mathcal{X}$;
- lower semi-continuity: $\{X \in \mathcal{X} : \rho(X) \leq \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$.

A functional ρ is called a coherent risk measure, [1], if it satisfies monotonicity, translatability, positive homogeneity and subadditivity; it is a convex risk measure if it satisfies monotonicity, convexity, lower semi-continuity and $\rho(0) = 0$.

The monotonicity of risk measure captures a basic asymmetry in a financial position: the downside risk of a position is reduced if the payoff profile is increased, [16]. The translatability property shows a risk measure can be regarded as a capital requirement which, if added to the position and invested in a risk-free manner, makes the position acceptable, [18]. The convexity property implies diversification of investment strategies should not increase risk, [17]. Positive homogeneity decentralizes the task of managing the risk arising from a collection of different financial positions, [1]. Positivity means if a financial position always makes a profit, then its riskiness is smaller than the riskiness of the null position, [15]. Constancy shows that the riskiness of constant positions is simply the opposite of their net worth, [8]. Subadditivity encourages the diversification through portfolios of risks because the riskiness of a portfolio $(X+Y)$ should be smaller than the sum of the riskiness of the single positions X and Y. Convexity assures risk is not increased by diversification through portfolios originated by sums of single positions, [25].

1.2.2 Value-at-Risk

Currently Value-at-Risk, VaR, is a popular risk measure [15]. However, it suffers from various defects. Under normal market conditions and with no trading in the portfolio, Value-at-Risk measures the worst expected loss on a specific portfolio of financial assets. For a given portfolio and time horizon, Value-at-Risk is defined as the lowest quantile of the potential losses that can occur on the portfolio over the given time horizon. In the definition of Value-at-Risk, there are two major parameters that should be chosen in a way appropriate to the overall goal of risk measurement. One is the basic time period which can be a day, a month or a year. Another is the quantile (the confidence level) which is relatively high, typically either 95% or 99%. Once the two elements are given, Value-at-Risk gives an estimate of investment loss. There are three methods of calculating Value-at-Risk: the historical method, the variance-covariance method and the Monte Carlo simulation [10]. However, Value-at-Risk only controls the probability of a loss; it does not capture the size of such a loss if it occurs. Actually, it is far more important to worry about what happens when losses exceed Value-at-Risk. Because Value-at-Risk is not a convex measure of risk, it may penalize diversification instead of encouraging it. That means the Value-at-Risk of a combined portfolio can be larger than the sum of the Values-at-Risk of its components.

1.2.3 Conditional Value-at-Risk

Because of the undesirable mathematical characteristics of Value-at-Risk, such as the lack of convexity and subadditivity, Conditional Value-at-Risk is a better alternative risk measure than Value-at-Risk. It has the following properties: it is monotonic, convex, positively homogeneous and transition-equivariant. Conditional Value-at-Risk is the weighted average of Value-at-Risk and expected losses strictly exceeding Value-at-Risk. While a portfolio's Value-at-Risk is the maximum loss one expects to suffer at the confidence level by holding it over a period, a portfolio's Conditional Value-at-Risk is the average loss one expects to suffer, given that the loss is equal to or larger than its Value-at-Risk [14].

1.2.4 Dynamic risk measures and BSDE

A static risk measure as described above applies to a single-stage portfolio allocation problem. However, most investors make a sequence of portfolio allocations dynamically over time. Consequently they need time-consistent dynamic risk measures which are appropriate not only for the entire time horizon but also for the intermediate stages as the process evolves. We shall see that they can be defined using backward stochastic differential or difference equations.

A dynamic risk measure is a map such that:

- $\rho_t : \mathcal{X} \to \mathcal{L}^0(\mathcal{F}_t)$, for all $t \in [\,0, T\,]$;
- ρ_0 is a static risk measure;
- $\rho_T(X) = -X$ for all $X \in \mathcal{X}$;
- convexity: $\forall t \in [\,0, T\,], \rho_t$ is convex;
- positivity: $X \geq 0 \Rightarrow \forall t \in [\,0, T\,], \rho_t(X) \leq \rho_t(0)$;
- constancy: $\forall t \in [\,0, T\,], \forall c \in \mathbb{R}, \rho_t(c) = -c$;
- translatability: $\forall t \in [\,0, T\,], \forall X \in \mathcal{X}, \rho_t(X + a) = \rho_t(X) - a$;
- positive homogeneity: $\forall t \in [\,0, T\,], \forall \alpha \geq 0, \forall X \in \mathcal{X}, \rho_t(\alpha X) = \alpha \rho_t(X)$;
- subadditivity: $\forall t \in [\,0, T\,], \forall X, Y \in \mathcal{X}, \rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$.

Let $X = (X_t)_{t \geq 0}$ represent a stochastic process, $\mathcal{F}_t^X$ be the filtration generated by X during the time interval $[\,0, t\,]$, and $m\mathcal{F}_t^X$ be the set of all real valued $\mathcal{F}_t^X$–measurable random variables. Suppose $Y \in m\mathcal{F}_T^X$. Then $\mathcal{E}[\,Y|\mathcal{F}_t^X]$ is called an $\mathcal{F}_t$–consistent nonlinear expectation for each $0 \leq s \leq t \leq T$ and $Y, Z \in \mathcal{F}_T^X$ if it satisfies the following axioms [9]:

- monotonicity: $\mathcal{E}[\,Y|\mathcal{F}_t\,] \geq \mathcal{E}[\,Z|\mathcal{F}_t\,]$, a.s., if $Y \geq Z$, a.s.;
- constancy: $\mathcal{E}[\,Y|\mathcal{F}_t\,] = Y$, a.s.;
- time–consistency: $\mathcal{E}[\,\mathcal{E}[\,Y|\mathcal{F}_t\,] \mid \mathcal{F}_s\,] = \mathcal{E}[\,Y|\mathcal{F}_s\,]$, a.s.;

- zero–one law: for each $t, \mathcal{E}[\,1_A Y|\mathcal{F}_t\,] = 1_A\mathcal{E}[\,Y|\mathcal{F}_t\,]$, a.s.,$\forall A \in \mathcal{F}_t$;

- concavity: $\forall t \in [\,0, T\,], \mathcal{E}[\,Y|\mathcal{F}_t\,]$ is concave.

A dynamic risk measure $\rho : \mathcal{L}^2(\mathcal{F}_T) \to \mathbb{R}$ can be defined by setting $\rho_t(X) \triangleq \mathcal{E}[-X \,|\, \mathcal{F}_t]$, for all $X \in \mathcal{L}^2(\mathcal{F}_T)$, where $\mathcal{E}$ is a nonlinear expectation.

Risk measures can be defined using g–expectations which are nonlinear expectations given by solutions of backward stochastic differential equations depending on a function g [22, 23, 24].

Suppose g is an $\mathbb{R}$–valued, $\mathcal{F}_t$–adapted process

$$g = g(\omega, t, y, z) : \Omega \times [\,0, T\,] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}.$$

A backward stochastic differential equation (BSDE) is an equation of the form:

$$\begin{aligned} Y_t &= Y_0 - \int_{[0,t]} g(u, Y_u, Z_u)\, du + \int_{[0,t]} Z_u\, dM_u, \quad \forall t \in [0, T]; \\ Y_T &= X. \end{aligned} \tag{1.2.1}$$

In the work of Peng [23], M is a Brownian Motion. The solution (Y, Z) is required to be adapted to the forward filtration, and Z is required to be predictable. More general martingales M were considered by El Karoui and Huang [12]. Cohen and Elliott [5], [6] discussed backward stochastic differential and difference equations when the martingale term M is related to a finite state Markov chain or other finite state processes.

A solution of (1.2.1) is a pair (Y, Z) of adapted processes. In [21] it is shown that for a given terminal condition $X \in \mathcal{L}^2(\mathcal{F}_T)$ the equation (1.2.1) has a unique solution (Y, Z) if g satisfies the conditions:

- g is Lipschitz in (y, z), i.e. there exists a constant $C > 0$ such that $\forall t \in [\,0, T\,], \forall (y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^d, |g(t, y_0, z_0) - g(t, y_1, z_1)| \leq C(|y_0 - y_1| + \|z_0 - z_1\|)$.

- $g(\cdot, y, z) \in \mathcal{L}^2_{\mathcal{F}}(T; \mathbb{R})$ for any $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

- $g(t, y, 0) = 0$ for any $t \in [\,0, T\,]$ and $y \in \mathbb{R}$.

- $g(t, y, z)$ is continuous in t for any $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

- $g(t, \alpha y, \alpha z) = \alpha g(t, y, z)$ for any $t \in [\,0, T\,]$, $\alpha \geq 0$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

- $g(t, y_0+y_1, z_0+z_1) \leq g(t, y_0, z_0)+g(t, y_1, z_1)$ for any $t \in [\,0, T\,]$, $(y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^d$.

- g is convex in (y, z): $\forall t \in [\,0, T\,], \forall (y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^d, \forall \alpha \in (0, 1)$,
$g(t, \alpha y_0 + (1-\alpha)y_1, \alpha z_0 + (1-\alpha)z_1) \leq \alpha g(t, y_0, z_0) + (1-\alpha)g(t, y_1, z_1)$.

- g does not depend on y.

For any $X \in \mathcal{L}^2(\mathcal{F}_T)$, let $(Y_t^X, Z_t^X)_{t\in[\,0,T\,]} \in \mathcal{L}^2_{\mathcal{F}}(T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(T;\mathbb{R}^d)$ be the solution of the BSDE (1.2.1) with terminal condition X. The *g-expectation $\mathcal{E}_g$ of X* is defined by $\mathcal{E}_g[\,X\,] \triangleq Y_0^X$; for any $t \in [\,0, T\,]$ and the *conditional g–expectation of X given $\mathcal{F}_t$* is defined by $\mathcal{E}_g[\,X \mid \mathcal{F}_t\,] \triangleq Y_t^X$.

g–expectations then satisfy the following properties of nonlinear expectations [3]:

- If g is positively homogeneous in (y, z), then, for any $t \in [\,0, T\,]$,
$\mathcal{E}_g[\,\alpha X \mid \mathcal{F}_t\,] = \alpha \mathcal{E}_g[\,X \mid \mathcal{F}_t\,], \qquad \forall \alpha \geq 0, \forall X \in \mathcal{L}^2(\mathcal{F}_T)$.

- If g is sublinear in (y, z), then $\mathcal{E}_g$ and $\mathcal{E}_g[\,\cdot \mid \mathcal{F}_t\,]$ are sublinear.

- g is concave in (y, z) if and only if, for any $t \in [\,0, T\,], \mathcal{E}_g[\,\cdot \mid \mathcal{F}_t\,]$ is concave in $X \in \mathcal{L}^2(\mathcal{F}_T)$.

- g is positively homogeneous in (y, z) if and only if, for any $t \in [\,0, T\,]$,
$\mathcal{E}_g[\,\cdot \mid \mathcal{F}_t\,]$ is positively homogeneous in $X \in \mathcal{L}^2(\mathcal{F}_T)$.

- g is sublinear in (y, z) if and only if, for any $t \in [\,0, T\,], \mathcal{E}_g[\,\cdot \mid \mathcal{F}_t\,]$ is sublinear in $X \in \mathcal{L}^2(\mathcal{F}_T)$.

A dynamic risk measure can then be defined using a g–expectation by putting $\rho_g(x) = \mathcal{E}_g[-X \,|\, \mathcal{F}_t]$. The dynamic risk measure $(\rho_t)_{t\in[\,0,T\,]}$, which represents the riskiness at time t, monitors the riskiness of a position X at any intermediate t between the initial date 0 and the final T. In fact, two boundary conditions at times 0 and T are imposed on $(\rho_t)_{t\in[\,0,T\,]}$: ρ_0 has to be a static risk measure and ρ_T has to reduce to the opposite of the risky position, i.e. $\rho_T(X) = -X$. See [2].

A dynamic risk measure $(\rho_t)_{t\in[\,0,T\,]}$ is
— coherent if it satisfies positivity, translatability, positive homogeneity and subadditivity;
— convex if it satisfies dynamic convexity and $\rho(0) = 0$ for any $t \in [\,0, T\,]$;
— time-consistent if $\forall t \in [\,0, T\,], \forall X \in \mathcal{X}, \forall A \in \mathcal{F}_t, \rho_0[\,X1_A\,] = \rho_0[\,-\rho_t(X)1_A\,]$ [19].

Backward stochastic difference equations have been considered in [20], [4] and other works, particularly as numerical approximations and Monte Carlo simulation to continuous time processes. Cohen and Elliott [6] approach the backward stochastic difference equations related to a discrete time and finite state processes as entities in their own right, not as approximations and the related numerical methods.

1.3 Single jump process

1.3.1 The continuous finite time single jump process

A continuous time single jump process is defined as $X(\omega) = \{X_t(\omega), t \in [0, L]\}$, where L is a finite deterministic terminal time. $X(\omega)$ takes values in a measurable space $(\mathbb{E}, \mathcal{E})$ and remains at its initial point $z_0 \in \mathbb{E}$ until a random time $T(\omega)$, when it jumps to a new random position $z(\omega) \in \mathbb{E}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le L}, \mu)$ be a filtered probability space, where $\Omega = [0, L] \times \mathbb{E}$, $\mathcal{F} = \sigma\{\mathcal{B}([0, L]) \times \mathcal{E}\}$, $\mathcal{F}_t$ is the completed σ-field generated by $\{X_s\}, s \le t$ and $\mu : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$. We suppose that

$$\mu([0, L] \times \{z_0\}) = 0 = \mu(\{0\} \times \mathbb{E})$$

so that the probabilities of a zero jump and a jump at time zero are zero. A sample path of the single jump process X is

$$X_t(\omega) = \begin{cases} z_0, & 0 \le t < T(\omega) \le L; \\ z(\omega), & 0 < T(\omega) \le t \le L. \end{cases}$$

1.3.2 The discrete finite time single jump process

A discrete time single jump process is defined as $X(\omega) = \{X_t(\omega)$, where $t \in \{0, 1, \cdots, L\}\}$ and L is a finite deterministic terminal time. X takes values in a measurable space $(\mathbb{E}, \mathscr{E})$ and remains at its initial point $z_0 \in \mathbb{E}$ until a random time T, when it jumps to a new random position $z \in \mathbb{E}$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0 \le t \le L}, \mu)$ be the filtered probability space, where $\Omega = \{0, \cdots, L\} \times \mathbb{E}$, $\mathscr{F}$ is the σ–field generated by sets of the form $\{i\} \times A$, $i \in \{0, \cdots, L\}$, $A \in \mathscr{E}$, $\mathscr{F}_t$ is the completed σ-field generated by $\{X_s\}, s \le t$ and $\mu : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$. We again suppose that

$$\mu(\{0, 1, \cdots, L\} \times \{z_0\}) = 0 = \mu(\{0\} \times \mathbb{E})$$

so that the probabilities of a zero jump and a jump at time zero are zero. A sample path of the single jump process X in discrete time is

$$X_t(\omega) = \begin{cases} z_0, & \text{if } 0 \le t < T(\omega) \le L; \\ z(\omega), & \text{if } 0 < T(\omega) \le t \le L. \end{cases}$$

1.4 Structure of this thesis

This thesis consists of four published or submitted papers. Through these papers, the theories and applications of BSDEs with respect to the single jump process are developed and applied.

Part I of the thesis, Chapter 2 and 3, considers the theories of BSDE with respect to the single jump process. In Chapter 2, we define backward stochastic difference equations related to a discrete finite time single jump process. We prove the existence and uniqueness of solutions under some assumptions. A comparison theorem for these solutions is also given. Applications to the theory of nonlinear expectations are then investigated. In Chapter 3, we consider backward stochastic differential equations related to a finite continuous time single jump process. We prove the existence and uniqueness of solutions when the coefficients satisfy Lipschitz continuity conditions. A comparison theorem for these solutions is also given. Applications to the theory of nonlinear expectations are then investigated.

Part II of the thesis, Chapter 4 and 5, considers the applications of the theories established in Part I. In Chapter 4, some risk measures related to the solutions of backward stochastic difference equations with respect to a discrete time single jump process are defined and some simple numerical examples are given. In Chapter 5, we consider the question of an optimal transaction between two investors to minimize their risks. We define a dynamic entropic risk measure using backward stochastic differential equations related to a continuous time single jump process. The inf-convolution of dynamic entropic risk measures is a key transformation in solving the optimization problem.

Part I

Theory

Chapter 2

Discrete Time

Statement of Authorship

Backward Stochastic Difference Equations for a Single Jump Process

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Shen, Leo (Candidate)
Initially developed theory, proved results and wrote manuscript.

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Elliott, Robert J. (Supervisor)
Contributed to proofs of results, participated in discussions of work and manuscript evaluation.

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Chapter 3

Continuous Time

Statement of Authorship

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Initially developed theory, proved results and wrote manuscript.

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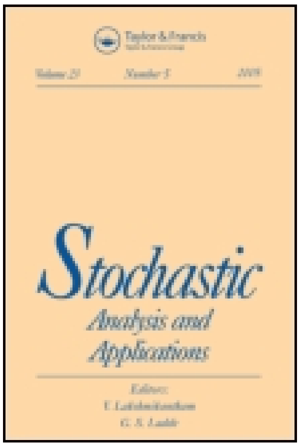

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Backward Stochastic Differential Equations for a Single Jump Process

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Part II

Applications

Chapter 4

Discrete Time

Statement of Authorship

How to measure risk

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Chapter 5

Continuous Time

Statement of Authorship

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01 May 2012

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Dear Shen:

Paper No: AP/14232
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By: LEO SHEN, ROBERT J. ELLIOTT

I am glad to say that the latest version of your paper (#AP/14232_2 received on 5 March 2012) has now been accepted and is scheduled for publication in *Journal of Applied Probability* Vol. 49 No. 4 (December 2012).

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OPTIMAL DESIGN OF DYNAMIC DEFAULT RISK MEASURES

LEO SHEN,* *University of Adelaide*

ROBERT ELLIOTT,** *University of Adelaide*

Abstract

We consider the question of an optimal transaction between two investors to minimize their risks. We define a dynamic entropic risk measure using backward stochastic differential equations related to a continuous time single jump process. The inf-convolution of dynamic entropic risk measures is a key transformation in solving the optimization problem.

Keywords: Inf-convolution; Dynamic entropic risk measure; Single jump process; Backward stochastic differential equation

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1. Introduction

This paper considers the optimal structure of a contract depending on a non-tradable risk related to a non-financial risk, such as natural catastrophe. The papers of Barrieu and El Karoui [1], [2] discuss a related problem in a continuous diffusion setting. In an earlier paper [13] we have constructed backward stochastic differential equations associated with a single jump process. This process might relate to a natural disaster or default. Our results could describe how risk should be optimally allocated between

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an insurer and the insured. In Section 2 below we first review risk measures used in mathematical finance, including static and dynamic risk measures. We next recall results relating to backward stochastic differential equations (BSDEs) associated with a finite horizon continuous time, single jump process developed in [13]. Then we introduce the dynamic entropic risk measure based on the solution of a BSDE and generate new dynamic risk measures as the inf-convolution of dynamic entropic risk measures. Finally we solve the problem of the optimal structure.

2. Static and dynamic risk measures

Our random variables and processes will be defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Our processes will be defined on $[0, T]$, where T is finite and deterministic.

2.1. Static risk measures

Suppose $\mathcal{X}$ denotes a set of financial positions, that is, $\mathcal{X}$ is the set of bounded, $\mathcal{F}_T$ measurable random variables. Following [9], a *static risk measure* $\rho(\cdot)$ is a mapping $\rho : \mathcal{X} \to \mathbb{R}$, which satisfies some of the following properties for all X, Y in $\mathcal{X}$:

- monotonicity: $\rho(X) \geq \rho(Y)$, if $X \leq Y$ a.s.;
- convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\forall \alpha \in (0, 1)$;
- positivity: $X \geq 0$ a.s. $\Rightarrow \rho(X) \leq \rho(0)$;
- constancy: $\rho(\alpha) = -\alpha, \forall \alpha \in \mathbb{R}$;
- translatability: $\rho(X + \beta) = \rho(X) - \beta, \forall \beta \in \mathbb{R}$;
- subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
- lower semi-continuity: $\{X \in \mathcal{X} : \rho(X) \leq \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$.

A functional ρ is called a convex risk measure if it satisfies monotonicity, convexity, lower semi-continuity and $\rho(0) = 0$. The convexity property implies diversification of investment strategies should not increase risk, [10].

Example 2.1. For any X in $\mathcal{X}$, an important example of a convex risk measure is the *entropic risk measure*:

$$e^{\gamma}(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \left(\mathbb{E}_{\mathbb{Q}}(-X) - \gamma h(\mathbb{Q}\,|\,\mathbb{P})\right) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{1}{\gamma}X\right)\right). \tag{1}$$

Here γ is the risk tolerance coefficient, $\mathcal{M}_1$ is the set of all probability measures on the considered space and $h(\mathbb{Q}\,|\,\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ with respect to the probability of $\mathbb{P}$, which is defined as

$$h(\mathbb{Q}\,|\,\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right), & \text{if } \mathbb{Q} \ll \mathbb{P}; \\ +\infty, & \text{otherwise.} \end{cases}$$

Another convex risk measure is the inf-convolution of convex functionals. This is established in the following theorem. For the proof see [1]:

Theorem 2.1. *Let ρ_1 and ρ_2 be two convex risk measures. $\rho_{1,2}$, the inf-convolution of ρ_1 and ρ_2, is defined as:*

$$\rho_{1,2}(X) = \rho_1 \Box \rho_2(X) = \inf_{S\in\mathcal{X}}\{\rho_1(X-S) + \rho_2(S)\}.$$

We assume that $\rho_{1,2}(0) > -\infty$. Then $\rho_{1,2}$ is a convex risk measure, which is finite for all $X \in \mathcal{X}$.

2.2. Dynamic risk measures

A static risk measure as described above applies to a single-stage portfolio allocation problem. However, most investors make portfolio allocations dynamically over time. Consequently they need time-consistent dynamic risk measures which are appropriate not only for the final time horizon but also for intermediate times as the process evolves. In fact, dynamic risk measures can be defined using backward stochastic differential, or in discrete time difference, equations.

A dynamic risk measure is a map satisfying some of the conditions:

- $\rho_t : \mathcal{X} \to \mathcal{L}^0(\mathcal{F}_t)$, for all $t \in [\,0, T\,]$;
- ρ_0 is a static risk measure;
- $\rho_T(X) = -X$ for all $X \in \mathcal{X}$;
- convexity: for all $t \in [\,0, T\,], \rho_t$ is a convex risk measure;
- positivity: $X \geq 0 \Rightarrow$ for all $t \in [\,0, T\,], \rho_t(X) \leq \rho_t(0)$ a.s.;

- constancy: for all $t \in [0, T]$ and all $c \in \mathbb{R}, \rho_t(c) = -c$;
- translatability: for all $t \in [0, T]$ and all $X \in \mathcal{X}, \rho_t(X + a) = \rho_t(X) - a$ a.s.;
- subadditivity: for all $t \in [0, T], X, Y \in \mathcal{X}, \rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$.

3. Backward stochastic differential equations for the single jump process

Suppose g is an $\mathbb{R}$–valued, $\mathcal{F}_t$–adapted process

$$g = g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$$

satisfying suitable conditions.

A backward stochastic differential equation (BSDE) is an equation of the form:

$$\begin{aligned} Y_t &= Y_0 - \int_{[0,t]} g(u, Y_u, Z_u)\, du + \int_{[0,t]} Z_u\, dM_u, \quad \forall t \in [0, T]; \\ Y_T &= X. \end{aligned} \tag{2}$$

In the work of Peng [12] and Pardoux and Peng [11], M is a Brownian Motion. A solution of (2) is a pair (Y, Z) of adapted processes. In [11] it is shown that for a given terminal condition $X \in \mathcal{L}^2(\mathcal{F}_T)$ the equation (2) has a unique solution (Y, Z) if g satisfies some regular conditions. The solution (Y, Z) is required to be adapted to the forward filtration, and Z is required to be predictable. More general martingales M were considered by El Karoui and Huang [7]. Cohen and Elliott [4], [5] discussed backward stochastic differential and difference equations when the martingale term M is related to a finite state Markov chain or some other finite state processes.

For appropriate coefficients g, a general dynamic risk measure ρ can be defined using the solutions of the BSDE (2) by putting $\rho_t(X) = -Y_t$. The dynamic risk measure $(\rho_t)_{t \in [0,T]}$ then provides a measure of risk of a position X at intermediate times t between the initial time 0 and the final T. Depending on the properties of g, ρ_t will be a dynamic risk measure. ρ_T will be the opposite of the final risky position, i.e. $\rho_T(X) = -X$. See [3].

3.1. The continuous finite time single jump process

Consider a continuous finite time single jump process $W(\omega) = \{W_t(\omega), t \in [0, L]\}$, where L is a finite deterministic terminal time. In fact, $W(\omega)$ remains at 0 until a random time $\tau(\omega)$ (where $0 < \tau(\omega) \leq L$ a.s.), when it jumps to 1. τ might model the time of an insurance event or a default.

Then W can be defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq L}, \mu)$, where $\Omega = [0, L]$, $\mathcal{F} = \mathcal{B}([0, L])$, $\mathcal{F}_t$ is the completed σ-field generated by $\{W_s, s \leq t\}$ and $\mu : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$. We suppose that the probability of a jump at time zero is zero.

Write F_t for the probability that $\tau \in]t, L]$. Then F_t is monotonic and non-increasing. We suppose F_t is continuous.

For $t \in]0, L]$, write

$$p_t = \mathbb{I}_{\tau(\omega)\leq t},$$
$$\tilde{p}_t = \int_{]0,\tau(\omega)\wedge t]} \frac{1}{F_s} d(-F_s).$$

Then $q_t = p_t - \tilde{p}_t$ is an $\mathcal{F}_t$ martingale. ([6], [8])

Consider the set $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ of functions such that $\int_\Omega |f|d\mu < +\infty$. For $g \in \mathcal{L}^1(\mu)$ we define the Stieltjes integrals, with $\Omega = [0, L]$,

$$\int_{[0,L]} g(u)dp_u = g(\tau(\omega)),$$
$$\int_{[0,L]} g(u)d\tilde{p}_u = \int_{]0,\tau(\omega)]} g(u)\frac{1}{F_u} d(-F_u).$$

Put

$$\int_{[0,L]} g(u)dq_u = \int_{[0,L]} g(u)dp_u - \int_{[0,L]} g(u)d\tilde{p}_u$$

and

$$\int_{]0,t]} g(u)dq_u = \int_{[0,L]} \mathbb{I}_{u\leq t} g(u)dq_u.$$

We have the following Martingale Representation Theorem ([6], [8]).

Theorem 3.1. *For any $\mathcal{F}_t$ martingale $\mathcal{M}_t$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq L}, \mu)$, there exists $g \in \mathcal{L}^1(\mu)$ such that $\mathcal{M}_t = \mathcal{M}_0 + \mathcal{M}_t^g$ a.s., where $\mathcal{M}_t^g = \int_{]0,t]} g(u)dq_u$.*

3.2. Backward stochastic differential equations

A backward stochastic differential equation (BSDE) based on the martingale random measure q is an equation of the form

$$Y_t + \int_{]t,L]} H(\omega, u, Z_u(\cdot))d(-F_u) + \int_{]t,L]} Z_u dq_u = Q \tag{3}$$

for $t \in [0, L]$.

Here, H is an adapted function $H : \Omega \times [0, L] \times \mathbb{R} \to \mathbb{R}$. A solution of the BSDE (3) is a pair of adapted processes (Y, Z) which satisfies (3) with $Y_L(\omega) = Q(\omega)$ for $\omega \in \Omega$. We assume Y_u is left continuous. Also we suppose $H(\omega, u, Z_u(\cdot)) \in \mathcal{L}^2(\mathcal{F}_u)$ for all u.

Theorem 3.2. *Assume that H is Lipschitz continuous as follows: there exists $c \in \mathbb{R}^+$ such that for all $u \in [0, L]$*

$$\left|H(\omega, u, Z_u^1(\cdot)) - H(\omega, u, Z_u^2(\cdot))\right| \leq c \left|Z_u^1 - Z_u^2\right|. \tag{4}$$

Then for any $\mathcal{F}_L$-measurable terminal condition Q, the BSDE (3) *has an adapted unique solution* (Y, Z). (See Shen and Elliott [13].)

4. Optimal design problem

In the following, we focus on an optimal transaction between two economic agents, respectively denoted by A and B, who exist in an uncertain universe modeled by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq L}, \mu)$. In the work of Barrieu and El Karoui [2], a problem is considered where uncertainty is modeled by a Brownian filtration.

Suppose agent A invests a dollars in a defaultable zero-coupon bond with maturity T $(0 < T < L)$ at time 0. Agent A is exposed towards a non-hedgeable risk associated with the possible default. Default might occur at a random time τ, (where τ is defined on the above probability space $(\Omega, \mathcal{F}, \mu)$ and $0 < \tau \leq L$). For $t \in]0, T]$, the time-t value X of the defaultable zero-coupon bond, with maturity T, deterministic interest rate $(r(s); s \geq 0)$ and constant rebate δ $(0 < \delta < 1)$, is defined as:

- The payment of $a \exp(\int_{]0,T]} r(s)ds)$ at time T if default τ has not occurred before time T.
- A payment of $a\delta \exp(\int_{]0,T]} r(s)ds)$, made at maturity, if the default time $\tau \leq T$.

That is,

$$X = a \exp\left(\int_{]0,T]} r(s)ds\right)(\mathbb{I}_{\tau > T} + \delta \mathbb{I}_{\tau \leq T}).$$

Agent A wishes to issue a financial product $S(\tau)$ and sell it to agent B for a forward price paid at time T, denoted by π to reduce his exposure.

4.1. Optimal structure in a static framework

Suppose that both agents assess the risk associated with their respective positions using an entropic risk measure as defined by (1), denoted respectively e^{γ} and $e^{\gamma'}$. Here agent A (resp. B) has risk tolerance γ (resp. γ').

The issuer, agent A, wants to determine the structure (S, π) as to minimize his global risk measure

$$\inf_{S,\pi} e^{\gamma}(X - S + \pi)$$

with the constraint

$$e^{\gamma'}(S - \pi) \leq e^{\gamma'}(0) = 0.$$

Using the translatability property in Section 2.1 and binding the constraint at the optimum, the pricing rule of the S-structure is fully determined by the buyer as

$$\pi^* = -e^{\gamma'}(S).$$

Using again the translatability property, the optimization program simply becomes

$$\inf_{S} \left(e^{\gamma}(X - S) + e^{\gamma'}(S) \right).$$

4.2. Solving the inf-convolution in a dynamic framework

We extend the notion of static entropic risk measure defined by (1) to a dynamic one on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq L}, \mu)$.

For $t \in]0, T]$, consider the martingale

$$M_t^{\gamma} = \mathbb{E}\left[\exp\left(-\frac{X - S}{\gamma} \right) \middle| \mathcal{F}_t \right],$$

where the risk tolerance coefficient γ is strictly positive. Define the dynamic entropic risk measure associated with receiving X and paying S at time T

$$e_t^{\gamma}(X - S) = \gamma \log M_t^{\gamma}.$$

We now prove the following result:

Theorem 4.1. $(-e_t^{\gamma}(X - S), Z_t^{\gamma})$ *is the solution of the following BSDE*

$$-e_t^{\gamma}(X - S) + \int_{]t,T]} H^{\gamma}(\omega, u, Z_u^{\gamma}(\cdot))d(-F_u) + \int_{]t,T]} Z_u^{\gamma} dq_u = X - S \tag{5}$$

where

$$H^{\gamma}(\omega, t, Z_t^{\gamma}(\cdot)) = \frac{\mathbb{I}_{t \leq \tau}}{F_t} \left(Z_t^{\gamma} + \gamma \exp\left(-\frac{Z_t^{\gamma}}{\gamma} \right) - \gamma \right).$$

Proof. We shall prove $-e_t^\gamma(X-S)$ is the solution of the BSDE (5).

Clearly, $e_T^\gamma(X-S) = -(X-S)$.

By the Martingale Representation Theorem for the single jump process ([6], [8]), there exists a unique $\varphi^\gamma \in \mathcal{L}^1(\mu)$, such that for $t \in]0,T]$,

$$M_t^\gamma = M_0^\gamma + \int_{]0,t]} \varphi_s^\gamma dq_s.$$

By the Itô formula ([8]),

$$\begin{aligned}
e_t^\gamma(X-S) &= \gamma \log M_0^\gamma + \gamma \int_{]0,t]} \frac{1}{M_{u-}^\gamma} \varphi_u^\gamma (dp_u - d\tilde{p}_u) \\
&+ \gamma \sum_{0<u\le t} (\log M_u^\gamma - \log M_{u-}^\gamma - \frac{1}{M_{u-}^\gamma} \triangle M_u^\gamma) \\
&= \gamma \log M_0^\gamma + \gamma \left(\mathbb{I}_{\tau\le t} \frac{\varphi_\tau^\gamma}{M_{\tau-}^\gamma} - \int_{]0,\tau\wedge t]} \frac{\varphi_u^\gamma}{M_{u-}^\gamma} \frac{1}{F_u} d(-F_u) \right) \\
&+ \gamma \mathbb{I}_{\tau\le t} \left(\log\left(1 + \frac{\varphi_\tau^\gamma}{M_{\tau-}^\gamma}\right) - \frac{\varphi_\tau^\gamma}{M_{\tau-}^\gamma} \right) \\
&= \gamma \log M_0^\gamma - \gamma \int_{]0,\tau\wedge t]} \frac{\varphi_u^\gamma}{M_{u-}^\gamma} \frac{1}{F_u} d(-F_u) \\
&+ \gamma \int_{]0,t]} \log\left(1 + \frac{\varphi_u^\gamma}{M_{u-}^\gamma}\right) dp_u \\
&= \gamma \log M_0^\gamma + \gamma \int_{]0,\tau\wedge t]} \log\left(1 + \frac{\varphi_u^\gamma}{M_{u-}^\gamma}\right) \frac{1}{F_u} - \frac{\varphi_u^\gamma}{M_{u-}^\gamma} \frac{1}{F_u} d(-F_u) \\
&+ \gamma \int_{]0,t]} \log\left(1 + \frac{\varphi_u^\gamma}{M_{u-}^\gamma}\right) dq_u.
\end{aligned}$$

Define

$$Z_u^\gamma = -\gamma \log\left(1 + \frac{\varphi_u^\gamma}{M_{u-}^\gamma}\right), \tag{6}$$

then

$$\frac{\varphi_u^\gamma}{M_{u-}^\gamma} = \exp\left(-\frac{Z_u^\gamma}{\gamma}\right) - 1.$$

For the expression for φ^γ, see Appendix A.

Hence,

$$\begin{aligned}e_t^\gamma(X-S) &= \gamma \log M_0^\gamma - \int_{]0,\tau\wedge t]} \left(\frac{Z_u^\gamma}{F_u} + \frac{\gamma}{F_u}\left(\exp\left(-\frac{Z_u^\gamma}{\gamma}\right) - 1\right)\right) d(-F_u) \\ &- \int_{]0,t]} Z_u^\gamma dq_u \\ &= \gamma \log M_0^\gamma - \int_{]0,t]} \frac{\mathbb{I}_{u\leq\tau}}{F_u}\left(Z_u^\gamma + \gamma\exp\left(-\frac{Z_u^\gamma}{\gamma}\right) - \gamma\right) d(-F_u) \\ &- \int_{]0,t]} Z_u^\gamma dq_u.\end{aligned}$$

Write

$$H^\gamma(\omega, u, Z_u^\gamma(\cdot)) = \frac{\mathbb{I}_{u\leq\tau}}{F_u}\left(Z_u^\gamma + \gamma\exp\left(-\frac{Z_u^\gamma}{\gamma}\right) - \gamma\right),$$

then

$$e_t^\gamma(X-S) = \gamma \log M_0^\gamma - \int_{]0,t]} H^\gamma(\omega, u, Z_u^\gamma(\cdot)) d(-F_u) - \int_{]0,t]} Z_u^\gamma dq_u.$$

Since

$$\begin{aligned}e_T^\gamma(X-S) &= -(X-S) \\ &= \gamma \log M_0^\gamma - \int_{]0,T]} H^\gamma(\omega, u, Z_u^\gamma(\cdot)) d(-F_u) - \int_{]0,T]} Z_u^\gamma dq_u,\end{aligned}$$

then

$$-e_t^\gamma(X-S) + \int_{]t,T]} H^\gamma(\omega, u, Z_u^\gamma(\cdot)) d(-F_u) + \int_{]t,T]} Z_u^\gamma dq_u = X - S.$$

By Theorem 3.2, $(-e_t^\gamma(X-S), Z_t^\gamma)$ is the unique solution of BSDE (5) with terminal condition $X - S$. □

We now discuss the inf-convolution of two entropic risk measures.

Similarly to the above, for γ' define:

$$M_t^{\gamma'} = \mathbb{E}\left[\exp\left(-\frac{S}{\gamma'}\right)\middle|\mathcal{F}_t\right]$$

and

$$e_t^{\gamma'}(S) = \gamma' \log M_t^{\gamma'}.$$

Then as above there exists a unique $\varphi^{\gamma'} \in \mathcal{L}^1(\mu)$, such that for $t \in]0,T]$,

$$M_t^{\gamma'} = M_0^{\gamma'} + \int_{]0,t]} \varphi_s^{\gamma'} dq_s.$$

For the expression for $\varphi^{\gamma'}$, see Appendix A.

Also from Theorem 4.1

$$-e_t^{\gamma'}(S) + \int_{]t,T]} H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot))d(-F_u) + \int_{]t,T]} Z_u^{\gamma'} dq_u = S, \tag{7}$$

where

$$Z_u^{\gamma'} = -\gamma' \log\left(1 + \frac{\varphi_u^{\gamma'}}{M_{u-}^{\gamma'}}\right), \tag{8}$$

and

$$H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)) = \frac{\mathbb{I}_{u \le \tau}}{F_u}\left(Z_u^{\gamma'} + \gamma' \exp\left(-\frac{Z_u^{\gamma'}}{\gamma'}\right) - \gamma'\right).$$

$e_t^{\gamma'}(S)$ is the dynamic entropic risk measure of S when the risk tolerance is γ'.

We now study for any $t \in]0, T]$ the inf-convolution of the dynamic entropic risk measures e_t^{γ} and $e_t^{\gamma'}$. As in Theorem 2.1 this is defined as

$$\left(e^{\gamma} \square e^{\gamma'}\right)_t (X) = \inf_S \left(e_t^{\gamma}(X - S) + e_t^{\gamma'}(S)\right).$$

This quantity describes the optimum minimal total remaining risk for the two investors if A buys an insurance product of value S from B.

Write $Z_u = Z_u^{\gamma} + Z_u^{\gamma'}$, then we have

$$\begin{aligned} &H^{\gamma}(\omega, u, Z_u^{\gamma}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)) \\ &\qquad = H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot)). \end{aligned}$$

Define

$$H^{\gamma} \square H^{\gamma'}(\omega, u, Z_u(\cdot)) = \inf_{Z_u^{\gamma'}} \left(H^{\gamma}(\omega, u, Z_u(\cdot) - Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega, u, Z_u^{\gamma'}(\cdot))\right).$$

We now prove the following theorem:

Theorem 4.2.

$$H^{\gamma} \square H^{\gamma'}(\omega, u, Z_u(\cdot)) = H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot)). \tag{9}$$

Also

$$\begin{aligned} \left(e^{\gamma} \square e^{\gamma'}\right)_t (X) &= \int_{]t,T]} H^{\gamma+\gamma'}(\omega, u, Z_u(\cdot))d(-F_u) \\ &\quad + \int_{]t,T]} Z_u dq_u - X \\ &= \left(e^{\gamma+\gamma'}\right)_t (X). \end{aligned} \tag{10}$$

Proof. Adding (5) and (7), we have

$$e_t^{\gamma}(X-S)+e_t^{\gamma'}(S)=\int_{]t,T]}\left(H^{\gamma}(\omega,u,Z_u^{\gamma}(\cdot))+H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\right)d(-F_u)$$
$$+\int_{]t,T]}\left(Z_u^{\gamma}+Z_u^{\gamma'}\right)dq_u-X.$$

With $Z_u=Z_u^{\gamma}+Z_u^{\gamma'}$, then

$$\begin{aligned}&e_t^{\gamma}(X-S)+e_t^{\gamma'}(S)\\&=\int_{]t,T]}\left(H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{\gamma'}(\cdot))+H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\right)d(-F_u)\\&+\int_{]t,T]}Z_u dq_u-X.\end{aligned}\tag{11}$$

Consider the functional

$$\begin{aligned}&H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{\gamma'}(\cdot))+H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\\&=\frac{\mathbb{I}_{u\le\tau}}{F_u}\left(Z_u+\gamma\exp\left(-\frac{Z_u-Z_u^{\gamma'}}{\gamma}\right)+\gamma'\exp\left(-\frac{Z_u^{\gamma'}}{\gamma'}\right)-\gamma-\gamma'\right).\end{aligned}\tag{12}$$

This is a convex function with respect to $Z_u^{\gamma'}$, since the second derivative of (12) with respect to $Z_u^{\gamma'}$ is, for each ω,

$$\frac{\mathbb{I}_{u\le\tau}}{F_u}\left(\frac{1}{\gamma}\exp\left(-\frac{Z_u-Z_u^{\gamma'}}{\gamma}\right)+\frac{1}{\gamma'}\exp\left(-\frac{Z_u^{\gamma'}}{\gamma'}\right)\right)\ge 0.$$

Therefore, for each ω, the minimum of (12) with respect to $Z_u^{\gamma'}$ occurs when the first derivative of (12) with respect to $Z_u^{\gamma'}$ is zero. That is when

$$\frac{\mathbb{I}_{u\le\tau}}{F_u}\left(\exp\left(-\frac{Z_u-Z_u^{\gamma'}}{\gamma}\right)-\exp\left(-\frac{Z_u^{\gamma'}}{\gamma'}\right)\right)=0.\tag{13}$$

Write $Z_u^{*\gamma'}$ for the value at which the minimum is attained. Clearly, $Z_u^{*\gamma'}$ is unique, and

$$Z_u^{*\gamma'}=\frac{\gamma'}{\gamma+\gamma'}Z_u.$$

Therefore

$$\begin{aligned}H^{\gamma}\Box H^{\gamma'}(\omega,u,Z_u(\cdot))&=\inf_{Z_u^{\gamma'}}\left(H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{\gamma'}(\cdot))+H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\right)\\&=H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{*\gamma'}(\cdot))+H^{\gamma'}(\omega,u,Z_u^{*\gamma'}(\cdot))\\&=\frac{\mathbb{I}_{u\le\tau}}{F_u}\left(Z_u+(\gamma+\gamma')e^{-\frac{Z_u}{\gamma+\gamma'}}-(\gamma+\gamma')\right)\\&=H^{\gamma+\gamma'}(\omega,u,Z_u(\cdot)).\end{aligned}\tag{14}$$

This establishes equation (9).

By (11) and (9), we obtain

$$e_t^\gamma(X-S)+e_t^{\gamma'}(S) \geq \int_{]t,T]} \left(H^{\gamma+\gamma'}(\omega,u,Z_u(\cdot))\right) d(-F_u) + \int_{]t,T]} Z_u dq_u - X,$$

therefore

$$\inf_S \left(e_t^\gamma(X-S)+e_t^{\gamma'}(S)\right) \geq \int_{]t,T]} \left(H^{\gamma+\gamma'}(\omega,u,Z_u(\cdot))\right) d(-F_u) + \int_{]t,T]} Z_u dq_u - X. \tag{15}$$

Take $S^* = \frac{\gamma'}{\gamma+\gamma'}X$. We shall show that

$$e_t^\gamma(X-S^*)+e_t^{\gamma'}(S^*) = \int_{]t,T]} \left(H^{\gamma+\gamma'}(\omega,u,Z_u(\cdot))\right) d(-F_u) + \int_{]t,T]} Z_u dq_u - X. \tag{16}$$

In fact, with $S^* = \frac{\gamma'}{\gamma+\gamma'}X$,

$$\frac{X-S^*}{\gamma} = \frac{S^*}{\gamma'}.$$

Therefore, the martingales

$$M_t^\gamma = \mathbb{E}\left[\exp\left(-\frac{X-S^*}{\gamma}\right)\middle|\mathcal{F}_t\right]$$

and

$$M_t^{*\gamma'} = \mathbb{E}\left[\exp\left(-\frac{S^*}{\gamma'}\right)\middle|\mathcal{F}_t\right]$$

are equal, as well the integrands $\varphi^\gamma, \varphi^{*\gamma'}$ in their martingale representations. Then with

$$Z_u^\gamma = -\gamma\log\left(1+\frac{\varphi_u^\gamma}{M_{u-}^\gamma}\right)$$

and

$$Z_u^{*\gamma'} = -\gamma'\log\left(1+\frac{\varphi_u^{*\gamma'}}{M_{u-}^{*\gamma'}}\right),$$

we have

$$\gamma Z_u^{*\gamma'} = \gamma' Z_u^\gamma.$$

With

$$Z_u = Z_u^{\gamma} + Z_u^{*\gamma'}$$
$$= \frac{\gamma+\gamma'}{\gamma'} Z_u^{*\gamma'},$$

we have

$$Z_u^{*\gamma'} = \frac{\gamma'}{\gamma+\gamma'} Z_u.$$

Consequently, as in (9) this $Z_u^{*\gamma'}$ is such that

$$\begin{aligned}
&\inf_{Z_u^{\gamma'}} \Big(H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\Big)\\
&= H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{*\gamma'}(\cdot)) + H^{\gamma'}(\omega,u,Z_u^{*\gamma'}(\cdot))\\
&= H^{\gamma}\Box H^{\gamma'}(\omega,u,Z_u(\cdot))\\
&= H^{\gamma+\gamma'}(\omega,u,Z_u(\cdot)).
\end{aligned}$$

This establishes (10).

We have established that for all $t \in]0,T]$, when $S = S^*$ and $Z_t^{\gamma'} = Z_t^{*\gamma'}$, we have

$$\begin{aligned}
&\inf_{S} \Big(e_t^{\gamma}(X-S) + e_t^{\gamma'}(S)\Big) \qquad (17)\\
&= e_t^{\gamma}(X-S^*) + e_t^{\gamma'}(S^*)\\
&= \int_{]t,T]} \inf_{Z_u^{\gamma'}} \Big(H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{\gamma'}(\cdot)) + H^{\gamma'}(\omega,u,Z_u^{\gamma'}(\cdot))\Big)\, d(-F_u)\\
&+ \int_{]t,T]} Z_u dq_u - X\\
&= \int_{]t,T]} H^{\gamma}(\omega,u,Z_u(\cdot)-Z_u^{*\gamma'}(\cdot)) + H^{\gamma'}(\omega,u,Z_u^{*\gamma'}(\cdot)) d(-F_u)\\
&+ \int_{]t,T]} Z_u dq_u - X.
\end{aligned}$$

By Theorem 3.2, $(-e_t^{\gamma'}(S^*), Z_t^{*\gamma'})$ is the unique solution of BSDE (7) with terminal condition S^*. We note that for any constant c,

$$e_t^{\gamma}(X-S-c) = e_t^{\gamma}(X-S) + c$$

and

$$e_t^{\gamma'}(S+c) = e_t^{\gamma'}(S) - c.$$

Therefore, $S^* + c$ is also optimal.

5. Conclusion

We obtained an optimal solution for the inf-convolution problem of the dynamic entropic risk measures. This is the minimum remaining risk if investor A buys an insurance product of value S from B.

Appendix A. The expression for φ^{γ} and $\varphi^{\gamma'}$

Clearly, S is $\mathcal{F}_T$-measurable, therefore, S is defined as

$$S = h(\tau)\mathbb{I}_{\tau \leq T} + b\mathbb{I}_{\tau > T}$$

where $h \in \mathcal{L}^1(\mu)$ and b is constant.

As in Davis's paper [6], for all $t \in]0,T]$, the integrands have the form

$$\begin{aligned}\varphi_t^{\gamma} &= \exp\left(-\frac{1}{\gamma}\left(a\delta\exp\left(\int_{]0,T]} r(s)ds\right) - h(t)\right)\right)\\ &\quad - \frac{1}{F_t}\int_{]t,T]} \exp\left(-\frac{1}{\gamma}\left(a\delta\exp\left(\int_{]0,T]} r(s)ds\right) - h(u)\right)\right) d(-F_u)\\ &\quad - \frac{F_T}{F_t}\exp\left(-\frac{1}{\gamma}\left(a\exp\left(\int_{]0,T]} r(s)ds\right) - b\right)\right),\end{aligned}$$

and

$$\varphi_t^{\gamma'} = \exp\left(-\frac{h(t)}{\gamma'}\right) - \frac{1}{F_t}\int_{]t,T]} \exp\left(-\frac{h(u)}{\gamma'}\right) d(-F_u) - \frac{F_T}{F_t}\exp\left(-\frac{b}{\gamma'}\right).$$

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