



# Homomorphisms of Semi-Holonomic Verma Modules : An Exceptional Case

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
2.1	Almost Hermitian Symmetric Structures . . . . .	3
2.2	The Weight System of $E_6$ . . . . .	3
2.3	The Weyl Group . . . . .	6
2.4	$ 1 $ -graded Lie Algebras . . . . .	7
2.5	Decomposition of Tensor Products . . . . .	9
2.6	Homogeneous Vector Bundles . . . . .	13
<b>3</b>	<b>Invariant Differential Operators</b>	<b>15</b>
3.1	Invariant Differential Operators and Jet Bundles . . . . .	15
3.2	Semi-Holonomic Jets . . . . .	16
3.3	Invariant Operators in Curved Space . . . . .	18
3.4	Verma Modules . . . . .	19
3.5	The Structure of Verma Modules . . . . .	21
3.6	The Central Character of a Representation . . . . .	24
3.7	Semi-Holonomic Verma Modules . . . . .	26
<b>4</b>	<b>Holonomic Case and Translation</b>	<b>31</b>
4.1	Classification of Homomorphisms of Verma Modules . . . . .	31
4.2	The Translation Principle: A Preliminary Example . . . . .	33
4.3	The Translation Principle: The General Description . . . . .	35
4.4	Criteria for the Non-vanishing of the Translated Operator . . . . .	37
4.5	Applications of Theorem 4.4.2 . . . . .	40
4.5.1	Regular Standard Operators . . . . .	41
4.5.2	Singular Standard Operators . . . . .	46
4.5.3	Singular Non-Standard Operators . . . . .	48
4.6	One Way Translation . . . . .	54
4.7	Operators Not Obtainable by Translation . . . . .	64
<b>5</b>	<b>Semi-Holonomic Case</b>	<b>67</b>
5.1	Lifting of the Initial Data . . . . .	67
5.2	The Translation Principle in the Semi-Holonomic Case . . . . .	68
5.3	Non-existence of Lifts of the First Exceptional Family . . . . .	71
<b>6</b>	<b>Conclusions and Outlook</b>	<b>75</b>

<b>A</b>	<b>Lowering Operators for <math>E_6</math> with Commutation Rules</b>	<b>79</b>
<b>B</b>	<b>Composition Series for the Fundamental Representations</b>	<b>81</b>
<b>C</b>	<b>Weyl Group Orbits of Highest Weights</b>	<b>83</b>
	C.1 Weights of $\mathbb{W}$ . . . . .	83
	C.2 Weights of $\mathbb{W}^*$ . . . . .	84
<b>D</b>	<b>Classifying Patterns</b>	<b>85</b>
	D.1 The Hasse Diagram . . . . .	85
	D.2 The Regular Patterns . . . . .	86
	D.3 The De Rham Sequence . . . . .	87
	D.4 The Singular Patterns . . . . .	88

## Abstract

Verma modules play an important part in the theory of invariant operators on homogeneous spaces. If  $G$  is a semisimple Lie group and  $P$  a parabolic subgroup of  $G$ , then there is often a differential geometry for which the homogeneous space  $G/P$  represents the flat model. An example is conformal geometry, where  $G$  is the special orthogonal group  $SO(n, \mathbb{C})$ . A Verma module homomorphism will correspond to an invariant operator on the flat space. The obvious question is: how can we generalize these operators to cases where there is curvature?

In this thesis we will look at a variation of Verma modules called *semi-holonomic* Verma modules, introduced by Eastwood and Slovák. They have studied the conformal case in detail, but here we will investigate instead the exceptional case of  $G = E_6$ . We will investigate when a Verma module homomorphism lifts to a semi-holonomic Verma module homomorphism. When this happens, we can deduce that there is a curved analogue of the corresponding invariant operator.