Homomorphisms of Semi-Holonomic Verma Modules : An Exceptional Case

Justin Sawon
Department of Pure Mathematics

University of Adelaide

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Contents

Abstract

Verma modules play an important part in the theory of invariant operators on homogeneous spaces. If G is a semisimple Lie group and P a parabolic subgroup of G, then there is often a differential geometry for which the homogeneous space G/P represents the flat model. An example is conformal geometry, where G is the special orthogonal group $\mathrm{SO}(n, \mathbb{C})$. A Verma module homomorphism will corresponds to an invariant operator on the flat space. The obvious question is: how can we generalize these operators to cases where there is curvature?

In this thesis we will look at a variation of Verma modules called *semi-holonomic* Verma modules, introduced by Eastwood and Slovák. They have studied the conformal case in detail, but here we will investigate instead the exceptional case of $G = E_6$. We will investigate when a Verma module homomorphism lifts to a semi-holonomic Verma module homomorphism. When this happens, we can deduce that there is a curved analogue of the corresponding invariant operator.

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Justin Sawon
16th September, 1996

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1 Introduction

Verma modules play an important part in the theory of invariant operators on homogeneous spaces. If G is a semisimple Lie group and P a parabolic subgroup of G, then there is often a differential geometry for which the homogeneous space G/P represents the flat model. A Verma module homomorphism then corresponds to an invariant operator on this flat space. When (G, P) is a general Hermitian symmetric pair, these Verma module homomorphisms are well understood (see [4, 7, 8], where the classical and exceptional cases are all discussed). The corresponding differential geometries are the almost Hermitian symmetric (AHS) geometries of Baston [1]. Minkowski space fits into this picture, by taking G to be $\mathrm{GL}(4, \mathbb{C})$ and P to be the appropriate parabolic (such that G/P is the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$, which can be identified with complexified compactified Minkowski space). This example is discussed in detail in [10]. Furthermore, this article also looks at invariant operators on general *curved* space-times. This is really four dimensional conformal geometry. Invariant operators for higher dimensional conformal geometries are looked at in [9], but we will follow more closely the approach presented in [11].

In this article, a variation of Verma modules called *semi-holonomic Verma modules* are introduced. Although Verma module homomorphisms correspond to invariant operators in the flat case, this is not always so for general curved manifolds. However, a homomorphism of semi-holonomic Verma modules does give an invariant operator on curved space. So our aim is to *lift* holonomic Verma module homomorphisms to the semi-holonomic case. For conformal geometry, the existence and non-existence of these lifts is completely classified in [11]. In this thesis we will turn our attention to the exceptional case when $G = E_6$ (and P is chosen appropriately).

So the results presented herein are completely algebraic in nature, but are motivated by the geometric considerations mentioned above. Although we don't quite arrive at a complete classification of which Verma module homomorphisms lift to the semi-holonomic case, we do achieve a great deal. In fact, we show that the majority of the Verma module homomorphisms do lift; there are just five exceptional families for which we are unable to find lifts. Furthermore, for one of these families we prove that no lifts exist. For the remaining four families, the problem can be reduced to proving the existence or otherwise of a lift of just one initial case (i.e. one from each family). These four cases could be decided by a direct assault, involving a large (but finite) calculation, but it seems there should be an easier way.

Of course, our ultimate goal would be to arrive at a complete classification of the invariant operators on the general curved spaces, or rather which invariant operators on the flat spaces admit curved analogues. As yet this is still an outstanding problem, even for conformal geometry. By lifting Verma module homomorphisms to the semi-holonomic case, we can show that a lot of invariant operators do admit curved analogues. However, when a lift does not exist, this does not necessarily imply that no curved analogue exists. Indeed there are cases where a curved analogue exists despite the fact that the Verma module homomorphism does not lift to the semi-holonomic case. For example, in [13] it is shown that various powers of the Laplacian are conformally invariant on general curved manifolds; these include the *long* operator in even dimensions, for which a lift to the semi-holonomic case is known not to exist. In general, showing that a curved analogue does not exist is a very delicate matter.

We begin in Section 2 with some preliminaries. We describe the basic properties of E_6 that will be necessary in later chapters (the weight system, the Weyl group, the $|1|$−grading), as well as describing some of the more general objects and results that we will need (AHS structures, homogeneous vector bundles, decomposing tensor products of representations).

In Section 3 we introduce invariant differential operators, the main subject of our study, and show how in the flat case they are related to Verma module homomorphisms. We give some motivation of why it is necessary to look at semi-holonomic jets and semi-holonomic Verma modules when studying operators on general curved manifolds. In fact, we show how a lift of a Verma module homomorphism to a semi-holonomic Verma module homomorphism gives rise to a curved analogue of the corresponding operator. We also look at the structure of Verma modules and semi-holonomic Verma modules, and give an example of a (holonomic) homomorphism and a lift to the semi-holonomic case.

We present a theorem which classifies the Verma module homomorphisms in Section 4. We then introduce the translation principle, and using the above theorem, are able to find out a great deal about which operators can be obtained from others by translating.

Finally in Section 5, we show that the translation principle remains valid in the semi-holonomic case. Then using the results of Chapter 4, we can obtain lifts to the semi-holonomic case of most invariant operators, after (trivially) lifting just a few. There remains just a few exceptional families of operators, and we prove that one of these families does not admit lifts to the semi-holonomic case.

We conclude in Section 6 with some speculation concerning the other exceptional families of operators. The appendices contain various tables and diagrams which are referred to throughout the thesis.

2 Preliminaries

2.1 Almost Hermitian Symmetric Structures

The following work is inspired by conformal differential geometry. Recall that the flat model of conformal geometry is a sphere S^n. This can be written as G/P where $G = \mathrm{SO}_0(n+1,1)$ is the group of conformal motions of S^n and P is the isotropy subgroup of some basepoint in S^n. In general, if (G, P) is a Hermitian symmetric pair, there is a differential geometry for which G/P represents the flat model. A general curved manifold M in this differential geometry will come equipped with a principle P-bundle, which in the flat case is simply given by $G \to G/P$. These are the almost Hermitian symmetric (AHS) manifolds introduced by Baston [1, 2].

Whilst most of the following results apply for general semisimple Lie group G and parabolic subgroup P, we shall primarily be concerned with the case when G is the exceptional Lie group E_6, and P is the parabolic subgroup corresponding to the Lie subalgebra with Dynkin diagram


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where we are using the same notation as that which appears in [3]. Note that we will let E_6 denote the Lie *group* corresponding to the exceptional Lie algebra, which we shall write as $\mathfrak{e}_6$. However, we will usually continue to write simply G, $\mathfrak{g}$, and P, and it will be clear when we are using specific properties of E_6.

Observe that P has a reductive subgroup consisting of a direct sum of the Lie group SO(10) and a one-dimensional abelian part. The irreducible representations of P are obtained from representations of this subgroup (see below). The homogeneous space G/P is sixteen-dimensional.

2.2 The Weight System of E_6

Most of our work will be carried out with the Lie algebras $\mathfrak{g}$ and $\mathfrak{p}$ of G and P, but sometimes we will need to refer to the groups themselves. We can pass freely from a representation of the group, $\rho : G \to \mathrm{GL}(V)$, to the corresponding representation of the Lie algebra. For simplicity we will use the same notation, namely $\rho : \mathfrak{g} \to \mathrm{End}(V)$ (and the same treatment will apply to representations of $\mathfrak{p}$ and P). We present here some information about the weight system of the Lie algebra of E_6.

Recall that the Lie algebra of E_6 has Dynkin diagram


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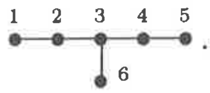
Its root system is generated by simple roots $\alpha_1, \ldots, \alpha_6$ satisfying

$$\frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = a_{ij},$$

where $\{a_{ij}\}$ is the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

These roots correspond to the nodes of the diagram as follows



This root system can be thought of as sitting inside $\mathbb{C}^8$ if we let

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\epsilon_1 + \epsilon_8 - \epsilon_2 - \ldots - \epsilon_7) \\ \alpha_2 &= \epsilon_2 - \epsilon_1 \\ \alpha_3 &= \epsilon_3 - \epsilon_2 \\ \alpha_4 &= \epsilon_4 - \epsilon_3 \\ \alpha_5 &= \epsilon_5 - \epsilon_4 \\ \alpha_6 &= \epsilon_1 + \epsilon_2, \end{aligned}$$

where $\{\epsilon_1, \ldots, \epsilon_8\}$ is a basis of $\mathbb{C}^8$. The fundamental weights of E_6 are given by the change of basis

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix}.$$

Writing the fundamental weights in terms of $\epsilon_1, \ldots, \epsilon_8$, we find

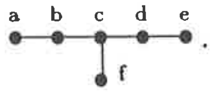
$$\begin{aligned}
\lambda_1 &= \frac{2}{3}(-\epsilon_6 - \epsilon_7 + \epsilon_8) \\
\lambda_2 &= \frac{1}{6}(-3\epsilon_1 + 3\epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + 3\epsilon_5 - 5\epsilon_6 - 5\epsilon_7 + 5\epsilon_8) \\
\lambda_3 &= \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8 \\
\lambda_4 &= \frac{1}{3}(3\epsilon_4 + 3\epsilon_5 - 2\epsilon_6 - 2\epsilon_7 + 2\epsilon_8) \\
\lambda_5 &= \frac{1}{3}(3\epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8) \\
\lambda_6 &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8).
\end{aligned}$$

In particular, the lengths of these weights are

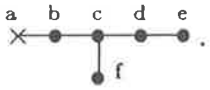
$$\begin{aligned}
|\lambda_1| &= \sqrt{4/3} \\
|\lambda_2| &= \sqrt{10/3} \\
|\lambda_3| &= \sqrt{6} \\
|\lambda_4| &= \sqrt{10/3} \\
|\lambda_5| &= \sqrt{4/3} \\
|\lambda_6| &= \sqrt{2}.
\end{aligned}$$

We will need these results later on.

An irreducible representation $\mathbb{V}$ of $\mathfrak{g}$ will have highest weight $a\lambda_1 + \ldots + f\lambda_6$ where $a, \ldots, f$ are non-negative integers, and we will write this weight as



We will call such a weight $\mathfrak{g}$−dominant. It is also common to use this same notation to denote $\mathbb{V}$ itself. However, to denote irreducible representations we will use instead minus the lowest weight, which is the highest weight of the dual representation. The reason for this choice will become apparent later on, but note that it is really just a notational convenience. An irreducible representation of $\mathfrak{p}$ is given by an irreducible representation of the subalgebra $\mathfrak{so}(10)$ (corresponding to the nodes in the Dynkin diagram) and an irreducible representation of the abelian subalgebra (corresponding to the cross). The representation of $\mathfrak{so}(10)$ will have minus lowest weight $b\lambda_2 + \ldots + f\lambda_6$, where $b, \ldots, f$ are non-negative integers, and the representation of the abelian part will have minus lowest weight given simply by a complex number a. The direct sum of these subalgebras gives a reductive subalgebra of $\mathfrak{p}$, and irreducible representations of $\mathfrak{p}$ are given by irreducible representations of this subalgebra extended trivially to $\mathfrak{p}$. Overall, we can write such a representation as



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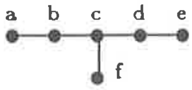
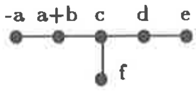
Regarding this as a weight now, we call it $\mathfrak{p}$−dominant, meaning that it is minus the lowest weight of some irreducible $\mathfrak{p}$−module.

2.3 The Weyl Group

The Weyl group $\mathcal{W}$ of E_6 is the group generated by the simple reflections in the weight space, i.e. by the reflections in the walls perpendicular to the simple roots. For example, reflection in the wall perpendicular to the first simple root α_1 will take the weight

$\lambda =$ 

to

$\lambda - \langle \lambda, \alpha_1 \rangle \alpha_1 =$ 

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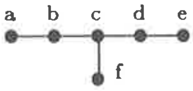
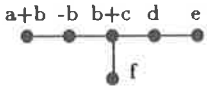
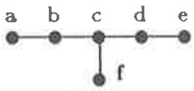
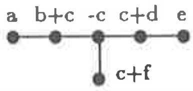
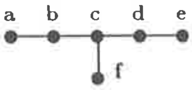
We call this reflection s_1. Similarly, we get

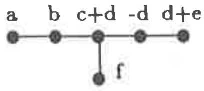
s_2 : 

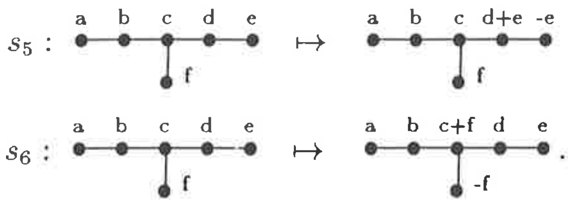
$\mapsto$ 

s_3 : 

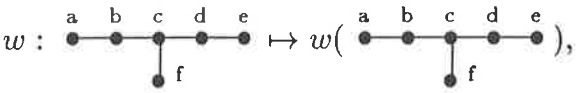
$\mapsto$ 

s_4 : 

$\mapsto$ 



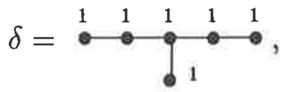
For a general element $w \in \mathcal{W}$, we write



or simply $w : \lambda \mapsto w\lambda$.

The walls perpendicular to the roots divide the weight space into chambers. The Weyl group acts transitively on these chambers. In particular, the *fundamental Weyl chamber* consists of the $\mathfrak{g}$–dominant weights, and every other weight is related by a unique Weyl group reflection to a unique $\mathfrak{g}$–dominant weight.

The Weyl group can also act *affinely* on weights. This means we first translate the weight by the sum of the fundamental weights



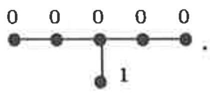
then allow the Weyl group element to act, and finally translate back by subtracting δ. We will denote this by a dot; thus the affine action of $w \in \mathcal{W}$ on the weight λ is

$$w.\lambda = w(\lambda + \delta) - \delta.$$

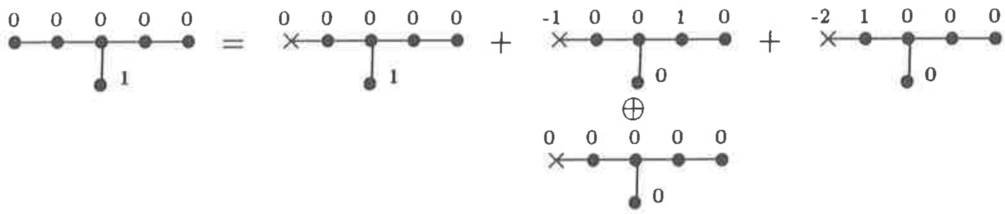
In general, we will often talk of properties of weights when we are really referring to properties of the translated weight. For example, we may call a weight λ $\mathfrak{g}$–dominant, when we really mean that $\lambda + \delta$ is in the fundamental Weyl chamber. We may also call λ regular, when what we mean is that $\lambda + \delta$ does not lie on the wall of a Weyl chamber. We will usually clarify such statements, especially when the meaning is not clear from the context.

2.4 |1|-graded Lie Algebras

For $\mathfrak{g} = \mathfrak{e}_6$, the adjoint representation of $\mathfrak{g}$ on itself is the representation



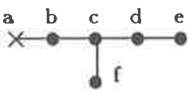
As a representation of the subalgebra $\mathfrak{p}$, this decomposes into the following *composition series* (we will describe precisely what is meant by *composition series* a little later)



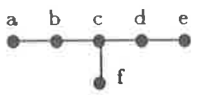
which we write as $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Note that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (where $\mathfrak{g}_{\pm 2} = 0$); in particular, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are abelian. We call such a Lie algebra $|1|$−graded. The subalgebra $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$. Then $\mathfrak{g}_0$ is just the Levi part $\mathfrak{l}$ of $\mathfrak{p}$ and this decomposes into a semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$ (namely $\mathfrak{so}(10)$) and a one-dimensional centre.

Both $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have dimension 16, and $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{so}(10)$ has dimension 45. In Appendix A there is a table of lowering operators $y_1, Y_2, \ldots, Y_{21}, y_{22} \ldots, y_{36}$ for $\mathfrak{e}_6$, i.e. these elements generate the negative root spaces of the Lie algebra. Similarly, the generators of the positive root spaces could be explicitly written out (we will assume they are labelled symmetrically with y_i's, so that x_i and y_i belong to opposite root spaces for each $i = 1, \ldots, 36$). Also, let $h_1, \ldots, h_6$ be generators of a Cartan subalgebra of $\mathfrak{e}_6$. Then $\mathfrak{g}_{-1}$ is generated by $\{y_1, y_{22}, \ldots, y_{36}\}$, $\mathfrak{g}_1$ is generated by $\{x_1, x_{22}, \ldots, x_{36}\}$, $[\mathfrak{l}, \mathfrak{l}]$ is generated by $\{h_2, \ldots, h_6, X_2, \ldots, X_{21}, Y_2, \ldots, Y_{21}\}$, and the one-dimensional centre of $\mathfrak{g}_0$ is generated by $H = \frac{1}{3}(4h_1 + 5h_2 + 6h_3 + 4h_4 + 2h_5 + 3h_6)$.

We find that the action of the one-dimensional centre is invariant on each term, and we have in fact chosen H so that $[H, \mathfrak{g}_i] = i\mathfrak{g}_i$. Indeed, on an irreducible $\mathfrak{p}$−module

$\mathbb{E} =$  ,

H will have eigenvalue $-\frac{1}{3}(4a + 5b + 6c + 4d + 2e + 3f)$ on the entire representation. We will call this linear functional on *weights* ℓ, i.e.

$\ell :$  $\mapsto \dfrac{1}{3}(4a + 5b + 6c + 4d + 2e + 3f).$

It is important to note that we do not include the minus sign when calculating ℓ on an individual weight, but we do when letting ℓ act on a $\mathfrak{p}$−module (since such a module is denoted by *minus* the lowest weight). Since the weights of $\mathbb{E}^*$ are minus those of $\mathbb{E}$, on the dual representation $\mathbb{E}^*$ we observe that

$$\ell(\mathbb{E}^*) = -\ell(\mathbb{E}).$$

Consider a general irreducible $\mathfrak{g}$-module $\mathbb{V}$. We can decompose $\mathbb{V}$ into eigenspaces of H, i.e. into components with distinct values of ℓ. Indeed, if we do this we get a series

$$\mathbb{V} = \mathbb{V}_\alpha + \mathbb{V}_{\alpha+1} + \ldots + \mathbb{V}_{\alpha+n},$$

where α is some number and we call n the *length* of the composition series. The action of H on $\mathbb{V}_{\alpha+j}$ is by multiplication by $\alpha + j$. Recall that $[H, \mathfrak{g}_i] = i\mathfrak{g}_i$. This implies that the lowering operators of $\mathfrak{g}_{-1}$ take $\mathbb{V}_{\alpha+j}$ to $\mathbb{V}_{\alpha+j-1}$ (or zero), the raising operators of $\mathfrak{g}_1$ take $\mathbb{V}_{\alpha+j}$ to $\mathbb{V}_{\alpha+j+1}$ (or zero), and the elements of $\mathfrak{g}_0$ take $\mathbb{V}_{\alpha+j}$ to itself. Thus each $\mathbb{V}_{\alpha+i}$ is a $\mathfrak{g}_0$-module (not necessarily irreducible), and $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ move us backwards and forwards (respectively) along the series.

Restricting $\mathbb{V}$ to be a representation of $\mathfrak{p}$, we get the same series, only now we can only move forwards along the series (not backwards as no lowering operators from $\mathfrak{g}_{-1}$ lie in $\mathfrak{p}$). This is what we mean by a *composition series*. It is only a direct sum if we regard the representations as $\mathfrak{g}_0$-modules. However, we get an inclusion

$$\mathbb{V}_{\alpha+n} \to \mathbb{V} = \mathbb{V}_\alpha + \mathbb{V}_{\alpha+1} + \ldots + \mathbb{V}_{\alpha+n}$$

of the last term, which is $\mathfrak{p}$-invariant as $\mathbb{V}_{\alpha+n}$ must be killed by the raising operators of $\mathfrak{g}_1$, and we get a projection

$$\mathbb{V} = \mathbb{V}_\alpha + \mathbb{V}_{\alpha+1} + \ldots + \mathbb{V}_{\alpha+n} \to \mathbb{V}_\alpha$$

onto the first term, also $\mathfrak{p}$-invariant as the raising operators of $\mathfrak{g}_1$ will take $\mathbb{V}_\alpha$ to $\mathbb{V}_{\alpha+1}$, which will be mapped to zero under the projection. Composition series for the six fundamental representations of E_6, restricted to P, are given in Appendix B.

2.5 Decomposition of Tensor Products

Given two irreducible representations $\mathbb{E}$ and $\mathbb{F}$ of $\mathfrak{g}$, we would like to decompose their tensor product into a direct sum of representations. Suppose $\mathbb{E}$ has minus lowest weight μ and the set of all weights of $\mathbb{F}$ is $\Psi_{\mathbb{F}}$, with $\nu \in \Psi_{\mathbb{F}}$ occurring with multiplicity m_ν. Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$. Then it follows from Kostant's Theorem (see, for example, [15]) that

$$\mathbb{E} \otimes \mathbb{F} = \bigoplus_{\nu \in \Psi_{\mathbb{F}}} (-1)^{\mathbf{length}(w_\nu)} m_\nu \mathbb{M}_{w_\nu.(\mu-\nu)},$$

where $w_\nu \in \mathcal{W}$ is the Weyl group element (acting affinely) which takes $\mu - \nu$ into the dominant Weyl group chamber, if this is possible (otherwise we omit that term), length(w_ν) is the length of w_ν, or in other words the minimum number of simple affine reflections required to move $\mu - \nu$ into the dominant chamber, and

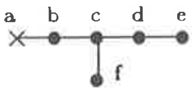
$$\mathbb{M}_{w_\nu.(\mu-\nu)}$$

is the representation with minus lowest weight $w_\nu.(\mu - \nu)$. In other words we begin with minus the lowest weight μ of $\mathbb{E}$ and add minus each of the weights ν of $\mathbb{F}$ to it in turn. Then we apply simple affine reflections to $\mu - \nu$ to move it into the dominant chamber (if possible), giving us $w_\nu.(\mu - \nu)$. Then

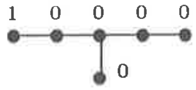
$$\mathbb{M}_{w_\nu.(\mu-\nu)}$$

occurs in the decomposition with multiplicity $(-1)^{\text{length}(w_\nu)} m_\nu$.

As an example, consider what happens when we take the tensor product of



and



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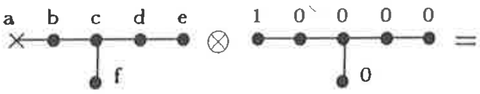
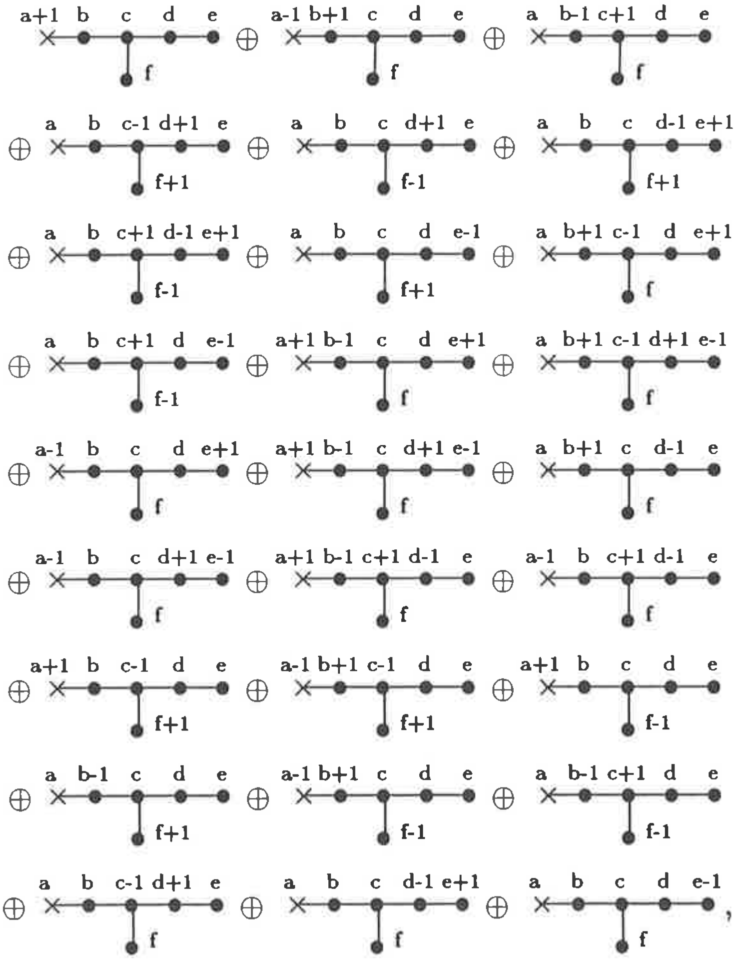
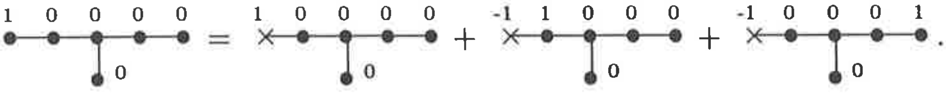
both considered as representations of $\mathfrak{g}_0$. The latter representation we will call $\mathbb{W}$, and its weights appear in Appendix C.1. It is actually a representation of $\mathfrak{g}$, but we will regard it as a representation of $\mathfrak{g}_0$ by restriction. As such, it is not irreducible, but we would get the same set of weights if we first decomposed it into a direct sum of irreducible representations and then decomposed each tensor product separately, combining them at the end. Taking μ to be minus the lowest weight of the first representation, consider what we get when we add minus each weight ν of $\mathbb{W}$ to μ. This sum could be $\mathfrak{g}_0-$dominant (i.e. each of the integers over the nodes is non-negative, not counting the cross as a node of course), but if it is not then it must be because one or more of these integers is equal to -1 (with the others non-negative). Then an affine reflection in one of the nodes with -1 above it will fix the weight, and furthermore, it will not be possible to move the weight into the dominant Weyl chamber by using affine reflections. According to the above result we would omit these weights. So overall, the decomposition consists of representations with minus lowest weights equal to those $\mu - \nu$ which are $\mathfrak{g}_0-$dominant. More explicitly,



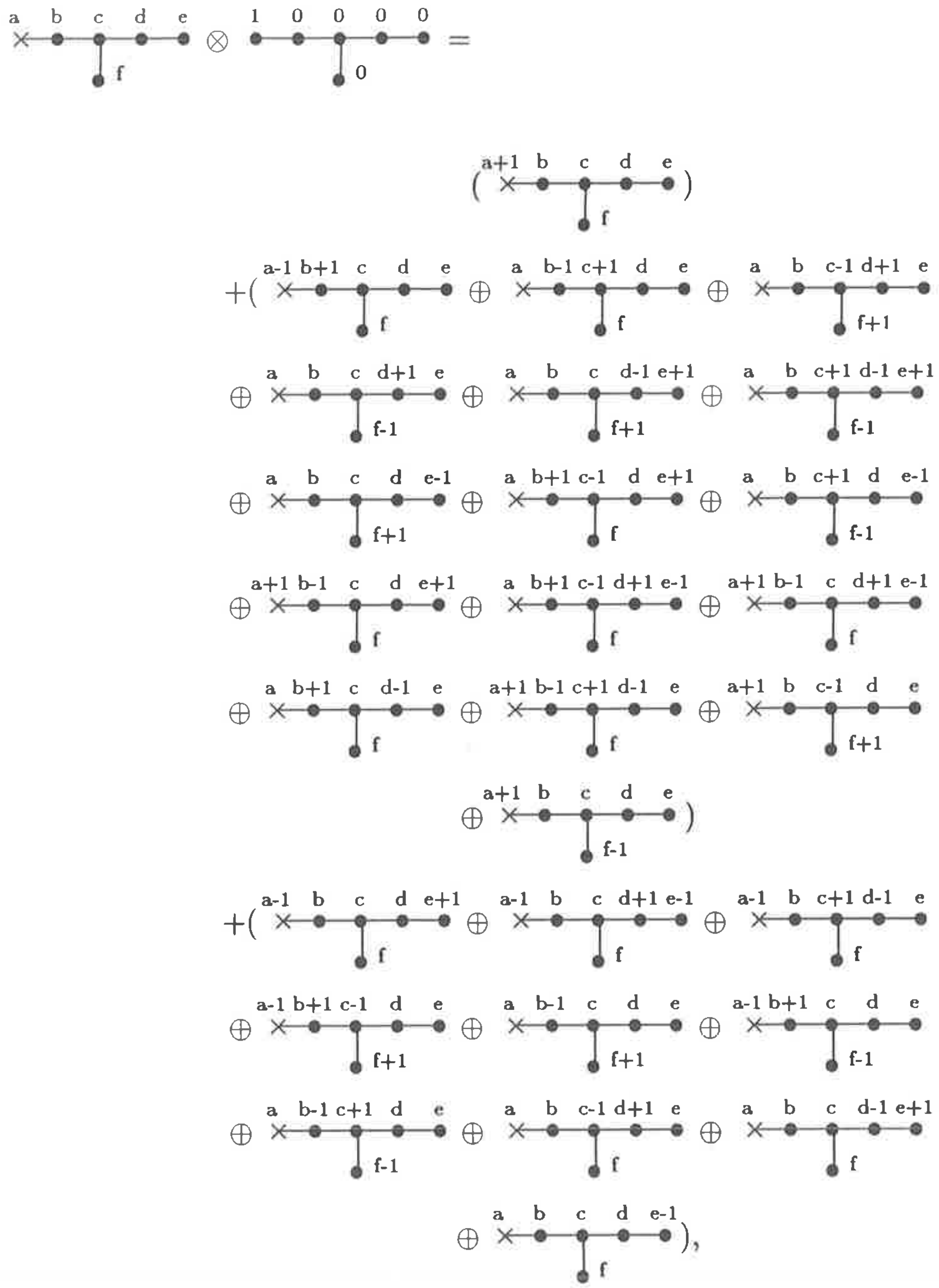


where any term with a -1 above a node would be omitted. We can get a similar decomposition with the representation $\mathbb{W}$ replaced by its dual $\mathbb{W}^*$, whose weights appear in Appendix C.2.

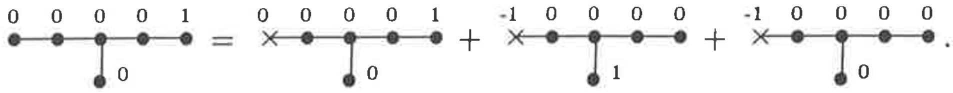
Now what we really want is similar decomposition for representations of $\mathfrak{p}$. However, as a $\mathfrak{p}$-module, $\mathbb{W}$ has composition series



It follows that we also get a composition series for the above tensor product, i.e.



as representations of $\mathfrak{p}$. Of course, we get a similar result using $\mathbb{W}^*$, which has composition series

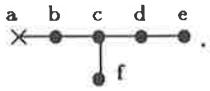


2.6 Homogeneous Vector Bundles

Suppose we have a finite-dimensional representation $\rho : P \to \mathrm{Aut}(\mathbb{E})$ of P. Then we can form a vector bundle

$$E := G \times_P \mathbb{E} = \frac{G \times \mathbb{E}}{(g, e) \sim (gp, \rho(p^{-1})e)}$$

on G/P (see [5]). There is an action of G on this vector bundle which is compatible with the action on G/P and is linear on the fibres. In general, a *homogeneous vector bundle* $E \to G/P$ is any vector bundle over G/P with an action of G on E which is compatible with the action on G/P and linear on the fibres. Thus the above bundle is a homogeneous bundle. In fact, given a homogeneous bundle $E \to G/P$, we can get a representation $\mathbb{E}$ of P by taking the fibre over the identity coset. Then we find that $E \equiv G \times_P \mathbb{E}$, so there is actually a one-to-one correspondence between the finite dimensional representations of P and the homogeneous vector bundles on G/P of finite rank. Recall that we can write an irreducible representation of P as



We will denote the corresponding homogeneous vector bundle on G/P by this same diagram.

3 Invariant Differential Operators

3.1 Invariant Differential Operators and Jet Bundles

We are interested in invariant differential operators on the homogeneous space G/P. Let $D : \mathcal{E} \to \mathcal{F}$ be a differential operator, where $\mathcal{E}$ and $\mathcal{F}$ are sheaves of germs of sections of the vector bundles E and F respectively. Then D takes a local section of E to a local section of F. If E and F are homogeneous vector bundles then there is an induced action of G on local sections. A differential operator is invariant if it commutes with this action of G.

Suppose $D : \mathcal{E} \to \mathcal{F}$ is a differential operator of order k. This means that the operator factors through the operator $\mathcal{E} \to \mathcal{J}^k\mathcal{E}$ which takes a germ of a section to the germ of its $k-$jet. Furthermore, the operator $\mathcal{J}^k\mathcal{E} \to \mathcal{F}$ comes from a vector bundle homomorphism $\bar{D} : J^kE \to F$, where J^kE is the k^{th} associated jet bundle of E (it's fibres consist of germs of sections of E up to order k). Since E is a homogeneous vector bundle, so is J^kE (i.e. we get an induced action of G on the jet bundles, and this action takes fibres to fibres and is linear on fibres). We denote the corresponding representation of P by $J^k\mathbb{E}$. There are natural projections $J^{k+1}E \to J^kE$, and hence surjective $P-$module homomorphisms $J^{k+1}\mathbb{E} \to J^k\mathbb{E}$. Taking the projective limit of these surjections, we arrive at

$$J^\infty\mathbb{E} \to \ldots \to J^{k+1}\mathbb{E} \to J^k\mathbb{E} \to J^{k-1}\mathbb{E} \to \ldots \to J^1\mathbb{E} \to \mathbb{E}.$$

Furthermore, $J^\infty\mathbb{E}$ is actually a $\mathfrak{g}-$module; elements of $\mathfrak{g}_{-1}$ act like 'derivatives', taking something in $J^k\mathbb{E} \subset J^\infty\mathbb{E}$ into a lower jet bundle $J^{k-1}\mathbb{E} \subset J^\infty\mathbb{E}$. The action of elements in $\mathfrak{g}_0$ and $\mathfrak{g}_1$ is of course induced from the action of P on each $J^k\mathbb{E} \subset J^\infty\mathbb{E}$ (as $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$).

Now if D is invariant under G, then $\bar{D}$ is actually induced by a homomorphism of $P-$modules, $\tilde{D} : J^k\mathbb{E} \to \mathbb{F}$. Furthermore, this homomorphism induces $P-$module homomorphisms $J^{k+m}\mathbb{E} \to J^m\mathbb{F}$ for all m, and hence $J^\infty\mathbb{E} \to J^\infty\mathbb{F}$. In fact, this final map is really a homomorphism of $\mathfrak{g}-$modules. (This final observation is often called *Frobenius reciprocity*.)

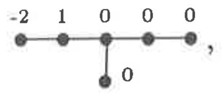
There is an exact sequence of bundles

$$0 \to \bigodot^k \Lambda^1 \otimes E \to J^kE \to J^{k-1}E \to 0$$

where Λ^1 is the cotangent bundle on G/P. This sequence is induced by the exact sequence of $P-$modules

$$0 \to \bigodot^k \mathfrak{g}_1 \otimes \mathbb{E} \to J^k\mathbb{E} \to J^{k-1}\mathbb{E} \to 0,$$

as $\mathfrak{g}_1$ is the $P-$representation with minus lowest weight



which induces the cotangent bundle $\bigwedge^1$. The composition

$$\bigodot^k \bigwedge^1 \otimes E \to J^k E \to F$$

is called the *symbol* of the operator D. If D is $G-$invariant, then this vector bundle homomorphism is also induced by a homomorphism of $P-$modules

$$\bigodot^k \mathfrak{g}_1 \otimes \mathbb{E} \to J^k \mathbb{E} \to \mathbb{F}.$$

3.2 Semi-Holonomic Jets

This subsection is somewhat non-rigorous, but the discussion has been included to motivate the development of the semi-holonomic jets, and describe why it would be necessary to consider such objects when we study invariant operators on general curved manifolds.

The exact sequence of bundles

$$0 \to \bigodot^k \bigwedge^1 \otimes E \to J^k E \to J^{k-1} E \to 0$$

mentioned above involves the symmetric product of the cotangent bundle. This symmetric product in some way reflects the fact that in the flat case derivatives commute; i.e. in local coordinates $\{q_1, \ldots, q_{16}\}$ in a neighbourhood on G/P, if $f(q_1, \ldots, q_{16})$ is a local section of E then

$$\frac{\partial^m f}{\partial q_{i_1} \ldots \partial q_{i_m}} = \frac{\partial^m f}{\partial q_{i_{\sigma(1)}} \ldots \partial q_{i_{\sigma(m)}}},$$

for σ a permutation of $\{1, \ldots, m\}$. In the curved case, we would replace these derivatives with a connection. In general, the connection will not commute, because curvature terms will arise (eg. $\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{abc}{}^d V_d$). We want to alter the jet bundles to reflect this. To be more precise, we would like to somehow define bundles $\bar{J}^k E$ such that we have an exact sequence of bundles

$$0 \to \bigotimes^k \bigwedge^1 \otimes E \to \bar{J}^k E \to \bar{J}^{k-1} E \to 0.$$

We define $\bar{J}^k E$ as follows.

Firstly, $\bar{J}^0 E = E$ and $\bar{J}^1 E = J^1 E$. We define the rest inductively, so suppose we have $\bar{J}^k E$ for some k. We will define $\bar{J}^{k+1} E$ as a subbundle of $J^1 \bar{J}^k E$. There are two maps $J^1 \bar{J}^k E \to J^1 \bar{J}^{k-1} E$. The first is obtained by applying J^1 to the natural projection $\bar{J}^k E \to \bar{J}^{k-1} E$, and the second is obtained by taking the composition of the projection $J^1 \bar{J}^k E \to \bar{J}^k E$ with the inclusion $\bar{J}^k E \hookrightarrow J^1 \bar{J}^{k-1} E$. We define $\bar{J}^{k+1} E$ to be the subbundle of $J^1 \bar{J}^k E$ on which these two maps agree. The bundles $\bar{J}^k E$ are known as *semi-holonomic* jet bundles (ordinary jet bundles are holonomic).

To see why this definition gives us the exact sequence we were hoping for, let us first look at the case $k = 2$. Elements of the bundle $J^1 J^1 E$ look like $(f, f_a^{(1)}, f_a^{(2)}, f_{ab}^{(12)})$. The two maps to $J^1 E$ take the above element to $(f, f_a^{(1)})$ and $(f, f_a^{(2)})$ respectively. So on $\bar{J}^2 E$, where these maps agree, we have $f_a^{(1)} = f_a^{(2)}$. Then $\bigotimes^2 \Lambda^1 \otimes E$ is essentially included into the $f_{ab}^{(12)}$ part of $\bar{J}^2 E$, and the remaining part, $(f, f_a^{(1)})$, is projected onto $\bar{J}^1 E$. Note that for $J^2 E$ we would still have $\bigodot^2 \Lambda^1 \otimes E$ included into the $f_{ab}^{(12)}$ part, but we would also require $f_{ab}^{(12)}$ to be symmetric.

In general, elements of $J^1 \ldots J^1 E$ (k times) look like

$$(f, f_{i_1}^{(1)}, \ldots, f_{i_1}^{(k)}, f_{i_1 i_2}^{(12)}, \ldots, f_{i_1 i_2}^{(k-1,k)}, f_{i_1 i_2 i_3}^{(123)}, \ldots, f_{i_1 i_2 \ldots i_k}^{(12\ldots k)}).$$

Elements of $\bar{J}^k E$ would have $f_{i_1}^{(1)} = \ldots = f_{i_1}^{(k)}$, $f_{i_1 i_2}^{(12)} = \ldots = f_{i_1 i_2}^{(k-1,k)}$, etc. Then $\bigotimes^k \Lambda^1 \otimes E$ is essentially included into the $f_{i_1 i_2 \ldots i_k}^{(12\ldots k)}$ part of $\bar{J}^k E$, and the remaining part is projected onto $\bar{J}^{k-1} E$. Elements of $J^k E$ would also require $f_{i_1 i_2}^{(12)}$, $f_{i_1 i_2 i_3}^{(123)}$, …, $f_{i_1 i_2 \ldots i_k}^{(12\ldots k)}$ to be symmetric; $\bigodot^k \Lambda^1 \otimes E$ would be included into the $f_{i_1 i_2 \ldots i_k}^{(12\ldots k)}$ part with the remaining part projected onto $J^{k-1} E$.

So we see that we do get the required exact sequence

$$0 \to \bigotimes^k \bigwedge^1 \otimes E \to \bar{J}^k E \to \bar{J}^{k-1} E \to 0$$

from this definition. More importantly, these semi-holonomic jet bundles were defined entirely by using first jets (i.e. taking first jets of first jets, etc.), and this fact shall have useful consequences later on.

Since $\bar{J}^k E$ is a homogeneous vector bundle, we can associate a $P-$module $\bar{J}^k \mathbb{E}$ to it. Similarly to the holonomic case, there is a projective limit of surjective $P-$module homomorphisms

$$\bar{J}^\infty \mathbb{E} \to \ldots \to \bar{J}^{k+1} \mathbb{E} \to \bar{J}^k \mathbb{E} \to \bar{J}^{k-1} \mathbb{E} \to \ldots \to \bar{J}^1 \mathbb{E} \to \mathbb{E}.$$

There is also a canonical differential operator $J^k E \to \bar{J}^k E$, which is essentially just inclusion.

3.3 Invariant Operators in Curved Space

We stated earlier that there is a class of differential geometries such that G/P represents the flat model. In the flat case, $G \to G/P$ is a principal $P-$bundle; furthermore, in the curved case there also exists a principal $P-$bundle, $X \to M$, where M is a curved manifold. This means that given a representation $\mathbb{E}$ of P, we can construct a vector bundle E over the curved space M. We have already seen this construction for the flat case, where $E = G \times_P \mathbb{E}$ is called a homogeneous bundle. In the curved case we just replace G by the principal $P-$bundle X, to get $E = X \times_P \mathbb{E}$.

Once we have constructed the bundle E, we could take its first jet bundle J^1E. Alternatively, there is the representation $J^1\mathbb{E}$ of P (since J^1E is a homogeneous bundle in the flat case, it comes from a representation of P; this representation is $J^1\mathbb{E}$). We could use this representation to construct a vector bundle in the curved case as well. However, it is a fact that these two vector bundles can be canonically identified (see [6]), that is,

$$J^1E \equiv X \times_P J^1\mathbb{E}.$$

(For the curved geometry associated with E_6, there is an invariant torsion whose vanishing characterizes flat space. Even with torsion, this canonical identification still holds.) Thus a general $P-$module homomorphism $J^1\mathbb{E} \to \mathbb{F}$ gives rise to a homomorphism of vector bundles, $J^1E \to F$, and hence we get a first order invariant differential operator in the curved case. For higher orders this does not always work. For example, in [12] Graham has shown that in four dimensional conformal geometry the cube of the Laplacian (a sixth order operator) does not have a curved analogue. So in this case a canonical identification

$$J^kE \equiv X \times_P J^k\mathbb{E}$$

cannot exist for $k = 6$. In general, there are likely to be many other values of k for which such a canonical identification will not exist.

However, the semi-holonomic jets are constructed entirely by iterating the first jet construction. Hence the vector bundle constructed from $\bar{J}^k\mathbb{E}$ in the curved case really will be the semi-holonomic jet bundle $\bar{J}^kE$. This mean that a $P-$module homomorphism $\bar{J}^k\mathbb{E} \to \mathbb{F}$ will give us a homomorphism of vector bundles $\bar{J}^kE \to F$. By composing with the operator $J^kE \to \bar{J}^kE$, we obtain an invariant differential operator $J^kE \to F$.

So suppose we have an invariant operator $J^kE \to F$ in the flat case. If we can lift the $P-$module homomorphism to the semi-holonomic jets

$$\begin{tikzcd} J^k\mathbb{E} \arrow[d] \arrow[dr] & \\ & \mathbb{F} \\ \bar{J}^k\mathbb{E} \arrow[ur] & \end{tikzcd}$$

then we can construct a curved analogue (i.e. an invariant operator with the same symbol as in the flat case). Our aim in this thesis is to completely classify which homomorphisms lift to the semi-holonomic jets, thereby giving curved analogues of the corresponding operators. Of course, we have not shown that the existence of a curved analogue necessarily implies that the P–module homomorphism $J^k\mathbb{E} \to \mathbb{F}$ must lift to the semi-holonomic jet. In fact, this is not true in general; there exist some curved operators which arise even when a lift to the semi-holonomic jets does not exist. However, this is rather an exceptional occurrence, and we shall see that the majority of operators do lift to the semi-holonomic case and hence do have curved analogues.

3.4 Verma Modules

From now on when considering G–modules (respectively P–modules) we will immediately pass to the corresponding representations of the Lie algebra $\mathfrak{g}$ (respectively $\mathfrak{p}$). Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $\rho^* : \mathfrak{p} \to \mathrm{End}(\mathbb{E}^*)$ be the dual of the representation $\rho : \mathfrak{p} \to \mathrm{End}(\mathbb{E})$. Consider the $\mathfrak{U}(\mathfrak{g})$–module $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{E}^*$, where $\mathfrak{g}$ acts trivially on $\mathbb{E}^*$. Now factor out the left $\mathfrak{U}(\mathfrak{g})$–submodule generated by

$$\{p \otimes e - 1 \otimes \rho^*(p)e\},$$

where $p \in \mathfrak{p}$. The resulting $\mathfrak{U}(\mathfrak{g})$–module is called a generalized Verma module, denoted $V(\mathbb{E})$.

The grading of $\mathfrak{U}(\mathfrak{g})$ induces a grading of $V(\mathbb{E})$

$$V(\mathbb{E}) \supset \ldots \supset V_{k+1}(\mathbb{E}) \supset V_k(\mathbb{E}) \supset V_{k-1}(\mathbb{E}) \supset \ldots \supset V_1(\mathbb{E}) \supset V_0(\mathbb{E}) = \mathbb{E}^*.$$

As $\mathfrak{p}$–modules, these are duals of the sequence of jets

$$\ldots \to J^{k+1}\mathbb{E} \to J^k\mathbb{E} \to J^{k-1}\mathbb{E} \to \ldots \to J^1\mathbb{E} \to \mathbb{E}.$$

Indeed, as $\mathfrak{U}(\mathfrak{g})$–modules, $J^\infty\mathbb{E}$ is the dual of $V(\mathbb{E})$.

In this dual picture, the exact sequence of $\mathfrak{p}$–modules

$$0 \to \bigodot^k \mathfrak{g}_1 \otimes \mathbb{E} \to J^k\mathbb{E} \to J^{k-1}\mathbb{E} \to 0$$

becomes

$$0 \to V_{k-1}(\mathbb{E}) \to V_k(\mathbb{E}) \to \bigodot^k \mathfrak{g}_{-1} \otimes \mathbb{E}^* \to 0,$$

recalling that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual $\mathfrak{p}$−modules.

Suppose that $\mathbb{F}$ is another $\mathfrak{p}$−module. We saw before that an invariant differential operator $D : \mathcal{E} \to \mathcal{F}$ induces a $\mathfrak{g}$−module homomorphism $J^\infty\mathbb{E} \to J^\infty\mathbb{F}$. Dually, this corresponds to a $\mathfrak{U}(\mathfrak{g})$−module homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$. Conversely, a $\mathfrak{U}(\mathfrak{g})$−module homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ will give us an invariant differential operator $D : \mathcal{E} \to \mathcal{F}$.

Suppose further that the operator is k^{th} order. Then we have a homomorphism of $\mathfrak{p}$−modules, $J^k\mathbb{E} \to \mathbb{F}$, or dually, $\mathbb{F}^* \to V_k(\mathbb{E})$. The symbol

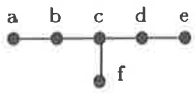
$$\bigodot^k \mathfrak{g}_1 \otimes \mathbb{E} \to \mathbb{F}$$

looks like

$$\mathbb{F}^* \to \bigodot^k \mathfrak{g}_{-1} \otimes \mathbb{E}^*$$

when we take its dual.

Recall that in Subsection 2.4 we introduced ℓ, a linear functional on weights given by

$\ell :$ 

$$\mapsto \frac{1}{3}(4a + 5b + 6c + 4d + 2e + 3f).$$

In fact, ℓ was defined as the eigenvalue of the one-dimensional centre of $\mathfrak{g}_0$, generated by H, on a weight vector in that weight space. On an irreducible $\mathfrak{p}$−module, H will have fixed eigenvalue (as $\mathfrak{g}_1$ must act trivially, and H is central in $\mathfrak{g}_0$). Furthermore, this eigenvalue will be invariant under homomorphisms, i.e. ℓ is a homomorphism invariant on irreducible $\mathfrak{p}$−modules. So consider the $\mathfrak{p}$−module homomorphism $\mathbb{F}^* \to V_k(\mathbb{E})$. As a $\mathfrak{g}_0$−module,

$$V_k(\mathbb{E}) = \bigoplus_{j=0}^{k} \bigodot^j \mathfrak{g}_{-1} \otimes \mathbb{E}^*,$$

and since $\ell(\mathfrak{g}_{-1}) = -1$, we have

$$\ell(\bigodot^j \mathfrak{g}_{-1} \otimes \mathbb{E}^*) = -j - \ell(\mathbb{E}).$$

In particular, we must have $-\ell(\mathbb{F}) = -j - \ell(\mathbb{E})$ for some $j \in \{0, \ldots, k\}$. In fact it must be for $j = k$, otherwise the operator would have order less than k. It follows that the order of the operator is given by

$$k = \ell(\mathbb{F}) - \ell(\mathbb{E}).$$

Furthermore, the image of $\mathbb{F}^*$ lies in $\bigodot^k \mathfrak{g}_{-1} \otimes \mathbb{E}^* \subset V_k(\mathbb{E})$. So the $\mathfrak{p}$−module homomorphism $\mathbb{F}^* \to V_k(\mathbb{E})$ is really just the symbol $\mathbb{F}^* \to \bigodot^k \mathfrak{g}_{-1} \otimes \mathbb{E}^*$, and indeed the entire homomorphism of Verma modules $V(\mathbb{F}) \to V(\mathbb{E})$ is determined by this symbol.

3.5 The Stucture of Verma Modules

Suppose we have a Verma module

$$V(\mathbb{E}) = \frac{\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{E}^*}{\langle p \otimes e - 1 \otimes \rho^*(p)e \rangle}.$$

An element of this module looks like

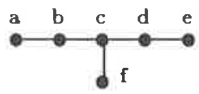
$$\sum_{v \in \mathbb{E}^*} p_v(y_1, y_{22}, \ldots, y_{36}, Y_2, \ldots, Y_{21}, h_1, \ldots, h_6, X_2, \ldots, X_{21}, x_1, x_{22}, \ldots, x_{36}) \otimes v$$

where p_v is some 'polynomial' type expression in the generators of $\mathfrak{g}$, but note that the order of elements in the p_v is important. However, we can commute all of $Y_2, \ldots, Y_{21}$, $h_1, \ldots, h_6$, $X_2, \ldots, X_{21}$, $x_1, x_{22}, \ldots, x_{36} \in \mathfrak{p}$ past the y_i and then allow them to act on v according to ρ^*. Thus we can rewrite the above element as

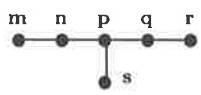
$$\sum_{w \in \mathbb{E}^*} q_w(y_1, y_{22}, \ldots, y_{36}) \otimes w$$

where q_w is a genuine polynomial in y_i (i.e. order no longer matters as the y_i commute with each other).

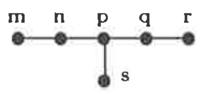
Now let $\mathbb{E}$ and $\mathbb{F}$ be two representations of $\mathfrak{p}$ with minus lowest weights



and



respectively. Of course, these are the highest weights of the dual representations which appear in the Verma modules (this is why it is more convenient from the outset to adopt the convention whereby we denote representations by minus their lowest weights). In order to find a Verma module homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ we need to find an element of $V(\mathbb{E})$ which has weight



and is killed by all raising operators $X_2, \ldots, X_{21}$, $x_1, x_{22}, \ldots, x_{36}$, i.e. a maximal weight vector. Then we simply map the highest weight vector of $V(\sigma)$ to this maximal weight vector and this will extend to give us the desired homomorphism. We know that the elements of $V(\mathbb{E})$ look like

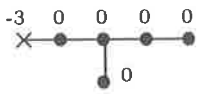
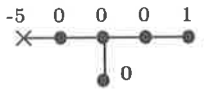
$$\sum_{w \in \mathbb{E}^*} q_w(y_1, y_{22}, \ldots, y_{36}) \otimes w,$$

so we just need to choose the q_w so that this element has the appropriate weight and is killed by all raising operators (to see how a raising operator acts on the above element we commute it past the y_i and then allow it to act on w). In fact, if the element is killed by the raising operators $X_2, \ldots, X_6$, then it will also be killed by $X_7 = [X_2, X_3]$, $X_8 = [X_3, X_4]$, …, $X_{21} = [X_4, X_{20}]$; i.e. by all the raising operators in $\mathfrak{p}$. Then what we get is a symbol

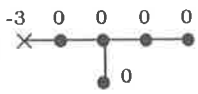
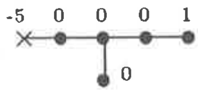
$$\bigodot^k \bigwedge^1 \otimes E \to F.$$

If we then allow x_1 to act on the element, we get the obstruction to this symbol lifting to an invariant operator, for if the element is killed by x_1 and $X_2, \ldots, X_{21}$, then it will clearly be killed by $x_{22} = [x_1, X_2]$, $x_{23} = [x_1, X_7]$, …, $x_{36} = [X_6, x_{35}]$ as well.

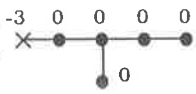
As an example, consider the second order operator

 $\to$ .

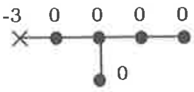
In terms of Verma modules, this becomes

$V($  $) \leftarrow V($  $)$.

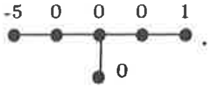
Now the $\mathfrak{p}$-module

$($  $)^*$

is one-dimensional, generated by w say. So we need to find a maximal weight vector

$u = q(y_1, y_{22}, \ldots, y_{36}) \otimes w \in V($  $)$

with weight



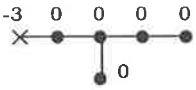
The restriction on the weight immediately forces u to look like

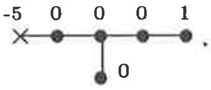
$$(Ay_1y_{28} + By_{22}y_{27} + Cy_{23}y_{26} + Dy_{24}y_{25}) \otimes w,$$

where A, B, C, and D are constants. Using the commutation table in Appendix A, we find

$$\begin{aligned}
X_2u &= (Ay_1y_{27} - By_1y_{27}) \otimes w, \\
X_3u &= (By_{22}y_{26} - Cy_{22}y_{26}) \otimes w, \\
X_4u &= (Cy_{23}y_{25} - Dy_{23}y_{25}) \otimes w, \\
X_5u &= 0, \\
X_6u &= (Cy_{23}y_{24} - Dy_{24}y_{23}) \otimes w.
\end{aligned}$$

Thus choosing $A = B = C = D$ gives us a weight vector u which is killed by all the raising operator in $\mathfrak{p}$, and so we get a symbol

$\bigodot^2 \bigwedge^1 \otimes$ 

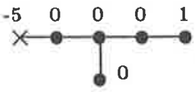
$\to$ 

.

The obstruction to this being the symbol of an invariant operator is x_1u. However, we find that

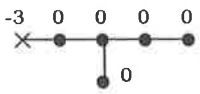
$$\begin{aligned}
x_1u &= A(h_1y_{28} + Y_2y_{27} + y_{22}Y_{14} + Y_7y_{26} + y_{23}Y_{13} + Y_{10}y_{25} + y_{24}Y_{11}) \otimes w \\
&= A(h_1y_{28} + Y_2y_{27} + Y_7y_{26} + Y_{10}y_{25}) \otimes w \\
&= A(-3y_{28} + y_{28} + y_{28} + y_{28}) \otimes w \\
&= 0,
\end{aligned}$$

after applying the relevant commutation rules. Therefore taking the highest weight vector of

(

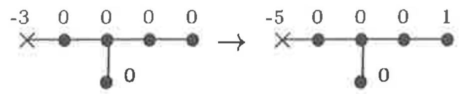
)*

to

$u = A(y_1y_{28} + y_{22}y_{27} + y_{23}y_{26} + y_{24}y_{25}) \otimes w \in V($ 

$)$

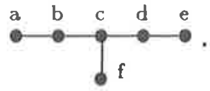
gives us a homomorphism of Verma modules (which is, of course, non-zero if we choose $A \neq 0$). It follows that there exists an invariant operator



on G/P.

3.6 The Central Character of a Representation

Suppose we have a representation $\mathbb{E}$ of $\mathfrak{p}$ with minus lowest weight



Then we can form the $\mathfrak{U}(\mathfrak{g})$-module $V(\mathbb{E})$. Recall that we can write elements of $V(\mathbb{E})$ as

$$\sum_{w \in \mathbb{E}^*} q_w(y_1, y_{22}, \dots, y_{36}) \otimes w,$$

where q_w is a polynomial in y_i. In fact, we can write elements $w \in \mathbb{E}^*$ as

$$w = \sum_j Y_{j_1} \dots Y_{j_r} u,$$

where u is a highest weight vector of $\mathbb{E}^*$. So in fact, the entire Verma module $V(\mathbb{E})$ can be generated by allowing the lowering operators of $\mathfrak{g}$ to act on the highest weight vector u, which is unique up to scale. We call such a $\mathfrak{U}(\mathfrak{g})$-module a *highest weight module*.

Let $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ be the centre of the algebra $\mathfrak{U}(\mathfrak{g})$, and let $z \in \mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$. Since z commutes with every element in $\mathfrak{U}(\mathfrak{g})$, we find that $h_i z.u = z h_i.u$ for $i = 1, \dots, 6$, so $z.u$ has the same weight as u. However, the highest weight vector u is, up to scale, the only weight vector in $V(\mathbb{E})$ with that weight (every other weight vector is obtained by lowering u, so must have a lower weight). Therefore it follows that $z.u = \phi(z)u$, where $\phi : \mathfrak{Z}(\mathfrak{U}(\mathfrak{g})) \to \mathbb{C}$ is some function. Indeed

$$\begin{aligned} \phi(z_1 z_2)u &= z_1 z_2.u \\ &= z_1.\phi(z_2)u \\ &= \phi(z_1)\phi(z_2)u, \end{aligned}$$

so ϕ is really a homomorphism of algebras, and since $\mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ is abelian, ϕ is called a character. Thus for each highest weight module $V(\mathbb{E})$ of $\mathfrak{U}(\mathfrak{g})$ we get a homomorphism $\phi : \mathfrak{Z}(\mathfrak{U}(\mathfrak{g})) \to \mathbb{C}$ which we call the *central character* of $V(\mathbb{E})$ (sometimes we will simply call it the central character of $\mathbb{E}$).

Again let $z \in \mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$. Since z commutes with all other elements in $\mathfrak{U}(\mathfrak{g})$, when it acts on $w \in \mathbb{E}^*$ we get

$$\begin{aligned} z.w &= \sum_j Y_{j_1} \dots Y_{j_r} z.u \\ &= \sum_j Y_{j_1} \dots Y_{j_r} \phi(z) u \\ &= \phi(z) w, \end{aligned}$$

and similarly, when it acts on any element of the Verma module $V(\mathbb{E})$ we get

$$z.(\sum_{w \in \mathbb{E}^*} q_w(y_1, \dots, y_{16}) \otimes w) = \phi(z)(\sum_{w \in \mathbb{E}^*} q_w(y_1, \dots, y_{16}) \otimes w).$$

Hence z really acts by scalar multiplication by $\phi(z)$ on the entire Verma module.

Now suppose we have an invariant operator $E \to F$ between two homogeneous bundles. This is equivalent to a Verma module homomorphism $D : V(\mathbb{F}) \to V(\mathbb{E})$. If v is some arbitrary element of $V(\mathbb{F})$, then by what we have seen above,

$$z.v = \phi_{\mathbb{F}}(z) v,$$

where $z \in \mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$ and $\phi_{\mathbb{F}}$ is the central character of $V(\mathbb{F})$. Now we have

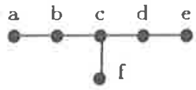
$$\begin{aligned} \phi_{\mathbb{E}}(z) D(v) &= z.D(v) \\ &= D(z.v) \\ &= D(\phi_{\mathbb{F}}(z) v) \\ &= \phi_{\mathbb{F}}(z) D(v), \end{aligned}$$

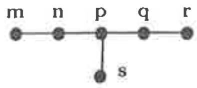
where we have used the facts that $D(v) \in V(\mathbb{E})$ and D is a $\mathfrak{U}(\mathfrak{g})$-module homomorphism. Since D is a non-zero homomorphism, it follows that we must have

$$\phi_{\mathbb{E}}(z) = \phi_{\mathbb{F}}(z)$$

for all $z \in \mathfrak{Z}(\mathfrak{U}(\mathfrak{g}))$, i.e. $V(\mathbb{E})$ and $V(\mathbb{F})$ must have the same central character if there exists a non-zero operator $E \to F$.

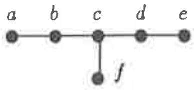
Let $\mathbb{E}$ and $\mathbb{F}$ have minus lowest weights

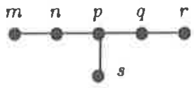


and


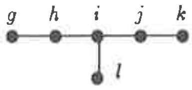
respectively. Then we have the following theorem due to Harish-Chandra (see, for example, [14]).

Theorem 3.6.1 *The highest weight modules $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character if and only if their highest weights*



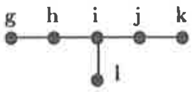
and


are in the same affine Weyl group orbit. In this case, we will use the notation $\chi_{g,h,i,j,k,l}$ to denote their central character, where



is the (unique) $\mathfrak{g}$−dominant weight in the affine Weyl group orbit of the highest weights.

Actually, we really have



$+\delta$

is $\mathfrak{g}$−dominant, which means that one or more of $g, \ldots, l$ may be equal to -1 if the highest weights are *singular* (i.e. if the weights plus δ lie on a wall of a Weyl chamber). However, even if they are singular, there will still be a *unique* $\mathfrak{g}$−dominant weight in their orbit, as after applying Weyl group reflections we can move the weights onto a wall of the dominant Weyl chamber. In particular, it could not end up on two distinct walls in the dominant Weyl chamber, as these adjacent walls are not related by a reflection.

3.7 Semi-Holonomic Verma Modules

The holonomic jets $J^\infty\mathbb{E}$ are duals of the generalized Verma modules $V(\mathbb{E})$. We would also like to construct *semi-holonomic Verma modules* $\bar{V}(\mathbb{E})$, such that the semi-holonomic jets $\bar{J}^\infty\mathbb{E}$ are duals of $\bar{V}(\mathbb{E})$. Recall that the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient of the tensor algebra $\mathfrak{T}(\mathfrak{g})$ by the ideal

$$\langle x \otimes y - y \otimes x - [x,y] | x, y \in \mathfrak{g}\rangle.$$

In particular, since $[y_i, y_j] = 0$ in the Lie algebra $\mathfrak{g}$, the y_i commute in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ (we have factored out by $\langle y_i \otimes y_j - y_j \otimes y_i\rangle$). Define $\bar{\mathfrak{U}}(\mathfrak{g})$ to be the quotient of $\mathfrak{T}(\mathfrak{g})$ by the ideal

$$\langle x \otimes y - y \otimes x - [x,y] | x \in \mathfrak{p}, y \in \mathfrak{g}\rangle.$$

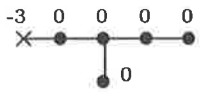
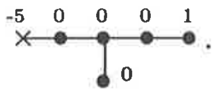
We have no longer factored out by $\langle y_i \otimes y_j - y_j \otimes y_i\rangle$ (since $y_i \in \mathfrak{g}\backslash\mathfrak{p}$), and hence the y_i do not commute in $\bar{\mathfrak{U}}(\mathfrak{g})$.

We define the semi-holonomic Verma module $\bar{V}(\mathbb{E})$ to be the quotient of the $\bar{\mathfrak{U}}(\mathfrak{g})$–module $\bar{\mathfrak{U}}(\mathfrak{g}) \otimes \mathbb{E}^*$ by the left $\bar{\mathfrak{U}}(\mathfrak{g})$–submodule generated by

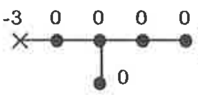
$$\{p \otimes e - 1 \otimes \rho^*(p)e\}.$$

The structure of these semi-holonomic Verma modules is the same as the holonomic Verma modules except that now the order of the y_i in the 'polynomials' q_w is important. Homomorphisms are found in the same way as before, but at all stages we must be wary that the y_i are not allowed to commute.

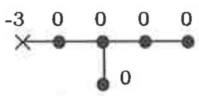
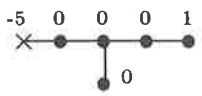
For example, let us try to find a semi-holonomic lift of the second order operator

 $\rightarrow$ .

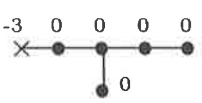
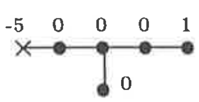
We saw in Subsection 3.5 that the maximal weight vector

$u = A(y_1y_{28} + y_{22}y_{27} + y_{23}y_{26} + y_{24}y_{25}) \otimes w \in V($  $)$

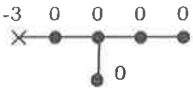
gives us a homomorphism of Verma modules

$V($  $) \leftarrow V($  $)$,

which is non-zero for $A \neq 0$. Now we wish to find a semi-holonomic Verma module homomorphism

$\bar{V}($  $) \leftarrow \bar{V}($  $)$

which is a lift of the above (holonomic) Verma module homomorphism. In other words, we want to find a maximal weight vector

$v = \bar{q}(y_1, y_{22}, \ldots, y_{36}) \otimes w \in \bar{V}($  $)$

with the appropriate weight, where $\bar{q}$ is not really a polynomial anymore since the y_i's are no longer allowed to commute. Furthermore, for this to be a lift, we must be able to recover u from v by allowing the y_i's to commute.

Now as before, weight considerations force v to look like

$$(A_1y_1y_{28}+A_2y_{28}y_1+B_1y_{22}y_{27}+B_2y_{27}y_{22}+C_1y_{23}y_{26}+C_2y_{26}y_{23}+D_1y_{24}y_{25}+D_2y_{25}y_{24})\otimes w,$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are constants. Applying raising operators we find

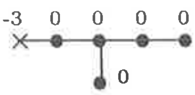
$$\begin{aligned}
X_2v &= (A_1y_1y_{27} + A_2y_{27}y_1 - B_1y_1y_{27} - B_2y_{27}y_1) \otimes w,\\
X_3v &= (B_1y_{22}y_{26} + B_2y_{26}y_{22} - C_1y_{22}y_{26} - C_2y_{26}y_{22}) \otimes w,\\
X_4v &= (C_1y_{23}y_{25} + C_2y_{25}y_{23} - D_1y_{23}y_{25} - D_2y_{25}y_{23}) \otimes w,\\
X_5v &= 0,\\
X_6v &= (C_1y_{23}y_{24} + C_2y_{24}y_{23} - D_1y_{24}y_{23} - D_2y_{23}y_{24}) \otimes w.
\end{aligned}$$

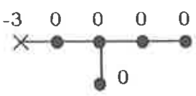
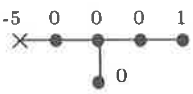
Thus we need to choose $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = D_1 = D_2$ for all these terms to vanish. Finally

$$\begin{aligned}
x_1v &= A_1(h_1y_{28} + y_{28}h_1 + Y_2y_{27} + y_{22}Y_{14} + Y_{14}y_{22} + y_{27}Y_2 + Y_7y_{26} + y_{23}Y_{13} + \\
&\qquad + Y_{13}y_{23} + y_{26}Y_7 + Y_{10}y_{25} + y_{24}Y_{11} + Y_{11}y_{24} + y_{25}Y_{10}) \otimes w\\
&= A_1(h_1y_{28} + y_{28}h_1 + Y_2y_{27} + Y_{14}y_{22} + Y_7y_{26} + Y_{13}y_{23} + Y_{10}y_{25} + Y_{11}y_{24}) \otimes w\\
&= A(-3y_{28} - 3y_{28} + y_{28} + y_{28} + y_{28} + y_{28} + y_{28} + y_{28}) \otimes w\\
&= 0,
\end{aligned}$$

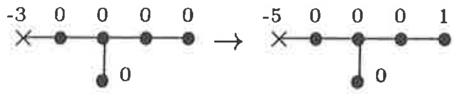
after commuting the Y_j's past the y_i's. Thus the maximal weight vector

$$u = A_1(y_1y_{28} + y_{28}y_1 + y_{22}y_{27} + y_{27}y_{22} + y_{23}y_{26} + y_{26}y_{23} + y_{24}y_{25} + y_{25}y_{24}) \otimes w$$

in $\bar{V}($  $)$ gives us a semi-holonomic Verma module homomorphism

$\bar{V}($  $) \leftarrow \bar{V}($  $).$

Choosing $A_1 = A/2$ gives us a lift of the holonomic Verma module homomorphism, and hence the operator



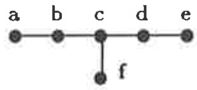
has a curved analogue.

4 Holonomic Case and Translation

4.1 Classification of Homomorphisms of Verma Modules

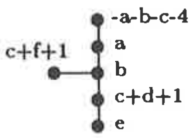
In the previous section we saw that the classification of all $G-$invariant differential operators on G/P is equivalent to the classification of all $\mathfrak{U}(\mathfrak{g})-$module homomorphisms $V(\mathbb{F}) \to V(\mathbb{E})$. We now wish to investigate these homomorphisms in the case of $G = E_6$. The classification is due to Boe and Collingwood [4], and involves certain patterns which appear in Appendix D. First we will describe how these patterns are obtained.

We begin with the Hasse diagram of the parabolic $\mathfrak{p} \subset \mathfrak{g}$, which appears in Appendix D.1. The Hasse diagram gives an inclusion of the quotient of Weyl groups $\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{p}}$ into the Weyl group $\mathcal{W}_{\mathfrak{g}}$ by representing cosets by minimal length elements. We allow the reflections in the Hasse diagram to act affinely on the weight

$\lambda =$ 

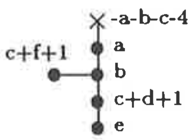
,

where $\lambda + \delta$ is $\mathfrak{g}-$dominant (i.e. $a, \ldots, f$ are all greater than or equal to -1). Then we take the corresponding Verma modules with highest weights given by the $\mathfrak{p}-$dominant weights in the diagram. If a weight in this diagram is not $\mathfrak{p}-$dominant, then no Verma module will occur in that place. For example, to get the fourth Verma module from the top, we would perform simple affine reflections on the nodes '3', '2', and '1', in that order (which we would write as '$s_1 s_2 s_3$'), to get the weight



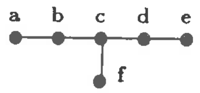
.

Then provided a, b, $c + d + 1$, e, and $c + f + 1$ are all non-negative, we would take the corresponding Verma module

$V($ 

$).$

Now the arrows represent Verma module homomorphisms. In fact, the patterns that appear in the appendix involve the corresponding vector bundles rather than the Verma modules. Hence the arrows are really invariant operators between vector bundles, and the corresponding Verma module homomorphisms would of course go in the opposite direction. The short arrows which act between adjacent levels are known as the *standard operators*, and the longer arrows are known as the *non-standard operators.* Just taking the standard operators gives us a resolution of the locally constant sheaf



known as the Bernstein-Gel'fand-Gel'fand resolution (BGG resolution). The case when $a = \ldots = f = 0$ is just the de Rham sequence on G/P, which is a resclution of the locally constant sheaf $\mathbb{C}$. In this case the standard operators are all first order, whereas the non-standard operators are always of higher order. Since these patterns are resolutions, and hence exact from level to level, it can be deduced that a lot of the compositions of standard operators result in zero. However, there are some compositions which are not zero, though we haven't explicitly labelled them on these patterns.

We mentioned above that we only include a Verma module in the pattern when we get a $\mathfrak{p}-$dominant highest weight. When $\lambda + \delta$ is inside the dominant Weyl chamber, i.e. when $a+1, \ldots, f+1$ are all strictly greater than zero, then all the weights that occur will be $\mathfrak{p}-$dominant. Thus we get a *complete* pattern, which we call *non-singular* or *regular*. If $\lambda + \delta$ is on the boundary of the dominant Weyl chamber, i.e. if at least one of $a+1, \ldots, f+1$ is zero, then some weights will not be $\mathfrak{p}-$dominant. When $\lambda+\delta$ lies on precisely one wall (precisely one of $a+1, \ldots, f+1$ is zero), we get some $\mathfrak{p}-$dominant weights. This means we get an *incomplete* pattern, which we call *singular*. When $\lambda+\delta$ lies on precisely two walls (two of $a+1, \ldots, f+1$ are zero), we still get some $\mathfrak{p}-$dominant weights. However, all that appears in the pattern in this instance is several identity operator between identical bundles. If $\lambda + \delta$ lies on more than two walls, then none of the weights will be $\mathfrak{p}-$dominant.

The classification of Verma module homomorphisms for $G = E_6$ is as follows. Of course, here we present the equivalent formulation in terms of invariant operators on the homogeneous space G/P.

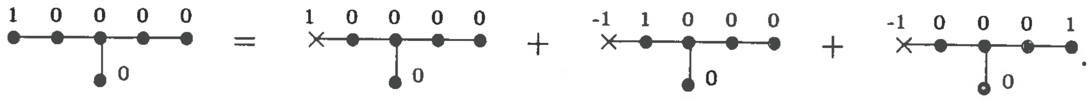
Theorem 4.1.1 *Taking the regular patterns in Appendix D.3 and the singular patterns in Appendix D.4 gives us all the invariant operators which exist on the homogeneous space G/P. Given a homogeneous vector bundle E, obtained from a $\mathfrak{p}-$module $\mathbb{E}$ with $\mathfrak{p}-$dominant minus lowest weight λ, we will find it in at most one of the above patterns, and can then read off precisely which invariant operators there are on E, or with E as their target.*

4.2 The Translation Principle: A Preliminary Example

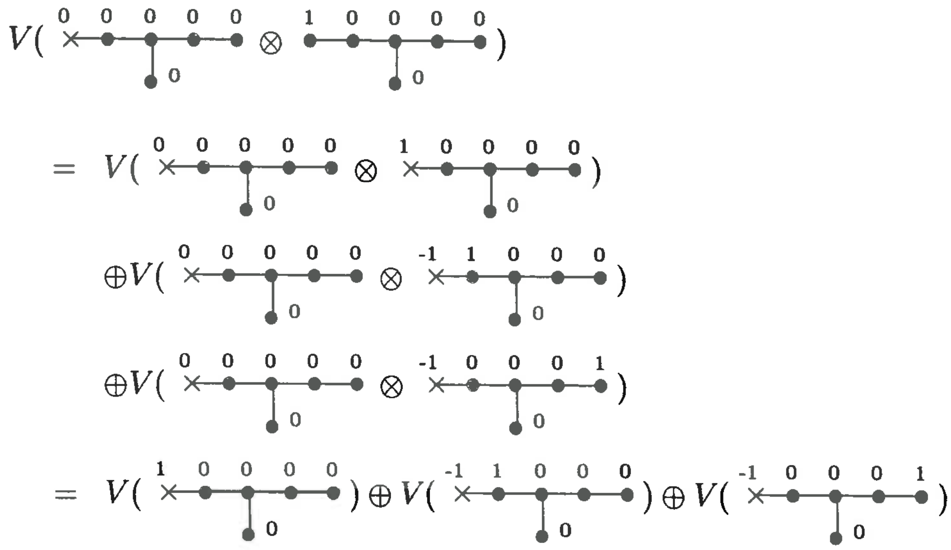
The translation principle is a method of constructing an invariant operator by beginning with one that we already know of, say $D : \mathcal{E} \to \mathcal{F}$, and then *translating* it to obtain a new invariant operator between new vector bundles, $D' : \mathcal{E}' \to \mathcal{F}'$. It is based on the Jantzmen-Zuckerman translation functor (see, for example, [10]). Our aim will be to construct as many operators as possible by translating just a few initial operators. We begin with a brief example of the translation principle; a more rigorous description follows in the subsequent subsections.

Suppose we have the first operator

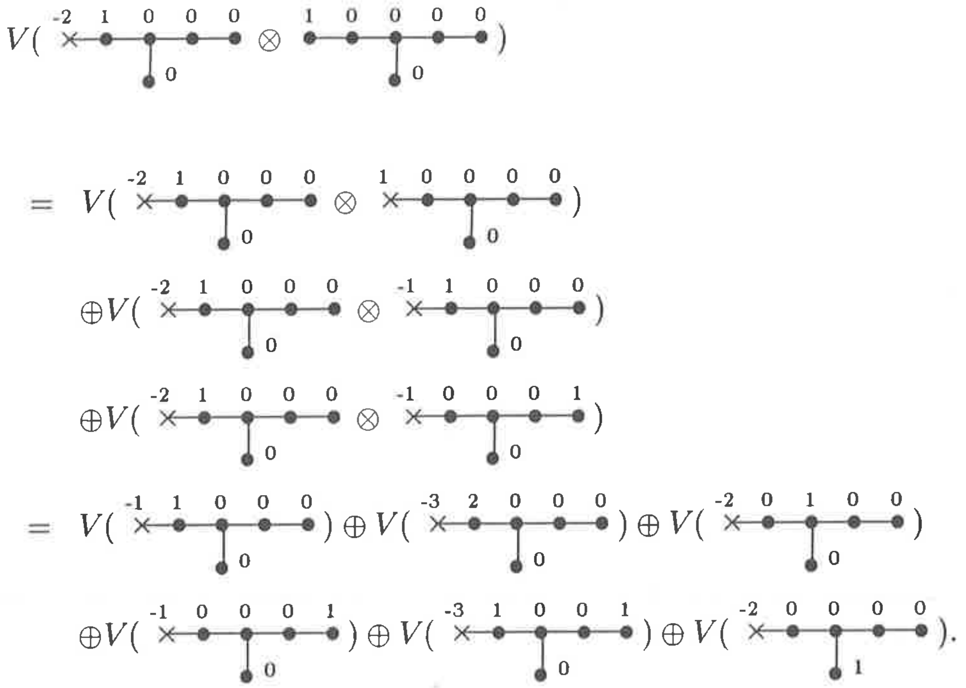
of the de Rham sequence and we wish to obtain the first operator in the BGG resolution with $a = 1$, $b = \ldots = f = 0$. We tensor with the $\mathfrak{g}$-module $\mathbb{W}$, which has composition series



The resulting Verma modules decompose into direct sums of other Verma modules according to central character, i.e.



and



We observe that

have the same central character $\chi_{1,0,0,0,0,0}$, and hence we deduce an operator (this operator will be non-zero, although not obviously so)

We can also tensor with the dual of $\mathbb{W}$, i.e. $\mathbb{W}^*$, which has composition series

This allows us to recover the original operator. In fact, this shows that the translated operator must have been non-zero, as claimed above.

This is the basic idea of how the translation principle applies; we will look at the underlying principles in the next subsections. For instance, it is not always obvious that the homomorphism resulting from a translation will be non-zero (i.e. we may not always be able to recover the original operator by translating with the dual), and in some cases it will be zero. However, we will prove a result which tells us that the resulting homomorphism must be non-zero in a large number of cases.

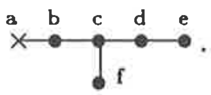
The main point is that by using the translation principle we can generate a large number of invariant operators beginning with just a few, rather than try to construct them all directly. To be more specific, we will take as our *initial data* the following operators (all of which will be first order):

- the standard operators in the case $a = \ldots = f = 0$, i.e. the de Rham sequence,
- the standard operators in each of the basic singular cases, namely $a = -1$, $b = \ldots = f = 0$, etc.

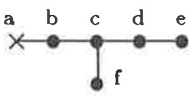
We will see that most of the invariant operators can be obtained just by translating these initial operators.

4.3 The Translation Principle: The General Description

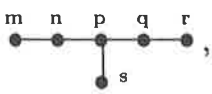
Recall that given a representation $\mathbb{E}$ of P (and hence of $\mathfrak{p}$), we get a homogeneous vector bundle on G/P which we denote by E. Let $\mathbb{E}$ be the representation



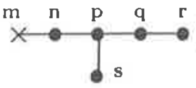
We will use this same notation to denote both $\mathbb{E}$ and E (precisely what we mean will be clear from context). We have also seen how to construct the (holonomic) Verma module $V(\mathbb{E})$, which we will also denote by

$V($  $).$

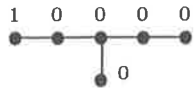
Let $\mathbb{F}$ be another representation of P (and $\mathfrak{p}$) with minus lowest weight



let F be the corresponding vector bundle and let $V(\mathbb{F})$ be the corresponding Verma module, also denoted by

$V($  $).$

Suppose we have an invariant differential operator from E to F, or equivalently, a $\mathfrak{U}(\mathfrak{g})$-module homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$. We wish to construct new Verma module homomorphisms (and hence new invariant operators from this one). Let $\mathbb{W}$ be the representation of G (and hence of $\mathfrak{g}$) with minus lowest weight



We can tensor $\mathbb{W}$ onto the representations $\mathbb{E}$ and $\mathbb{F}$, and then couple our Verma module homomorphism with the identity on $\mathbb{W}^*$, to get

$$V(\mathbb{F} \otimes \mathbb{W}) = V(\mathbb{F}) \otimes \mathbb{W}^* \to V(\mathbb{E}) \otimes \mathbb{W}^* = V(\mathbb{E} \otimes \mathbb{W}).$$

Now $\mathbb{F} \otimes \mathbb{W}$ and $\mathbb{E} \otimes \mathbb{W}$ will rarely be irreducible. However, we can decompose both of these $\mathfrak{p}$-modules into direct sums of $\mathfrak{g}_0$-modules (recall that $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1$), and hence

$$\mathbb{F} \otimes \mathbb{W} = \bigoplus_i \mathbb{F}_i$$

and

$$\mathbb{E} \otimes \mathbb{W} = \bigoplus_i \mathbb{E}_i,$$

where $\mathbb{F}_i$ and $\mathbb{E}_i$ are also $\mathfrak{p}$-modules, but the above tensor products only decompose into direct sums as $\mathfrak{g}_0$-modules.

Now suppose that $\mathbb{F}_1$ has different central character to all other $\mathbb{F}_i$ $(i \neq 1)$. Since the central character is an invariant under the action of $\mathfrak{g}$, it follows that we can write the Verma module $V(\mathbb{F} \otimes \mathbb{W})$ as

$$V(\mathbb{F}_1) \oplus V(\bigoplus_{i \neq 1} \mathbb{F}_i),$$

or in other words, there exists a $\mathfrak{U}(\mathfrak{g})$-invariant projection

$$V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$$

and a $\mathfrak{U}(\mathfrak{g})$-invariant inclusion

$$V(\mathbb{F} \otimes \mathbb{W}) \leftarrow V(\mathbb{F}_1).$$

Similarly, if $\mathbb{E}_1$ has different central character to all other $\mathbb{E}_i$ $(i \neq 1)$ then we also get a $\mathfrak{U}(\mathfrak{g})$-module splitting of $V(\mathbb{E} \otimes \mathbb{W})$, with $V(\mathbb{E}_1)$ splitting off. Combining these splittings with our coupled Verma module homomorphism, we get

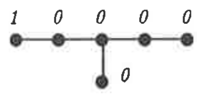
$$V(\mathbb{F}_1) \to V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{E}_1),$$

i.e. a new Verma module homomorphism and hence a new invariant operator from E_1 to F_1.

4.4 Criteria for the Non-vanishing of the Translated Operator

We have seen how to obtain a new invariant operator from an old one. However, it is possible that this new operator will vanish. We will describe here a general criteria which will ensure that the new operator is non-zero. This criteria will apply in a vast number of cases and will enable us to construct most of the invariant operators that we want, starting with just our initial data.

Lemma 4.4.1 *Let E and F be two homogeneous vector bundles on G/P. Let W be the trivial bundle with fibre $\mathbb{W}$, coming from the representation $\mathbb{W}$ of G with minus lowest weight*



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and let W^* be its dual bundle (i.e. the bundle obtained from the dual representation $\mathbb{W}^*$). Let $\mathrm{Diff}(E, F)$ be the vector space of invariant linear differential operators from E to F. Then there is a canonical isomorphism

$$\mathrm{Diff}(E, F \otimes W) = \mathrm{Diff}(E \otimes W^*, F).$$

Proof: We will use the notation F^A for $F \otimes W$, and E_A for $E \otimes W^*$. Let $e \mapsto K^A e$ be an operator in $\mathrm{Diff}(E, F^A)$ and let $f_B \mapsto L^B f_B$ be an operator in $\mathrm{Diff}(E_B, F)$. We will also write $e_B \mapsto K^A e_B$ (in $\mathrm{Diff}(E_B, F^A_B)$) for the above operator coupled with the identity on W^*, and similarly for other examples like this one.

Now define

$$\Phi : \mathrm{Diff}(E, F^A) \to \mathrm{Diff}(E_B, F)$$

by $(\Phi K^A)^B f_B = \delta_A{}^B K^A f_B$, and

$$\Psi : \mathrm{Diff}(E_B, F) \to \mathrm{Diff}(E, F^A)$$

by $(\Psi L^B)^A e = L^B(\delta_B{}^A e)$. We want to show that Φ and Ψ are inverses, and hence that we have a canonical isomorphism between $\mathrm{Diff}(E, F^A)$ and $\mathrm{Diff}(E_B, F)$.

Start with $e \mapsto K^A e$ in $\mathrm{Diff}(E, F^A)$. Applying Φ we get

$$f_B \mapsto (\Phi K^A)^B f_B = \delta_A{}^B K^A f_B$$

in $\mathrm{Diff}(E_B, F)$. Now applying Ψ we get back to

$$e \mapsto (\Psi(\Phi K^A)^B)^C e = (\Phi K^A)^B(\delta_B{}^C e) = \delta_A{}^B K^A \delta_B{}^C e = K^C e,$$

and hence $\Psi \circ \Phi = \mathrm{Id}$. Similarly, starting with $f_B \mapsto L^B f_B$ in $\mathrm{Diff}(E_B, F)$ and applying Ψ we get

$$e \mapsto (\Psi L^B)^A e = L^B(\delta_B{}^A e)$$

in $\mathrm{Diff}(E, F^A)$. Then when we apply Φ to this we get back

$$f_C \mapsto (\Phi(\Psi L^B)^A)^C f_C = \delta_A{}^C(\Psi L^B)^A f_C = \delta_A{}^C L^B(\delta_B{}^A f_C) = L^C f_C,$$

showing that $\Phi \circ \Psi = \mathrm{Id}$, which completes the proof. □

Remark: The above lemma will actually hold when $\mathbb{W}$ is any irreducible representation of G.

We can also rewrite this result in terms of Verma modules. Let $\mathrm{Hom}(V(\mathbb{F}), V(\mathbb{E}))$ be the space of $\mathfrak{U}(\mathfrak{g})$-module homomorphisms from $V(\mathbb{F})$ to $V(\mathbb{E})$. Then the above lemma says that

$$\mathrm{Hom}(V(\mathbb{F}) \otimes \mathbb{W}^*, V(\mathbb{E})) = \mathrm{Hom}(V(\mathbb{F}), V(\mathbb{E}) \otimes \mathbb{W}).$$

We will now describe a condition that will ensure that the translated operator is indeed non-zero.

Theorem 4.4.2 *Suppose that in the direct sum decomposition (as $\mathfrak{g}_0$-modules)*

$$\mathbb{E} \otimes \mathbb{W} = \bigoplus_i \mathbb{E}_i,$$

$\mathbb{E}_1$ has central character different to all other $\mathbb{E}_i$ $(i \neq 1)$, and similarly for $\mathbb{F}_1$ in the decomposition of $\mathbb{F} \otimes \mathbb{W}$. This allows us to obtain a new Verma module homomorphism $V(\mathbb{F}_1) \to V(\mathbb{E}_1)$ from the existing one $V(\mathbb{F}) \to V(\mathbb{E})$.

Suppose furthermore that $\mathbb{E}$ splits off from the tensor product $\mathbb{E}_1 \otimes \mathbb{W}^*$ with distinct central character, and so does $\mathbb{F}$ from the tensor product $\mathbb{F}_1 \otimes \mathbb{W}^*$. This would mean that we could also translate back to a homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ from $V(\mathbb{F}_1) \to V(\mathbb{E}_1)$. More importantly, if these conditions are all satisfied, then the homomorphisms we obtain from these translations will all be non-zero; i.e. translation in this instance will give us a new non-zero operator.

Proof: Denote by $\mathcal{M}$ the space of $\mathfrak{U}(\mathfrak{g})$-modules. Let $p : \mathcal{M} \to \mathcal{M}$ be projection to $\mathfrak{U}(\mathfrak{g})$-modules with central character equal to the central character of $\mathbb{E}$ (note that this is also the central character of $\mathbb{F}$, since there is a non-zero Verma module homomorphism from $V(\mathbb{F})$ to $V(\mathbb{E})$). Similarly, let $p_1 : \mathcal{M} \to \mathcal{M}$ be projection to $\mathfrak{U}(\mathfrak{g})$-modules with central character equal to the central character of $\mathbb{E}_1$ (and $\mathbb{F}_1$). Define two maps $\phi : \mathcal{M} \to \mathcal{M}$ and $\psi : \mathcal{M} \to \mathcal{M}$ by

$$\phi : \mathbb{M} \mapsto p_1(\mathbb{M} \otimes \mathbb{W})$$

and

$$\psi : \mathbb{M} \mapsto p(\mathbb{M} \otimes \mathbb{W}^*).$$

Note that we have

$$\phi(V(\mathbb{E})) = V(\mathbb{E}_1),$$
$$\phi(V(\mathbb{F})) = V(\mathbb{F}_1),$$
$$\psi(V(\mathbb{E}_1)) = V(\mathbb{E}),$$

and

$$\psi(V(\mathbb{F}_1)) = V(\mathbb{F}).$$

We also get an induced map on homomorphisms between $\mathfrak{U}(\mathfrak{g})$-modules. For example, if $D : \mathbb{M} \to \mathbb{N}$ is a homomorphism then $\phi(D) : \phi(\mathbb{M}) \to \phi(\mathbb{N})$ is given by $\phi(D) = p_1(D \otimes \mathrm{Id}_{\mathbb{W}})$, and similarly for $\psi(D)$.

Now by Lemma 4.4.1, we have

$$\begin{aligned} \mathrm{Hom}(V(\mathbb{E}_1), \phi(V(\mathbb{E}))) &= \mathrm{Hom}(V(\mathbb{E}_1), V(\mathbb{E}) \otimes \mathbb{W}) \\ &= \mathrm{Hom}(V(\mathbb{E}_1) \otimes \mathbb{W}^*, V(\mathbb{E})) \\ &= \mathrm{Hom}(\psi(V(\mathbb{E}_1)), V(\mathbb{E})), \end{aligned}$$

and similarly

$$\mathrm{Hom}(\phi(V(\mathbb{E})), V(\mathbb{E}_1)) = \mathrm{Hom}(V(\mathbb{E}), \psi(V(\mathbb{E}_1))).$$

The same equations hold when $\mathbb{E}$ is replaced by $\mathbb{F}$.

As a consequence of the above, we obtain

$$\mathrm{Hom}(\phi(V(\mathbb{E})), \phi(V(\mathbb{E}))) = \mathrm{Hom}(\psi\phi(V(\mathbb{E})), V(\mathbb{E})).$$

In particular, the identity on the left hand side must correspond to some non-zero homomorphism

$$\alpha : \psi\phi(V(\mathbb{E})) \to V(\mathbb{E})$$

on the right. Similarly, we can also find a non-zero homomorphism

$$\beta : V(\mathbb{E}) \to \psi\phi(V(\mathbb{E})).$$

The compositions

$$\alpha \circ \beta : V(\mathbb{E}) \to V(\mathbb{E})$$

and

$$\beta \circ \alpha : \psi\phi(V(\mathbb{E})) \to \psi\phi(V(\mathbb{E}))$$

must both be identities (up to scale), as the only non-zero homomorphisms of a Verma module to itself are multiples of the identity (the highest weight vector can only be mapped to a multiple of itself, and this generates the Verma module). Therefore, up to a constant multiple, α and β are inverses.

Furthermore, α induces isomorphisms

$$\mathrm{Hom}(V(\mathbb{E}), V(\mathbb{F})) \cong \mathrm{Hom}(\psi\phi(V(\mathbb{E})), V(\mathbb{F}))$$

and

$$\mathrm{Hom}(\psi\phi(V(\mathbb{E})), V(\mathbb{F})) \cong \mathrm{Hom}(\psi\phi(V(\mathbb{E})), \psi\phi(V(\mathbb{F}))),$$

which together imply an isomorphism

$$\mathrm{Hom}(V(\mathbb{E}), V(\mathbb{F})) \cong \mathrm{Hom}(\psi\phi(V(\mathbb{E})), \psi\phi(V(\mathbb{F}))).$$

It follows that the action of ϕ on $\mathrm{Hom}(V(\mathbb{E}), V(\mathbb{F}))$ is a bijection, with inverse given by ψ (up to a scalar multiple); i.e. translation (the action of ϕ) gives us an isomorphism

$$\mathrm{Hom}(V(\mathbb{E}), V(\mathbb{F})) \cong \mathrm{Hom}(\phi(V(\mathbb{E})), \phi(V(\mathbb{F}))) = \mathrm{Hom}(V(\mathbb{E}_1), V(\mathbb{F}_1)),$$

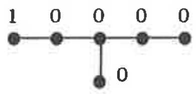
whose inverse is just translation by $\mathbb{W}^*$ (the action of ψ). In particular, when we translate an operator the new operator will indeed be non-zero. □

4.5 Applications of Theorem 4.4.2

We now wish to apply Theorem 4.4.2 in a number of different settings.

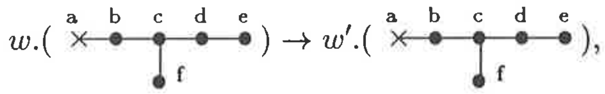
4.5.1 Regular Standard Operators

Our starting point will be the de Rham resolution, i.e. the standard operators in the BGG resolution when $a = \ldots = f = 0$. All the operators that occur in this pattern are first order, and the maximal weight vectors that give the corresponding Verma module homomorphisms could be easily found. We want to translate this initial set of operators to obtain the standard operators in all of the regular classifying patterns. This can be done merely by translating with the $\mathfrak{g}$-module $\mathbb{W}$ with minus lowest weight

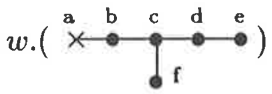


and with its dual $\mathbb{W}^*$. We will apply Theorem 4.4.2 to ensure that all these translations result in non-zero operators.

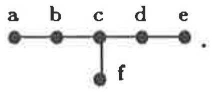
We will work inductively, so suppose that we have all the standard operators in the regular classifying pattern for some a, b, c, d, e and f (all non-negative integers). Our aim is to be able to increase each of these numbers by 1, then by induction we will have all the standard operators for every regular pattern. Let w and w' be two elements of the Weyl group $\mathcal{W}$ such that there is an operator



where

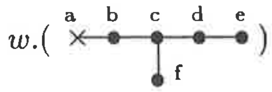


has the obvious meaning, i.e. the bundle obtained from the representation of $\mathfrak{p}$ with minus lowest weight given by w acting affinely on the weight

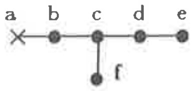


Translation is carried out by first tensoring with $\mathbb{W}$ or $\mathbb{W}^*$, then projecting out the part we are interested in. Firstly we will show that these translations will always fit into the criteria of the theorem, and secondly that we really can obtain all the operators we are after in this way.

According to our notation for labelling central characters,

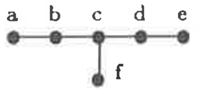


has the same central character as the $\mathfrak{p}$−module



,

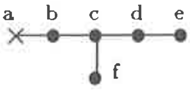
which has a $\mathfrak{g}$−dominant minus lowest weight. Indeed, they both have central character $\chi_{a,b,c,d,e,f}$. The weight

$w.($ 

$),$

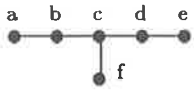
being regular, lies inside the Weyl chamber corresponding to $w \in \mathcal{W}$. To find the central character of any weight in this Weyl chamber, we merely need to apply w^{-1} to the weight so as to obtain the unique $\mathfrak{g}$−dominant weight with the same central character.

Now consider minus the lowest weights of the representations in the decomposition of the tensor product

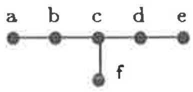
$w.($ 

$) \otimes \mathbb{W}.$

The highest weight of $\mathbb{W}$ is just the fundamental weight λ_6, which has Euclidean length $\sqrt{4/3}$, as we saw in Section 2. Indeed, all the weights of $\mathbb{W}$ are obtained by allowing the Weyl group to act on this highest weight (as can be seen from the diagram in Appendix C.1), so they must all have Euclidean length $\sqrt{4/3}$. This is also true for the weights of $\mathbb{W}^*$, which are obtained by allowing the Weyl group to act on the fundamental weight λ_1 (see Appendix C.2). Next consider the distance of the regular weight



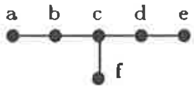
from the walls of the Weyl chamber. The walls are the planes perpendicular to the simple roots $\alpha_1, \ldots, \alpha_6$, and thus the distance to each of these walls will be

$\mathrm{dist}_i =$ 

$\cdot \frac{\alpha_i}{|\alpha_i|}$

$$= \frac{1}{\sqrt{2}}(a\delta_{1i} + \ldots + f\delta_{6i}),$$

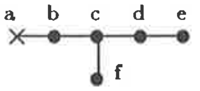
since the fundamental weights are defined so that $\lambda_i.\alpha_j = \delta_{ij}$. So a regular weight will be distance at least $1/\sqrt{2}$ from each wall (note that $a, \ldots, f$ are only non-negative, so they could be zero; what we really mean here is that the weight translated by δ will be distance at least $1/\sqrt{2}$ from each wall). Hence the distance between two regular weights in *distinct* Weyl chambers must be at least twice this, namely $\sqrt{2}$, which is greater than the lengths of the weights of $\mathbb{W}$ that we are translating with (i.e. greater than $\sqrt{4/3}$). In particular, when we add minus the weights of $\mathbb{W}$ to the minus lowest weight

$w.($ 

$)$

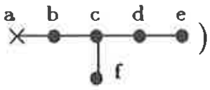
we cannot move into a new Weyl chamber. At worst we could move onto the boundary of the Weyl chamber we started in.

It follows from the above argument that to find the central characters of minus the lowest weights of the representations in the tensor product decomposition, we need to apply w^{-1} to each weight, to bring it back to the fundamental Weyl chamber (or onto the boundary of the fundamental Weyl chamber). Suppose then that two distinct minus lowest weights of representations in this decomposition, β_1 and β_2 say, have the same central character. Then we must have $w^{-1}.(\beta_1) = w^{-1}.(\beta_2)$, which would imply $\beta_1 = \beta_2$, a contradiction. It follows that all the $\mathfrak{p}$−modules in the decomposition of

$w.($ 

$) \otimes \mathbb{W}$

must have distinct central characters. Of course, the same is true for

$w'.($ 

$) \otimes \mathbb{W}$,

and when $\mathbb{W}$ is replaced by $\mathbb{W}^*$. Indeed whenever we translate one of these regular operators with $\mathbb{W}$ or $\mathbb{W}^*$, everything will split off with unique central character, and hence Theorem 4.4.2 will always be applicable.

This is one of the reasons why we only want to translate using $\mathbb{W}$ and $\mathbb{W}^*$, which have as minus lowest weights the fundamental weights λ_1 and λ_5. The other fundamental weights all have lengths greater than or equal to $\sqrt{2}$. So if we were to translate with representations which have as minus lowest weights the fundamental weights λ_2, λ_3, λ_4, or λ_6, then minus the lowest weights of some of the representations in the tensor product decomposition could move into a new Weyl chamber. It is then possible that two or more terms will occur with the same central character, making it impossible to apply Theorem 4.4.2.

Next we will show that we can increase each of a, b, c, d, e and f by 1 by performing these translations. Firstly, consider just a. In the orbit of the lowest weight

$$\begin{smallmatrix} -1 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$$

of $\mathbb{W}$ under the action of $\mathcal{W}$, we must get the weight

$$w\left(\begin{smallmatrix} -1 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right).$$

Therefore this weight will occur in $\mathbb{W}$. Then in the tensor product

$$w.\left(\begin{smallmatrix} a & b & c & d & e \\ & & f & & \end{smallmatrix}\right) \otimes \mathbb{W},$$

there will be a component of minus lowest weight

$$w.\left(\begin{smallmatrix} a & b & c & d & e \\ & & f & & \end{smallmatrix}\right) - w\left(\begin{smallmatrix} -1 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}\right) = w.\left(\begin{smallmatrix} a+1 & b & c & d & e \\ & & f & & \end{smallmatrix}\right).$$

Similarly,

$$w'.\left(\begin{smallmatrix} a & b & c & d & e \\ & & f & & \end{smallmatrix}\right) \otimes \mathbb{W}$$

will contain a component of minus lowest weight

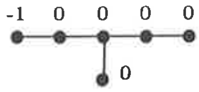
$$w'.\left(\begin{smallmatrix} a+1 & b & c & d & e \\ & & f & & \end{smallmatrix}\right).$$

Therefore we will obtain the operator with a increased by 1. By using $\mathbb{W}^*$ instead of $\mathbb{W}$, we can increase e by 1.

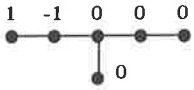
Next we want to increase b by 1. To do this, we first perform a translation so as to increase a by 1; then observe that the weight

$$\begin{smallmatrix} 1 & -1 & 0 & 0 & 0 \\ & & 0 & & \end{smallmatrix}$$

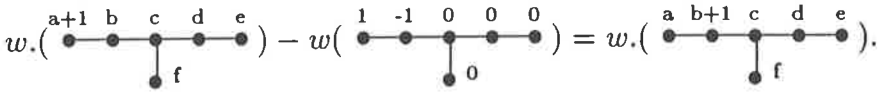
occurs in the orbit of the lowest weight



under the action of $\mathcal{W}$. Hence the weight

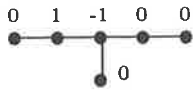
$w($  $)$

must also occur in this $\mathcal{W}$-orbit, and so when we translate again we will obtain a component of minus lowest weight

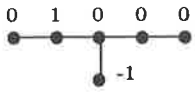


The same is true with w replaced by w', and hence we can increase b by 1. Similarly, by translating with $\mathbb{W}^*$ twice we can increase d by 1.

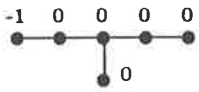
Finally, by observing that



and



both occur in the orbit of



under the action of $\mathcal{W}$, it follows that we can also increase c and f (respectively) by 1, by performing three translations with $\mathbb{W}$. In fact, we could obtain the same result by performing three translations with $\mathbb{W}^*$ instead, and it is also possible to increase f by translating once with $\mathbb{W}$ and once with $\mathbb{W}^*$.

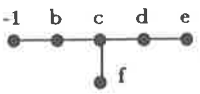
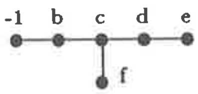
In conclusion, by induction we can obtain all standard operators in the regular patterns simply by starting with the de Rham resolution and applying the translation principle.

We should point out here that we haven't actually used any special property of the standard operators here, so the above arguments will also apply to the regular non-standard operators, provided we have the necessary initial operators (i.e. those that occur when $a = \ldots = f = 0$). However, these initial non-standard operators have orders four, six and eight; in particular, it is no trivial matter to try to find them directly by looking for maximal weight vectors in the appropriate Verma modules. Instead we will adopt a different approach. Later we will see that some of these initial regular non-standard operators can be obtained by translating singular non-standard operators, which in turn can be obtained by translating singular standard operators. In other words, the standard operators that we specified as our initial data will be all that is required to construct nearly all of the other operators. However, there will be some remaining operators that cannot be reached by translating simpler (lower order) operators, and we will see the consequences of this fact when we look at the semi-holonomic case and curved analogues.

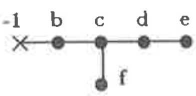
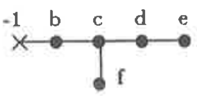
4.5.2 Singular Standard Operators

We have just seen how to obtain all the regular standard operators by beginning with the first order standard operators occurring in the de Rham sequence and using translation. We would now like to obtain a similar result for the singular standard operators. We will concentrate on the singular pattern obtained by putting $a = -1$, but the other cases can be done analogously. Recall that in this pattern we get a sequence of standard operators and one non-standard operator (see Appendix D.4). We shan't be concerned with the non-standard operator just yet. We will try to get all of the standard operators by translating the initial case of $b = \ldots = f = 0$, and using induction as before.

The main difference in the singular case is that minus the lowest weights are now on walls of Weyl chambers, rather than inside the Weyl chambers as in the regular case (of course, we really mean the weights translated by δ here). Therefore if we (affinely) reflect in the wall that the weight is on (call this reflection w_α), then the weight will be fixed. Consequently the weight can be written in two different ways

$w.($  $)$ and $w_\alpha w.($  $)$.

This means that an operator

$w.($  $) \to w'.($  $)$

between two bundles will also occur twice, because each bundle can be written in two different ways. This repetition can clearly be seen in the singular patterns in Appendix D.4. The reason that the bundles occur exactly twice is that the weight is on one wall only, so there is a unique reflection w_α which fixes it.

As before, we can increase e by tensoring with the dual representation $\mathbb{W}^*$. In the tensor product

$$w.\left(\begin{smallmatrix} -1 & b & c & d & e \\ & & f & & \end{smallmatrix}\right) \otimes \begin{smallmatrix} 0 & 0 & 0 & 0 & 1 \\ & & 0 & & \end{smallmatrix}$$

there will be a term with minus lowest weight

$$w.\left(\begin{smallmatrix} -1 & b & c & d & e \\ & & f & & \end{smallmatrix}\right) - w\left(\begin{smallmatrix} 0 & 0 & 0 & 0 & -1 \\ & & 0 & & \end{smallmatrix}\right) = w.\left(\begin{smallmatrix} -1 & b & c & d & e+1 \\ & & f & & \end{smallmatrix}\right).$$

We need to know that this is the unique term in the decomposition with this central character. We know that

$$w.\left(\begin{smallmatrix} -1 & b & c & d & e \\ & & f & & \end{smallmatrix}\right) \qquad \text{and} \qquad w.\left(\begin{smallmatrix} -1 & b & c & d & e+1 \\ & & f & & \end{smallmatrix}\right)$$

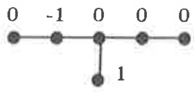
both lie on the same wall of a Weyl chamber. Minus the other lowest weights in the decomposition of the tensor product will either lie on this same wall, or lie inside or on a different wall of one of the adjacent Weyl chambers (this follows by considering the Euclidean lengths of the weights we are translating with, similarly to the regular case). In particular, since the Weyl group is generated by *reflections* in walls (not rotations), none of these minus lowest weights could possibly be in the same Weyl group orbit as

$$w.\left(\begin{smallmatrix} -1 & b & c & d & e+1 \\ & & f & & \end{smallmatrix}\right).$$

Therefore, this bundle splits off with unique central character. Of course, the same is true when we translate back using $\mathbb{W}$, and of course it is also true when we replace w by w'. So by Theorem 4.4.2, this translation results in a non-zero operator in both directions, and hence we can increase e by 1.

In fact, the above argument shows that the criteria of Theorem 4.4.2 are satisfied whenever we translate minus our lowest weight from one point on the wall to another. Next we need to show that we can actually increase each of b, c, d, and f by 1 just by translating with $\mathbb{W}$ and $\mathbb{W}^*$.

As in the regular case, we can increase d by 1 by translating with $\mathbb{W}^*$ twice, and c and f by translating with $\mathbb{W}^*$ three times. The difficulty lies with b. In the regular case we first increased a by 1 before increasing b, but in the singular case we cannot do this, as a is fixed at -1 (if we did try to increase a we would get back the regular case, which we don't want; furthermore, the conditions of Theorem 4.4.2 would not be satisfied). However, we observe that



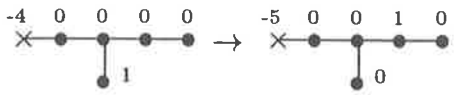
occurs amongst the weights of $\mathbb{W}^*$, and hence after translating with $\mathbb{W}^*$ three times to increase f by 1, we can translate with $\mathbb{W}^*$ once more to increase b by 1. Hence by induction we can obtain all the standard operators in the singular pattern $a = -1$.

Similarly, applying the same arguments to the five other singular patterns (each of $b = -1$, $c = -1$, $d = -1$, $e = -1$, and $f = -1$) enables us to obtain all the standard operators in the singular patterns by translation, starting only with our initial data.

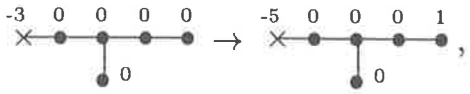
4.5.3 Singular Non-Standard Operators

So far we have looked only at standard operators. The non-standard operators in the singular patterns can also be obtained by translating the initial cases, and we will not repeat the argument here (i.e. it is the same as the argument for the singular standard operators). However, the initial cases are all of order greater than one, and so maximal weight vectors are considerably more difficult to find directly. Instead, we will see that all of the non-standard initial cases can be obtained by translating the standard initial data.

To begin with, let us show that when we translate the first order operator



using $\mathbb{W}$, then we get a non-zero second order operator



and we also get a non-zero operator if we translate back. In fact, this translation fits into the criteria of Theorem 4.4.2, as we shall see.

Firstly, we can decompose the tensor product

$$\begin{smallmatrix}-4&0&0&0&0\\&&1&&\end{smallmatrix} \otimes \mathbb{W} = \left(\begin{smallmatrix}-3&0&0&0&0\\&&1&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-5&1&0&0&0\\&&1&&\end{smallmatrix} \oplus \begin{smallmatrix}-4&0&0&1&0\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-3&0&0&0&0\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-5&0&0&0&1\\&&1&&\end{smallmatrix} \oplus \begin{smallmatrix}-5&1&0&0&0\\&&0&&\end{smallmatrix}\right).$$

Observe that only $\begin{smallmatrix}-3&0&0&0&0\\&&0&&\end{smallmatrix}$ has central character $\chi_{0,0,-1,0,0,0}$. Similarly, in

$$\begin{smallmatrix}-5&0&0&1&0\\&&0&&\end{smallmatrix} \otimes \mathbb{W} = \left(\begin{smallmatrix}-4&0&0&1&0\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-6&1&0&1&0\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-5&0&0&0&1\\&&1&&\end{smallmatrix} \oplus \begin{smallmatrix}-5&1&0&0&0\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-6&0&0&1&1\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-6&0&1&0&0\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-5&0&0&0&1\\&&0&&\end{smallmatrix}\right),$$

only $\begin{smallmatrix}-5&0&0&0&1\\&&0&&\end{smallmatrix}$ has central character $\chi_{0,0,-1,0,0,0}$. In the other direction, when we decompose

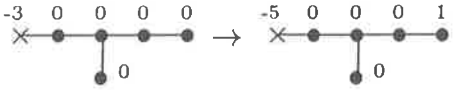
$$\begin{smallmatrix}-3&0&0&0&0\\&&0&&\end{smallmatrix} \otimes \mathbb{W}^* = \left(\begin{smallmatrix}-3&0&0&0&1\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-4&0&0&0&0\\&&1&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-4&0&0&0&0\\&&0&&\end{smallmatrix}\right),$$

we find that only $\begin{smallmatrix}-4&0&0&0&0\\&&1&&\end{smallmatrix}$ has central character $\chi_{0,0,0,-1,0,0}$, and in

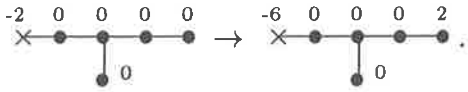
$$\begin{smallmatrix}-5&0&0&0&1\\&&0&&\end{smallmatrix} \otimes \mathbb{W}^* = \left(\begin{smallmatrix}-5&0&0&0&2\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-5&0&0&1&0\\&&0&&\end{smallmatrix} \oplus \begin{smallmatrix}-4&0&0&0&0\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-6&0&0&0&1\\&&1&&\end{smallmatrix} \oplus \begin{smallmatrix}-6&1&0&0&0\\&&0&&\end{smallmatrix}\right) + \left(\begin{smallmatrix}-6&0&0&0&1\\&&0&&\end{smallmatrix}\right),$$

only $\begin{smallmatrix}-5&0&0&1&0\\&&0&&\end{smallmatrix}$ has central character $\chi_{0,0,0,-1,0,0}$. Therefore by Theorem 4.4.2, we get a non-zero operator when we translate in either direction, but most importantly, we can obtain the second order operator from the first order one.

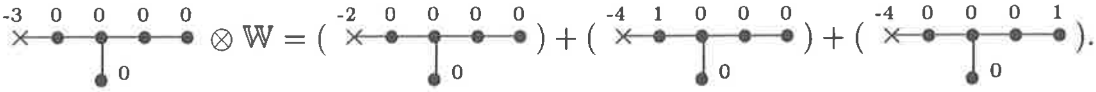
Next, we will show that by translating the second order operator

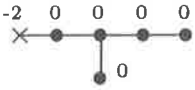


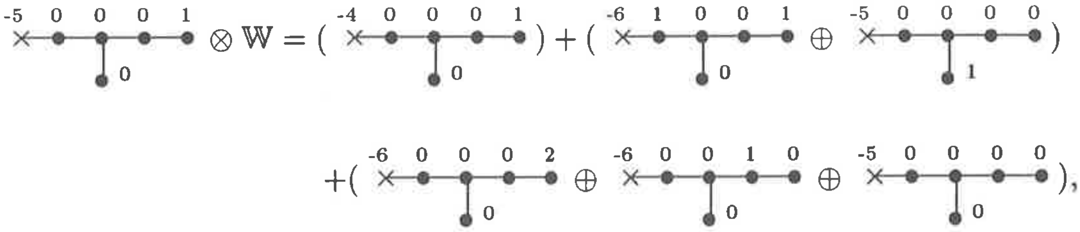
by $\mathbb{W}$, we get the fourth order operator

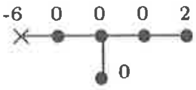


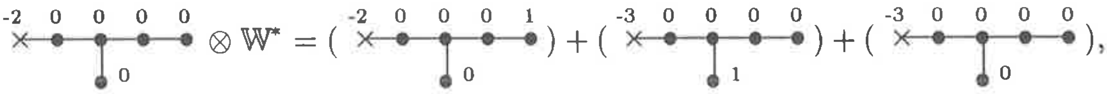
We decompose the tensor product

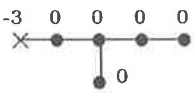


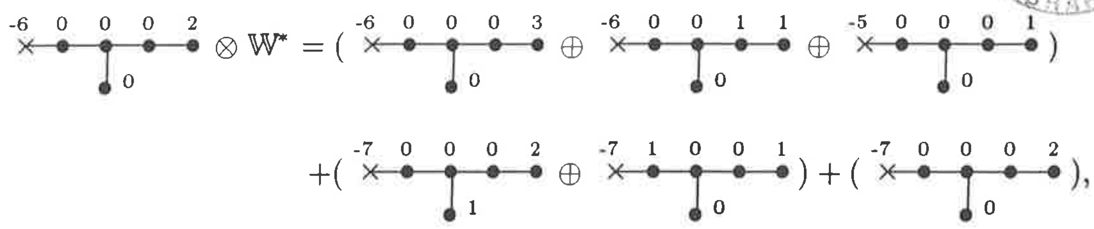
and observe that only  has central character $\chi_{0,-1,0,0,0,0}$. Similarly, in the decomposition

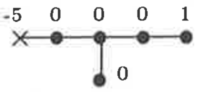


only  has central character $\chi_{0,-1,0,0,0,0}$. In the other direction, we decompose



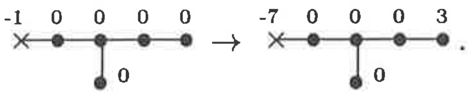
and only  has central character $\chi_{0,0,-1,0,0,0}$. Finally, in the decomposition



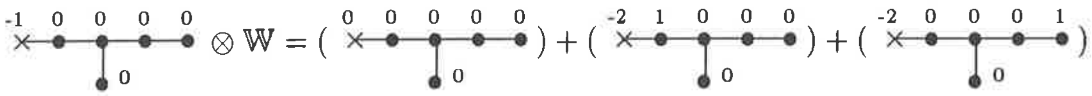
only 

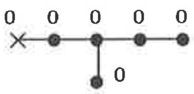
has central character $\chi_{0,0,-1,0,0,0}$. Therefore by Theorem 4.4.2, translation gives a non-zero operator in both directions. In particular, we can obtain the fourth order operator by translating the second order one.

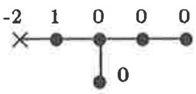
Similarly, we can translate the fourth order operator to obtain a sixth order operator



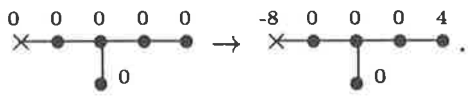
However, when we try to translate again we find that



contains two terms, 

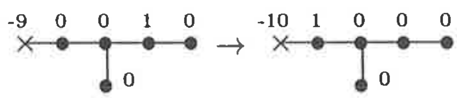
and 

, which both have central character $\chi_{0,0,0,0,0,0}$. Hence translation breaks down at this stage and we do not obtain the eighth order operator



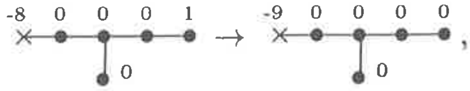
Indeed, this is a regular operator, and so we cannot reasonably expect to obtain it this easily.

If we look at the adjoint[1] operators, we find that we can also translate from the first order operator

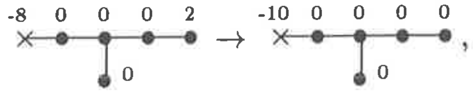


[1]There is a natural notion of adjointness between operators which we will not describe in detail. This essentially reflects the symmetry of the classifying patterns, with operators in the bottom half of the pattern being adjoints of the corresponding operators in the top half.

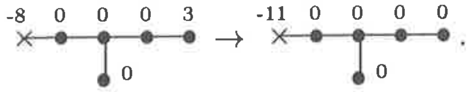
to the second order operator



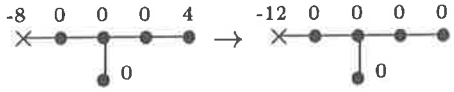
from this second order operator to the fourth order operator



and from this fourth order operator to the sixth order operator

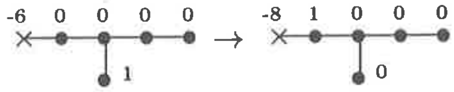


However, we cannot obtain the eighth order operator

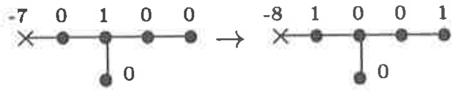


by translating the sixth order one.

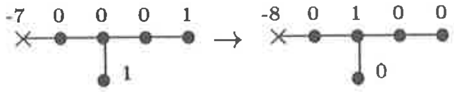
In a similar way we can get the second order operator



either by translating the first order operator

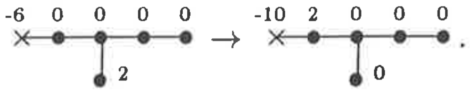


by $\mathbb{W}$, or by translating the first order operator



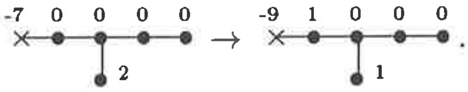
by $\mathbb{W}^*$.

We would then like to translate from this second order operator to the fourth order one



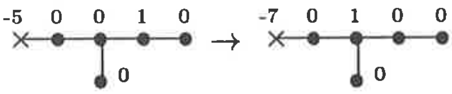
If we try to translate with $\mathbb{W}$ or $\mathbb{W}^*$, we find that it is not possible to get the fourth order operator in one step. However, we know that we can translate the second order operator to obtain a whole family of operators

In particular, putting $a = b = d = f = 0$ and $e = 1$, we get

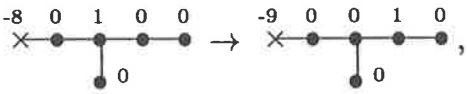


By translating this operator with $\mathbb{W}^*$, we can obtain the fourth order operator we were after.

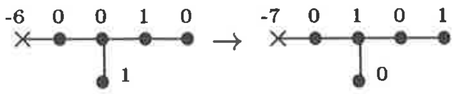
This leaves just two singular non-standard initial operators, namely



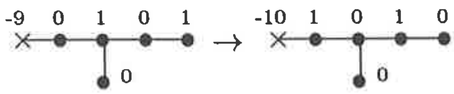
and



both of second order. These can be obtained by translating the first order operators



and



by $\mathbb{W}$ and $\mathbb{W}^*$ respectively.

4.6 One Way Translation

So far we have described a criteria which ensures that translation gives us a new non-zero operator when we translate in either direction, and we have used this result to obtain a large number of operators beginning with just our first order initial data. Our translation procedure relied on the fact that the appropriate Verma modules split off from the tensor products with *unique* central character. Theorem 4.4.2 also required that this be true in the reverse direction, in which case we could deduce that the new operator would be non-zero.

However, we can still perform translations in the case that the appropriate Verma modules don't split off from the tensor products with unique central character. Immediately it is clear that Theorem 4.4.2 will not apply, but this does not mean that the translated operator must necessarily be zero. Indeed, we will see that there are cases where the new operator will be non-zero, but we will get the zero operator if we try to translate back to the original operator. We shall refer to these translations as *one way* translations. The examples of these one way translations that we shall be concerned with is the translations from the singular non-standard operators to the regular non-standard operators. We shall begin with a general description of how this kind of translation works, and why it gives a non-zero operator.

First consider what happens when we try to translate in the other direction; i.e. suppose we have with a regular operator

$$V(\mathbb{E}) \leftarrow V(\mathbb{F})$$

that we would like to translate to get the singular operator

$$V(\mathbb{E}_1) \leftarrow V(\mathbb{F}_1).$$

When we consider the tensor products

$$V(\mathbb{E} \otimes \mathbb{W}) = V(\mathbb{E}) \otimes \mathbb{W}^*$$

and

$$V(\mathbb{F} \otimes \mathbb{W}) = V(\mathbb{F}) \otimes \mathbb{W}^*,$$

it is clear that $V(\mathbb{E}_1)$ and $V(\mathbb{F}_1)$ (respectively) each split off with unique central characters. This is because we are going from a regular highest weight (inside a Weyl chamber) to a singular highest weight (on a wall of that Weyl chamber); as all the highest weights for terms in the tensor product necessarily lie either in that Weyl chamber or on one of its walls, then they could not possibly be related to our singular weight by a Weyl reflection.

So the appropriate maps (inclusion and projection of Verma modules) exist for us to be able to perform the translation, and hence we get

$$\begin{array}{ccc} V(\mathbb{E}_1) & \leftarrow & V(\mathbb{E}\otimes\mathbb{W}) \\ & \nwarrow & \uparrow \\ V(\mathbb{F}_1) & \rightarrow & V(\mathbb{F}\otimes\mathbb{W}). \end{array}$$

However, we are going to show that this new operator is really zero, by looking at the diagonal operator that appears here and showing that it is zero.

The first step is to apply Lemma 4.4.1, which says that it is equivalent to consider the following diagonal operator

$$\begin{array}{ccc} V(\mathbb{E}_1\otimes\mathbb{W}^*) & \leftarrow & V(\mathbb{E}) \\ & \nwarrow & \uparrow \\ & & V(\mathbb{F}). \end{array}$$

We need to know how $V(\mathbb{E}_1\otimes\mathbb{W}^*)$ decomposes. Obviously $V(\mathbb{E})$ will occur, but not with unique central character. As we are translating a singular highest weight (on a wall of a Weyl chamber) to a regular highest weight (inside an adjacent Weyl chamber), we will also get a second regular highest weight with the same central character (i.e. the weight on the opposite side of the wall, related to the first regular weight by a reflection in that wall). If we call the second regular highest weight module $V(\mathbb{E}')$, then what we have is a composition series which looks like

$$V(\mathbb{E}_1\otimes\mathbb{W}^*) = (\ldots) + (V(\mathbb{E}')\oplus\ldots) + (V(\mathbb{E})\oplus\ldots),$$

or

$$V(\mathbb{E}_1\otimes\mathbb{W}^*) = (V(\mathbb{E}')\oplus\ldots) + (V(\mathbb{E})\oplus\ldots) + (\ldots),$$

where we have not shown modules with other central characters. This means that we have invariant operators, inclusion

$$V(\mathbb{E}') \to V(\mathbb{E}_1\otimes\mathbb{W}^*),$$

and projection

$$V(\mathbb{E}_1\otimes\mathbb{W}^*) \to V(\mathbb{E}).$$

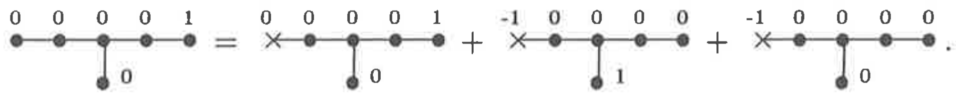
Note that the order of the composition series is important, as the two regular Verma modules do not split off completely; i.e. there does not exist an invariant projection

$$V(\mathbb{E}_1\otimes\mathbb{W}^*) \to V(\mathbb{E}'),$$

nor does there exist an invariant inclusion

$$V(\mathbb{E}) \to V(\mathbb{E}_1 \otimes \mathbb{W}^*).$$

The order is determined by where the $\mathfrak{p}$−modules $\mathbb{E}$ and $\mathbb{E}'$ occur in the decomposition of the tensor product $\mathbb{E}_1 \otimes \mathbb{W}^*$, bearing in mind that $\mathbb{W}^*$ has composition series



We have written the composition series for $V(\mathbb{E}_1 \otimes \mathbb{W}^*)$ with $V(\mathbb{E}')$ occuring to the left of $V(\mathbb{E})$ as it is what we require for our arguments (of course, when we apply this result to specific examples, it must be checked that the composition series really does follow this order).

Now consider the composition

$$\begin{tikzcd} V(\mathbb{E}) & V(\mathbb{E}_1 \otimes \mathbb{W}^*) \arrow[l] & V(\mathbb{E}) \arrow[l] \\ & & V(\mathbb{F}). \arrow[ul] \arrow[u] \end{tikzcd}$$

The composition of the top two operators must be zero, as there does not exist a differential splitting ($V(\mathbb{E})$ does not split off from the tensor product). It follows that the diagonal operator can depend only on $V(\mathbb{E}')$, i.e. it factors through $V(\mathbb{E}')$

$$\begin{tikzcd} V(\mathbb{E}_1 \otimes \mathbb{W}^*) & V(\mathbb{E}') \arrow[l] & \\ & & V(\mathbb{F}). \arrow[ul] \end{tikzcd}$$

Finally we will assume that there does not exist an operator

$$V(\mathbb{E}') \leftarrow V(\mathbb{F})$$

and so the diagonal operator must necessarily be zero. When we apply this result to specific examples, this assumption can easily be checked from the classification of homomorphisms of Verma modules, Theorem 4.1.1. It then follows that trying to translating the regular operator to the singular operator results in zero.

Next consider translating in the opposite direction, from the singular operator to the regular operator. We have already assumed that we have a composition series which looks like either

$$V(\mathbb{E}_1 \otimes \mathbb{W}^*) = (\ldots) + (V(\mathbb{E}') \oplus \ldots) + (V(\mathbb{E}) \oplus \ldots),$$

or

$$V(\mathbb{E}_1 \otimes \mathbb{W}^*) = (V(\mathbb{E}') \oplus \ldots) + (V(\mathbb{E}) \oplus \ldots) + (\ldots),$$

where we have not shown modules with other central characters. We also need to assume that the composition series for $V(\mathbb{F}_1 \otimes \mathbb{W}^*)$ looks like either

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (\ldots) + (V(\mathbb{F}) \oplus \ldots) + (V(\mathbb{F}') \oplus \ldots),$$

or

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (V(\mathbb{F}) \oplus \ldots) + (V(\mathbb{F}') \oplus \ldots) + (\ldots),$$

not showing terms with other central characters; i.e. we assume that $V(\mathbb{F})$ occurs to the left of $V(\mathbb{F}')$. Thus the necessary inclusions and projections of Verma modules will exist for us to be able to translate the singular operator to obtain

$$\begin{tikzcd} V(\mathbb{E}) & V(\mathbb{E}_1 \otimes \mathbb{W}^*) \arrow[l] \\ V(\mathbb{F}) \arrow[u] \arrow[r] & V(\mathbb{F}_1 \otimes \mathbb{W}^*). \arrow[u] \end{tikzcd}$$

Again, we apply Lemma 4.4.1 and consider instead

$$\begin{tikzcd} V(\mathbb{E} \otimes \mathbb{W}) & V(\mathbb{E}_1) \arrow[l] \\ & V(\mathbb{F}_1). \arrow[u] \arrow[ul] \end{tikzcd}$$

We know that $V(\mathbb{E}_1)$ splits off from $V(\mathbb{E}\otimes\mathbb{W})$ with unique central character. Suppose that the diagonal operator is zero. Then so too is the composition with

$$V(\mathbb{E}_1) \leftarrow V(\mathbb{E} \otimes \mathbb{W}),$$

i.e. the following composition gives zero

$$\begin{tikzcd} V(\mathbb{E}_1) & V(\mathbb{E} \otimes \mathbb{W}) \arrow[l] & V(\mathbb{E}_1) \arrow[l] \\ & & V(\mathbb{F}_1). \arrow[u] \arrow[ul] \end{tikzcd}$$

However, the composition of the top two operators is just the identity on $V(\mathbb{E}_1)$, as this is a differential splitting. It follows that the original singular operator

$$V(\mathbb{E}_1) \leftarrow V(\mathbb{F}_1)$$

that we are translating is zero, which is absurd. This contradiction implies that the diagonal operator cannot be zero.

Then by Lemma 4.4.1, it follows that the following diagonal operator is also non-zero

$$\begin{tikzcd} V(\mathbb{E}) & V(\mathbb{E}_1 \otimes \mathbb{W}^*) \arrow[l] \\ & V(\mathbb{F}_1 \otimes \mathbb{W}^*). \arrow[ul] \arrow[u] \end{tikzcd}$$

Finally, we have assumed that we have either

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (\ldots) + (V(\mathbb{F}) \oplus \ldots) + (V(\mathbb{F}') \oplus \ldots),$$

or

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (V(\mathbb{F}) \oplus \ldots) + (V(\mathbb{F}') \oplus \ldots) + (\ldots),$$

not showing terms with other central characters. Our last assumption will be that there does not exist an operator

$$V(\mathbb{E}) \leftarrow V(\mathbb{F}')$$

(again, in our examples this assumption will be easily checked from the classification of homomorphisms of Verma modules, Theorem 4.1.1), and, therefore, when we compose the diagonal operator with the inclusion

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) \leftarrow V(\mathbb{F})$$

we necessarily get a non-zero composition

$$V(\mathbb{E}) \leftarrow V(\mathbb{F}).$$

Thus we have shown that, assuming all the composition series occur in the appropriate orders, and assuming that we know of the non-existence of certain operators, we can translate the singular operator

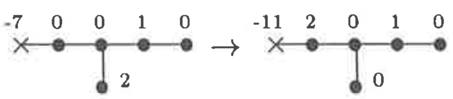
$$V(\mathbb{E}_1) \leftarrow V(\mathbb{F}_1)$$

to obtain the regular one

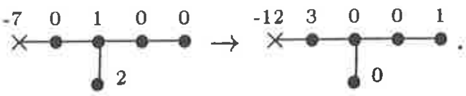
$$V(\mathbb{E}) \leftarrow V(\mathbb{F}),$$

but if we try to translate back we get the zero operator. Let us now apply this to a specific situation, to see how it works in practice.

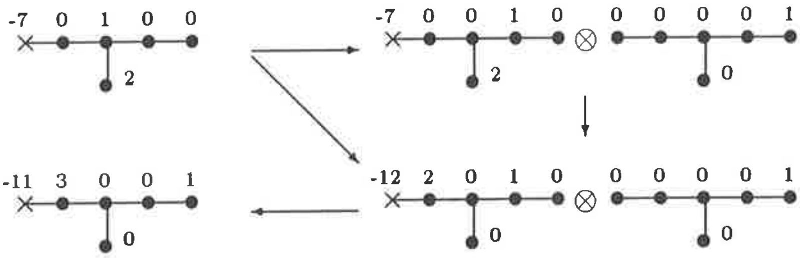
We would like to consider what will happen when we translate between the regular non-standard operator



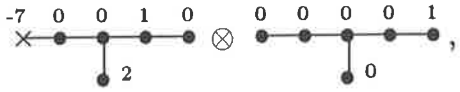
and the singular non-standard operator

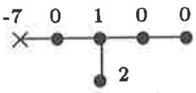


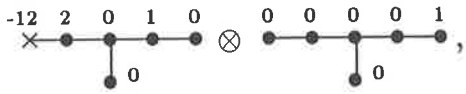
We will write down the vector bundles directly here, instead of the Verma modules. This will mean that all the arrows will be in the opposite directions. Consider first going from the regular operator to the singular one. The diagram looks like

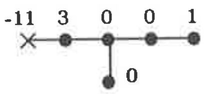


First we need to check that the horizontal operators really do exist. Decomposing

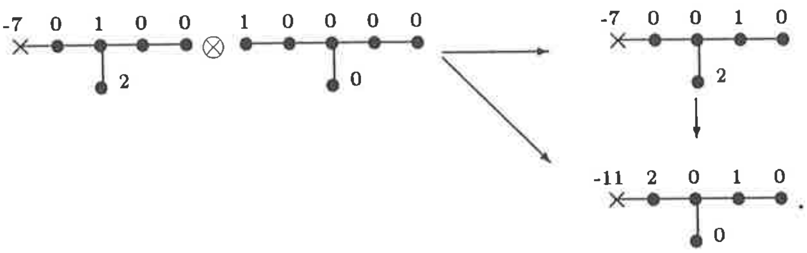


we see that  really does split off as the unique term with central character $\chi_{0,1,0,0,0,-1}$, and in

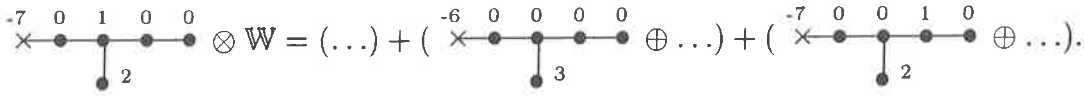


only  has central character $\chi_{0,1,0,0,0,-1}$. So the horizontal operators do exist, and we can perform the translation shown above.

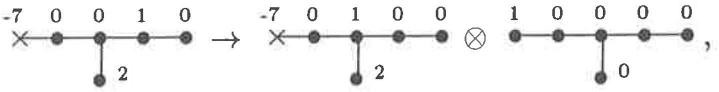
By Lemma 4.4.1, we consider the diagonal operator appearing in the diagram



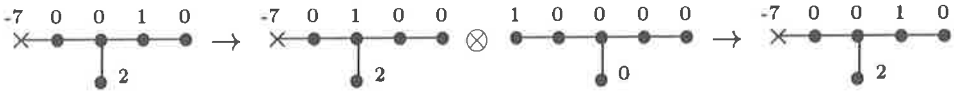
Decomposing the tensor product here, and only showing terms of central character $\chi_{0,0,0,0,0,0}$, we get the composition series

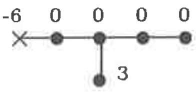


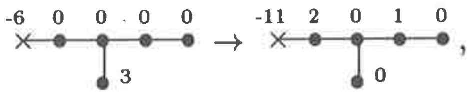
So we get an inclusion



but when we compose it with the diagonal operator above we get zero, as

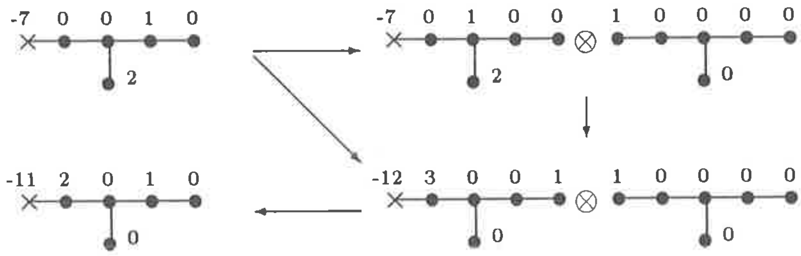


must be zero (there does not exist a differential splitting). Hence the diagonal operator can depend only on  (i.e. it factors through the projection onto this term). However, from the classification of homomorphisms of Verma modules, i.e. Theorem 4.1.1 (in particular, looking at the regular classifying pattern with $a = \ldots = f = 0$), we see that there is no non-zero operator



and hence the diagonal operator is zero also. It follows that translation applied to the regular operator will give us the zero operator.

Next we consider going from singular to regular.



We want to show that the diagonal composition is non-zero. Firstly, let us check that what we have written so far really does make sense. We have already seen the composition series for $\begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix} \otimes \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix}$, and we know that there exists an invariant inclusion

$$\begin{smallmatrix}-7&0&0&1&0\\&&2&&\end{smallmatrix} \to \begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix} \otimes \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix}.$$

Similarly, we get the composition series

$$\begin{smallmatrix}-12&3&0&0&1\\&&0&&\end{smallmatrix} \otimes \mathbb{W} = (\ldots) + (\begin{smallmatrix}-11&2&0&1&0\\&&0&&\end{smallmatrix} \oplus \ldots) + (\begin{smallmatrix}-12&3&0&0&0\\&&0&&\end{smallmatrix} \oplus \ldots),$$

not showing terms with central characters different to $\chi_{0,0,0,0,0,0}$. Hence there also exists an invariant projection

$$\begin{smallmatrix}-12&3&0&0&1\\&&0&&\end{smallmatrix} \otimes \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix} \to \begin{smallmatrix}-11&2&0&1&0\\&&0&&\end{smallmatrix},$$

so the operators we need in order to perform this translation really do exist.

By Lemma 4.4.1, we can consider the following composition.

$$\begin{tikzcd} \begin{smallmatrix}-7&0&0&1&0\\&&2&&\end{smallmatrix} \otimes \begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix} \arrow[r] \arrow[dr] & \begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix} \arrow[d] \\ & \begin{smallmatrix}-12&3&0&0&1\\&&0&&\end{smallmatrix} \end{tikzcd}$$

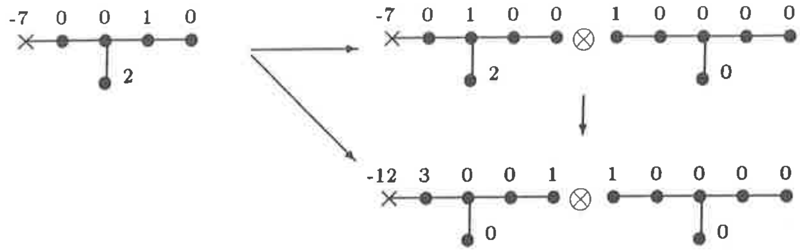
We have seen that $\begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix}$ splits off from the tensor product with unique central character, and hence there exists a differential splitting

$$\begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix} \to \begin{smallmatrix}-7&0&0&1&0\\&&2&&\end{smallmatrix} \otimes \begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix}.$$

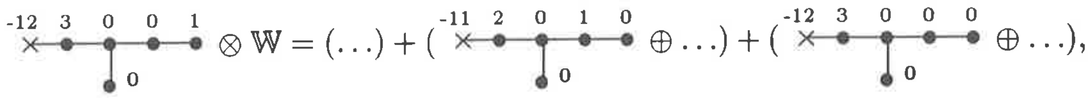
Thus if the diagonal operator above is zero, the composition

$$\begin{smallmatrix}-7&0&1&0&0\\&&2&&\end{smallmatrix} \to \begin{smallmatrix}-7&0&0&1&0\\&&2&&\end{smallmatrix} \otimes \begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix} \to \begin{smallmatrix}-12&3&0&0&1\\&&0&&\end{smallmatrix}$$

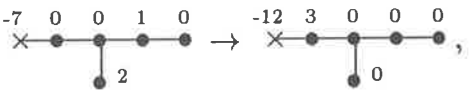
would also be zero, which is impossible since this is the singular operator that we began with. Hence the diagonal operator is non-zero, and then so is the following diagonal operator.



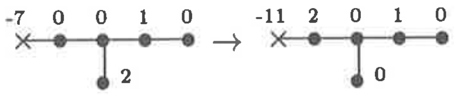
Finally, we have already seen that



not showing terms with central characters different to $\chi_{0,0,0,0,0,0}$. By the classification of homomorphisms of Verma modules, i.e. Theorem 4.1.1 (in this instance, by looking at the regular classifying pattern with $a = \ldots = f = 0$), we know that there is no non-zero operator

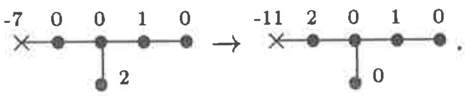


and therefore it follows that the regular operator

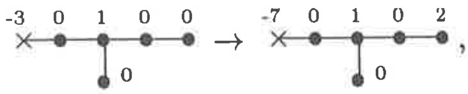


obtained by translating the singular one is non-zero. This is precisely the result we wanted.

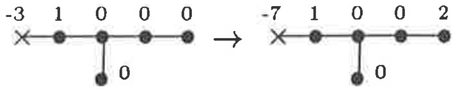
The usefulness of this one way translation is evident. By translating from the singular non-standard operators we can obtain many of the regular non-standard operators. In particular, we have just obtained the fourth order regular operator



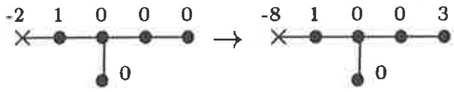
Using one way translation, we can also get the fourth order regular operator



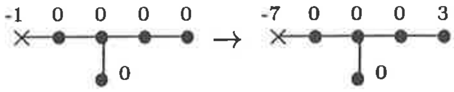
by translating the fourth order singular operator



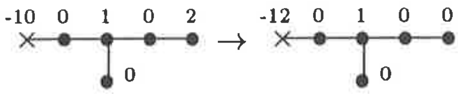
by $\mathbb{W}$, and the sixth order regular operator



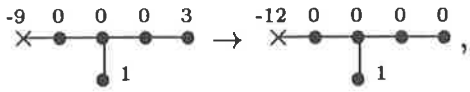
by translating the sixth order singular operator



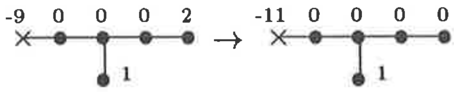
by $\mathbb{W}$. We can also obtain the adjoints of these operators, namely the fourth and sixth order regular operators



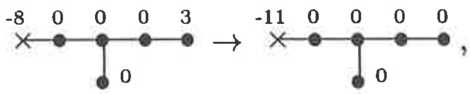
and



by translating the fourth and sixth order singular operators



and



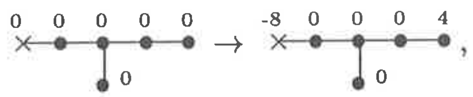
respectively, by $\mathbb{W}^*$. Then, as we mentioned at the end of Subsection 4.5.1, these regular non-standard operators form the initial operators which can be translated to obtain all of their corresponding families of operators. This just leaves several other families of non-standard regular operators which cannot be obtained by translating from simpler operators. We summarize these in the next subsection.

4.7 Operators Not Obtainable by Translation

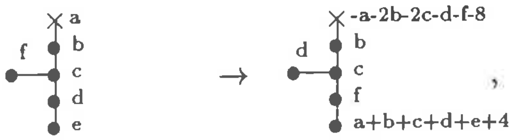
Beginning with our initial data, we have used translation to obtain all of the standard operators, in both the regular and the singular patterns, and all of the non-standard operators in the singular patterns. We have also obtained some of the non-standard operators in the regular patterns, but there are several that we have not been able to reach by translation. We will call these operators and their families *exceptional*, and summarise them below.

We have been unable to obtain the following operators by translation:

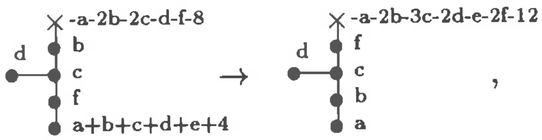
1. the eighth order operator



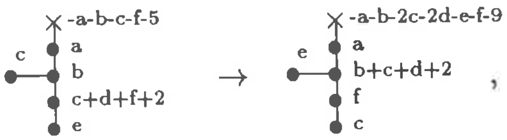
   and its associated family



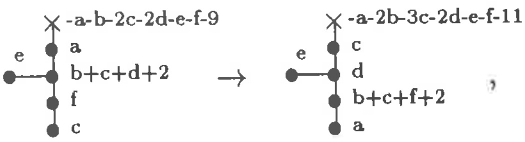
2. the adjoints of 1



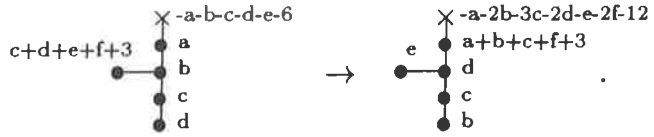
3. the family of operators



4. the adjoints of 3



5. the family of operators



We will refer to the above families as the *first exceptional family of operators*, the *second exceptional family of operators*, etc. On the other hand, when $a, \ldots, f$ are all zero, we will refer to the above operators as the *first exceptional operator*, the *second exceptional operator*, etc.

5 Semi-Holonomic Case

5.1 Lifting of the Initial Data

We want to investigate which operators admit lifts to the semi-holonomic case, allowing us to construct curved analogues. Firstly we will show that all low order operators must lift, where by low order we mean first or second order operators.

Since the first holonomic jet and the first semi-holonomic jet are the same thing, all of the first order operators automatically lift to the semi-holonomic case. This includes all of our initial data, namely the standard operators in the de Rham resolution and the standard operators in each of the basic singular patterns (i.e. each of $a = -1, b = \ldots = f = 0$, etc.). So we immediately acquire curved analogues of all these operators.

In fact, we can automatically get lifts of all the second order operators too. This follows from the following lemma.

Lemma 5.1.1 *There is a homomorphism of* $\mathfrak{p}-$*modules* $V_2(\mathbb{E}) \to \bar{V}_2(\mathbb{E})$ *that splits the projection* $\bar{V}_2(\mathbb{E}) \to V_2(\mathbb{E})$.

Proof: We define the homomorphism by the identity map on $V_1(\mathbb{E}) = \bar{V}_1(\mathbb{E}) \subset V_2(\mathbb{E})$ and by

$$XYe \mapsto \frac{1}{2}(XY + YX + [X,Y])e,$$

where $X, Y \in \mathfrak{g}$ and $e \in \mathbb{E}^*$. This is well-defined, as

$$\begin{aligned} XYe &= YXe + [X,Y]e \\ &\mapsto \frac{1}{2}(YX + XY + [Y,X])e + [X,Y]e \\ &= \frac{1}{2}(XY + YX - [X,Y])e + [X,Y]e \\ &= \frac{1}{2}(XY + YX + [X,Y])e. \end{aligned}$$

It is also $\mathfrak{p}-$equivariant, for if we allow $Z \in \mathfrak{p}$ to act on the left we get

$$\begin{aligned} Z(XYe) &= [Z,X]Ye + X[Z,Y]e + XY(Ze) \\ &\mapsto \frac{1}{2}([Z,X]Y + Y[Z,X] + [[Z,X],Y])e + \frac{1}{2}(X[Z,Y] + [Z,Y]X + \\ &\quad +[X,[Z,Y]])e + \frac{1}{2}(XY + YX + [X,Y])(Ze) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}([Z,X]Ye + X[Z,Y]e + XY(Ze)) + \frac{1}{2}([Z,Y]Xe + \\
&\quad +Y[Z,X]e + YX(Ze)) + \frac{1}{2}([[Z,X],Y]e + [X,[Z,Y]]e + [X,Y](Ze)) \\
&= \frac{1}{2}Z(XYe) + \frac{1}{2}Z(YXe) + \frac{1}{2}Z([X,Y]e) \\
&= Z(\frac{1}{2}(XY + YX + [X,Y])e),
\end{aligned}$$

where we have used the Jacobi identity $[Z,[X,Y]] = [[Z,X],Y] + [X,[Z,Y]]$. Finally, it is clear that this mapping splits the projection. □

If we have a second order operator $E \to F$, then we have a (holonomic) Verma module homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ which is determined by $\mathbb{F}^* \to V_2(\mathbb{E})$. Composing this with the above splitting gives a lift $\mathbb{F}^* \to \bar{V}_2(\mathbb{E})$ to the semi-holonomic case, so all second order operators automatically admit lifts as claimed.

Recall that we were able to obtain a great number of the remaining operators by translating the initial data. We will show that translation also works in the semi-holonomic case. Then since the initial data lifts to the semi-holonomic case, we can apply the translation principle as before to get lifts to the semi-holonomic case of all but the five families of exceptional operators.

We were unable to reach these families of exceptional operators by translating from the initial data. In fact, we will show that the first exceptional operator does not admit a lift to the semi-holonomic case at all, and hence neither does its corresponding family of operators. This means that we do not obtain curved analogues for this family of operators *by this method*. Whether or not they admit curved analogues at all is a much more difficult problem, and we will not discuss it here. For the other four families it is not even clear that the operators do not lift to the semi-holonomic case, and indeed it is possible that they may admit lifts. We discuss this further in the following section.

5.2 The Translation Principle in the Semi-Holonomic Case

We need to show that the translation principle is still applicable in the semi-holonomic case. The critical aspect of the translation principle that we need to look at is the differential splittings associated with terms in the decompositions of tensor products. This is the main ingredient of the translation principle, and we will show that because $\mathbb{W}$ and $\mathbb{W}^*$ have composition series of length two, then the splittings (invariant projections and inclusions) that occur when we tensor with these representations will be of order at most two. We know from the above discussion that second order operators automatically lift to the semi-holonomic case, and hence the invariant splittings must lift.

Consider a general irreducible $\mathfrak{g}$−module $\mathbb{V}$. Decomposing $\mathbb{V}$ into eigenspaces of the one-dimensional centre of $\mathfrak{g}_0$, as in Section 2, we get a composition series

$$\mathbb{V} = \mathbb{V}_\alpha + \mathbb{V}_{\alpha+1} + \ldots + \mathbb{V}_{\alpha+n}$$

of length n. Let $\mathbb{E}$ be a $\mathfrak{p}$−module, and suppose that $V(\mathbb{E}_1)$ splits off from $V(\mathbb{E}\otimes\mathbb{V})$. In fact, first suppose we just have an invariant inclusion

$$V(\mathbb{E}_1) \to V(\mathbb{E}\otimes\mathbb{V}),$$

and suppose that this is a k^{th} order operator. Then the symbol of this operator is

$$\mathbb{E}^* \otimes \mathbb{V}^* \to \bigodot^k \mathfrak{g}_{-1} \otimes \mathbb{E}_1^*.$$

Since this is a homomorphism of $\mathfrak{g}_0$−module, the action of H must be preserved. Now on the left, H will act by multiplication by $-\ell(\mathbb{E}) - \alpha - j$, for $j \in \{0, \ldots, n\}$. On the right, H will act by multiplication by $-k - \ell(\mathbb{E}_1)$, and hence

$$-\ell(\mathbb{E}) - \alpha - j = -k - \ell(\mathbb{E}_1)$$

for some $j \in \{0, \ldots, n\}$. In particular,

$$\begin{aligned} n - k &\geq j - k \\ &= \ell(\mathbb{E}_1) - \ell(\mathbb{E}) - \alpha. \end{aligned} \tag{1}$$

On the other hand, since $\mathbb{E}_1$ occurs in the decomposition (into a direct sum of $\mathfrak{g}_0$-modules)

$$\begin{aligned} \mathbb{E}\otimes\mathbb{V} &= \mathbb{E}\otimes\mathbb{V}_\alpha + \mathbb{E}\otimes\mathbb{V}_{\alpha+1} + \ldots + \mathbb{E}\otimes\mathbb{V}_{\alpha+n} \\ &= \bigoplus_i \mathbb{E}_i, \end{aligned}$$

we know that the action of H on $\mathbb{E}_1$ must be equal to the action of H on one of the terms $\mathbb{E}\otimes\mathbb{V}_{\alpha+m}$, i.e.

$$\ell(\mathbb{E}_1) = \ell(\mathbb{E}) + \alpha + m,$$

for some $m \in \{0, \ldots, n\}$. In particular,

$$\begin{aligned} 0 &\leq m \\ &= \ell(\mathbb{E}_1) - \ell(\mathbb{E}) - \alpha. \end{aligned} \tag{2}$$

Combining Equations 1 and 2 we get $n \geq k$. In other words, the order of the invariant inclusion

$$V(\mathbb{E}_1) \to V(\mathbb{E} \otimes \mathbb{V})$$

is not greater than the length of the composition series of $\mathbb{V}$. Using similar arguments, we can also show that the order of an invariant projection

$$V(\mathbb{E} \otimes \mathbb{V}) \to V(\mathbb{E}_1)$$

must also be less than or equal to the length of the composition series of $\mathbb{V}$.

Now suppose that we have an invariant inclusion

$$V(\mathbb{E}_1) \to V(\mathbb{E} \otimes \mathbb{W}).$$

Since $\mathbb{W}$ has a composition series of length two, this invariant operator must have order at most two. Therefore, by the discussion in the previous subsection, we automatically get a lift to the semi-holonomic case, i.e.

$$\bar{V}(\mathbb{E}_1) \to \bar{V}(\mathbb{E} \otimes \mathbb{W}).$$

Similarly, an invariant projection

$$V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{E}_1)$$

will also lift to the semi-holonomic case,

$$\bar{V}(\mathbb{E} \otimes \mathbb{W}) \to \bar{V}(\mathbb{E}_1),$$

and the same is true if we replace $\mathbb{W}$ by $\mathbb{W}^*$ in each situation above. We can now prove the following theorem.

Theorem 5.2.1 *Suppose that the Verma module homomorphism*

$$D : V(\mathbb{F}) \to V(\mathbb{E})$$

lifts to the semi-holonomic case,

$$\bar{D} : \bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E}).$$

Suppose further that we can translate the operator (in the holonomic case) by $\mathbb{W}$ (or by $\mathbb{W}^*$) to get a new operator

$$V(\mathbb{F}_1) \to V(\mathbb{E}_1).$$

Then translation in the semi-holonomic case will give us a lift of the new operator, i.e. there exists a lift

$$\bar{V}(\mathbb{F}_1) \to \bar{V}(\mathbb{E}_1).$$

Proof: The fact that we can translate the operator in the holonomic case means that there exists a non-zero composition

$$V(\mathbb{F}_1) \to V(\mathbb{F} \otimes \mathbb{W}) = V(\mathbb{F}) \otimes \mathbb{W}^* \to V(\mathbb{E}) \otimes \mathbb{W}^* = V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{E}_1),$$

where the first operator is an invariant inclusion, the second operator is $D \otimes 1$, and the third operator is an invariant projection. Since $\mathbb{W}$ has composition series of length two, we know that the invariant inclusion and projection must lift to the semi-holonomic case. Also, $\bar{D} \otimes 1$ is a lift to the semi-holonomic case of the operator $D \otimes 1$. Thus we get a commutative diagram

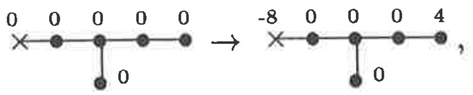
$$\begin{array}{ccccccccc}
\bar{V}(\mathbb{F}_1) & \to & \bar{V}(\mathbb{F} \otimes \mathbb{W}) & = \bar{V}(\mathbb{F}) \otimes \mathbb{W}^* & \to & \bar{V}(\mathbb{E}) \otimes \mathbb{W}^* & = \bar{V}(\mathbb{E} \otimes \mathbb{W}) & \to & \bar{V}(\mathbb{E}_1) \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
V(\mathbb{F}_1) & \to & V(\mathbb{F} \otimes \mathbb{W}) & = V(\mathbb{F}) \otimes \mathbb{W}^* & \to & V(\mathbb{E}) \otimes \mathbb{W}^* & = V(\mathbb{E} \otimes \mathbb{W}) & \to & V(\mathbb{E}_1).
\end{array}$$

Composition along the top row gives us a semi-holonomic lift of the translated operator. Of course, the same argument works when $\mathbb{W}$ is replaced by $\mathbb{W}^*$. □

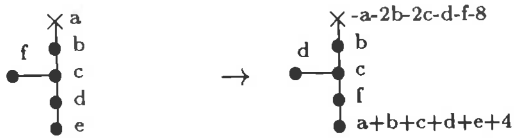
We have seen that all the initial data trivially lifts to the semi-holonomic case, as all these operators are first order. Hence what this theorem shows is that for every operator that could be obtained from the initial data by translation, there will exist a lift to the semi-holonomic case, which can be obtained by performing the same sequence of translations beginning with the semi-holonomic initial data. Therefore, at this stage we get lifts of all the invariant operators except for the five exceptional families mentioned at the end of the previous section. Next we will investigate the first of these families, and we will show that a lift to the semi-holonomic case does not exist for this exceptional family.

5.3 Non-existence of Lifts of the First Exceptional Family

We are going to show that the first exceptional operator



from the family

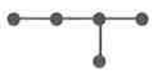


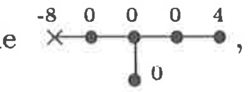
does not admit a lift to the semi-holonomic case. If any operator in this family admitted a lift to the semi-holonomic case, then by translating, we could obtain semi-holonomic lifts of all the operators in this family. Thus by showing that the operator above does not lift, it follows that none of the operators in this family admits a lift to the semi-holonomic case.

Notice that this operator looks remarkably like the *long operator* in the eight-dimensional conformal case (where $G = \mathrm{SO}(10,\mathbb{C})$). Indeed, if we ignore the fifth nodes of the terms in the classifying pattern then what we get is precisely the classifying pattern for the parabolic

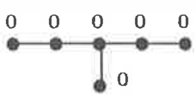
sitting inside $\mathrm{SO}(10,\mathbb{C})$, which is eight-dimensional conformal geometry (see, for example, [11]). This is hardly surprising, as $\mathrm{SO}(10,\mathbb{C})$ itself sits inside E_6. In terms of the corresponding Lie algebras, $\mathfrak{so}(10,\mathbb{C})$ can be realized inside $\mathfrak{e}_6$ as the root spaces whose roots do not include α_5; i.e. if $X \in \mathfrak{e}_6$ belongs to the root space $X_{a_1\alpha_1+\ldots+a_6\alpha_6}$, then the subalgebra $\{X \in \mathfrak{e}_6 | a_5 = 0\}$ with Cartan subalgebra $\langle h_1, h_2, h_3, h_4, h_6\rangle$ is isomorphic to $\mathfrak{so}(10,\mathbb{C})$. Furthermore, the parabolic

intersects this $\mathfrak{so}(10,\mathbb{C})$ subalgebra in the parabolic

 $\cap$  $=$  .

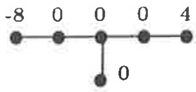
Now if we restrict our attention to the part of the classifying pattern (with $a = \ldots = f = 0$) which looks like the classifying pattern for $\mathrm{SO}(10,\mathbb{C})$, up to the vector bundle 

, then we see that there are no reflections in the fifth node. This is especially obvious if we look directly at the Hasse diagram in Appendix D.1. So as far as this operator is concerned, *all of the action* is essentially taking place in the $\mathfrak{so}(10,\mathbb{C})$ subalgebra. To make this clearer, let us rewrite minus the lowest weights of the relevant bundles in terms of the simple roots $\{\alpha_1, \ldots, \alpha_6\}$ instead of the fundamental weights $\{\lambda_1, \ldots, \lambda_6\}$. We get



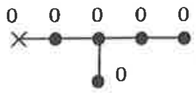
$= (0,0,0,0,0,0)$

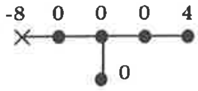
and



$= (-8, -8, -8, -4, 0, -4).$

Recall that to find a Verma module homomorphism

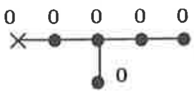
$V($ 

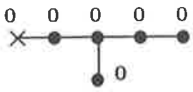
$) \leftarrow V($ 

$),$

we need to find a maximal (i.e. killed by all raising operators) element

$$v = q(y_1, y_{22}, \ldots, y_{36}) \otimes w$$

in $V($ 

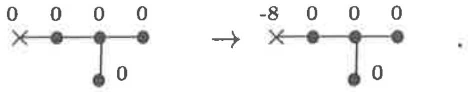
$)$ with the appropriate weight, where w is a highest weight vector for $($ 

$)^*$ and q is a polynomial in the y_i's (actually, not really a polynomial in the semi-holonomic case since the y_i's don't commute). Now from what we have seen above, choosing the appropriate weight will mean precisely that the polynomial q will involve only lowering operators y_i which belong to root spaces $X_{-a_1\alpha_1-\ldots-a_6\alpha_6}$ with $a_5 = 0$, i.e. q only involves lowering operators belonging to the subalgebra isomorphic to $\mathfrak{so}(10, \mathbb{C})$. Furthermore, to test that v is maximal we only need to act on it with each of the raising operators $x_1, X_2, \ldots, X_6$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_6$. Since all the lowering operators y_i in q belong to root spaces with $a_5 = 0$, we immediately know that X_5 will commute with them all, and hence commute with q. Therefore we automatically have

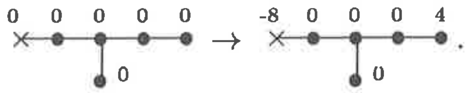
$$\begin{aligned} X_5 v &= X_5 q(y_1, y_{22}, \ldots, y_{36}) \otimes w \\ &= q(y_1, y_{22}, \ldots, y_{36}) \otimes X_5 w \\ &= 0, \end{aligned}$$

since w is a maximal weight vector. So we need to choose q in such a way that $x_1 v = X_2 v = X_3 v = X_4 v = X_6 v = 0$. It is now clear that *all of the action* really is taking place inside the $\mathfrak{so}(10, \mathbb{C})$ subalgebra.

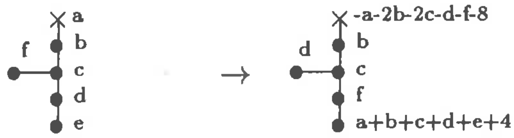
So suppose such a maximal element v exists, which gives a homomorphism of semi-holonomic Verma modules. Then we could take precisely the same 'polynomial' q and use it to construct a maximal weight vector in the eight-dimensional conformal case. Thus there would exist a lift to the semi-holonomic case of the so-called long operator



However, in [11] it is shown that the long operator does not admit a lift to the semi-holonomic case in any even-dimensional conformal geometry. This contradiction implies that there cannot exist a lift to the semi-holonomic case of the operator



From our discussion earlier, it follows that none of the operators



in the first exceptional family lift to the semi-holonomic case.

6 Conclusions and Outlook

The classification of invariant operators on the homogeneous space G/P is well understood. This is precisely Theorem 4.1.1, due to Boe and Collingwood ([4]). Our main aim in this thesis was to investigate which of these invariant operators have curved analogues which remain invariant on a general curved (AHS) manifold, and, ultimately, we would like to arrive at a complete classification. Since invariant operators on the flat space G/P correspond to homomorphisms of Verma modules, it is natural to try to extend these ideas to apply to the curved case. This is precisely what we have done by introducing semi-holonomic Verma modules.

Unfortunately we were unable to obtain a complete classification of which Verma module homomorphisms lift to the semi-holonomic case, thereby giving a curved analogue of the corresponding operator. However, we have shown by using the translation principle that most homomorphisms do lift, and there are just five exceptional families for which this approach fails. Furthermore, we were able to show that the first family does not lift, by using an analogous result from conformal geometry. We will now say a little about the remaining four families.

The second family of exceptional operators are the adjoints of the first family. Because of this, it would be absurd to suspect that they might lift to the semi-holonomic case when the first family do not. However, to make this into a rigorous argument, we really need to reformulate the notion of adjointness in terms of Verma module and *semi-holonomic* Verma module homomorphisms. We have made some partial progress with this approach; namely, given a maximal weight vector which induces a Verma module homomorphism, we know how to construct the maximal weight vector which induces the adjoint Verma module homomorphism. The next step is to duplicate this result for semi-holonomic Verma module homomorphisms. This would allow us to use the fact that the first family does not lift to conclude that the second family does not lift, a result which we strongly suspect is true.

We cannot be as certain about the fifth family. At first it appeared that this family was related to the family of long operators in six-dimensional conformal geometry. Then since this family of long operators does not lift to the semi-holonomic case, it seemed that neither should the fifth family. Indeed, this was to be our approach for all of the exceptional families, with the first and fifth families looking like the family of long operators in eight-dimensional geometry, and the third and fourth families like the family of long operators in four-dimensional conformal geometry. It worked for the first family, where we were able to show that *all of the action* was taking place inside of an $\mathfrak{so}(10,\mathbb{C})$ subalgebra. However, for the fifth family it is more complicated; the action is *not* simply taking place inside an $\mathfrak{so}(8,\mathbb{C})$ subalgebra. Similarly, for the third and fourth families the action is not taking place inside $\mathfrak{so}(6,\mathbb{C})$ subalgebras, and for the second family this approach fails for other reasons.

Of course, the fifth family also appears to be related to the second longest family of operators in eight-dimensional conformal geometry. Indeed, there is strong evidence to suggest that this is really the case. This evidence involves taking fourth direct images of sheaves on flag manifolds related to G/P. Again, to arrive at rigorous results we would need to reinterpret direct images in terms of Verma module and semi-holonomic Verma module homomorphisms, and it is not at all clear what they look like in this context. This approach could, in principle, show that the fifth family does lift to the semi-holonomic case, as the second longest family in eight-dimensional conformal geometry lifts.

It is also possible that the third and fourth families are related to the third longest operators in eight-dimensional conformal geometry. This may enable to show that these two family lift to the semi-holonomic case, as the third longest operators admit lifts in conformal geometry. However, this is more speculative than the fifth family, for which we actually have some evidence in terms of direct images. Finally, we should point out that formalising adjointness in terms of semi-holonomic Verma module homomorphisms would also allow us to conclude that the fourth family of exceptional operators lift to the semi-holonomic case if and only if the third family do, as these families are adjoint.

The problem of classifying which Verma module homomorphisms lift to semi-holonomic Verma module homomorphisms is only a small step towards the complete classification of curved analogues. Furthermore, we could decide this problem directly by looking for maximal weight vectors for the four remaining exceptional operators. Although this would involve some large calculations, there would only be a finite number of possibilities (so this is a possible 'last resort' approach). Once we have done this, we will know that curved analogues exist for all those operators that admit lifts to the semi-holonomic case. However, for those that do not admit lifts we can say very little. The existence or non-existence of curved analogues for these operators is a very delicate matter.

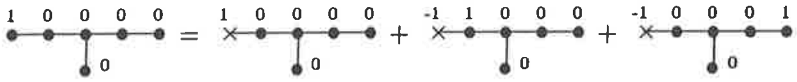
In conformal geometry, it is known that curved analogues can still exist when a lift to the semi-holonomic case does not; for example, the long operator has a curved analogue (see [13]) but does not lift (see [11]). Indeed, there is only one operator (in four-dimensional conformal geometry) for which we know that a curved analogue certainly does not exist (this was shown by Graham in [12]). We would hope to be able to deduce some results about curved analogues for the exceptional geometry discussed in this thesis by using results from conformal geometry. However, even in conformal geometry the classification of curved analogues is not completely understood.

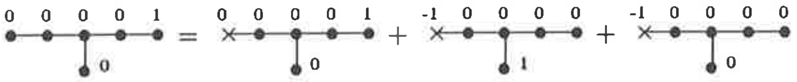
A Lowering Operators for E_6 with Commutation Rules

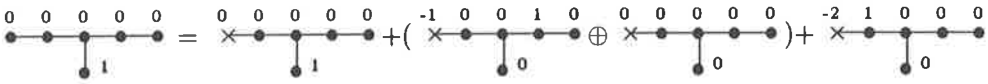
$y \in X_{-\alpha}$	α	Adjoint action of x					
		x_1	X_2	X_3	X_4	X_5	X_6
y_1	100000	h_1					
Y_2	010000		h_2				
Y_3	001000			h_3			
Y_4	000100				h_4		
Y_5	000010					h_5	
Y_6	000001						h_6
$Y_7 = [Y_2, Y_3]$	011000		Y_3	$-Y_2$			
$Y_8 = [Y_3, Y_4]$	001100			Y_4	$-Y_3$		
$Y_9 = [Y_3, Y_6]$	001001			Y_6			$-Y_3$
$Y_{10} = [Y_2, Y_8]$	011100		Y_8		$-Y_7$		
$Y_{11} = [Y_2, Y_9]$	011001		Y_9				$-Y_7$
$Y_{12} = [Y_4, Y_9]$	001101				Y_9		Y_8
$Y_{13} = [Y_2, Y_{12}]$	011101		Y_{12}		Y_{11}		Y_{10}
$Y_{14} = [Y_3, Y_{13}]$	012101			Y_{13}			
$Y_{15} = [Y_4, Y_5]$	000110				Y_5	$-Y_4$	
$Y_{16} = [Y_3, Y_{15}]$	001110			Y_{15}		$-Y_8$	
$Y_{17} = [Y_2, Y_{16}]$	011110		Y_{16}			$-Y_{10}$	
$Y_{18} = [Y_5, Y_{12}]$	001111					Y_{12}	$-Y_{16}$
$Y_{19} = [Y_2, Y_{18}]$	011111		Y_{18}			Y_{13}	$-Y_{17}$
$Y_{20} = [Y_3, Y_{19}]$	012111			Y_{19}		Y_{14}	
$Y_{21} = [Y_4, Y_{20}]$	012211				Y_{20}		
$y_{22} = [y_1, Y_2]$	110000	Y_2	$-y_1$				
$y_{23} = [y_1, Y_7]$	111000	Y_7		$-y_{22}$			
$y_{24} = [y_1, Y_{10}]$	111100	Y_{10}			$-y_{23}$		
$y_{25} = [y_1, Y_{11}]$	111001	Y_{11}					$-y_{23}$
$y_{26} = [y_1, Y_{13}]$	111101	Y_{13}			y_{25}		y_{24}
$y_{27} = [y_1, Y_{14}]$	112101	Y_{14}		y_{26}			
$y_{28} = [Y_2, y_{27}]$	122101		y_{27}				
$y_{29} = [y_1, Y_{17}]$	111110	Y_{17}				$-y_{24}$	
$y_{30} = [y_1, Y_{19}]$	111111	Y_{19}				y_{26}	$-y_{29}$
$y_{31} = [y_1, Y_{20}]$	112111	Y_{20}		y_{30}		y_{27}	
$y_{32} = [Y_2, y_{31}]$	122111		y_{31}			y_{28}	
$y_{33} = [y_1, Y_{21}]$	112211	Y_{21}			y_{31}		
$y_{34} = [Y_2, y_{33}]$	122211		y_{33}		y_{32}		
$y_{35} = [Y_3, y_{34}]$	123211			y_{34}			
$y_{36} = [Y_6, y_{35}]$	123212						y_{35}

B Composition Series for the Fundamental Representations

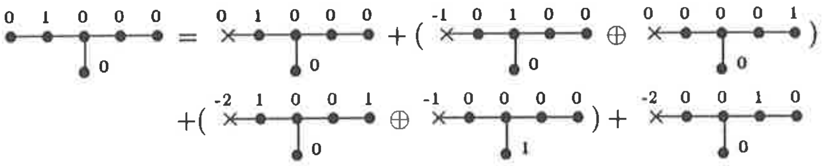
Length 2

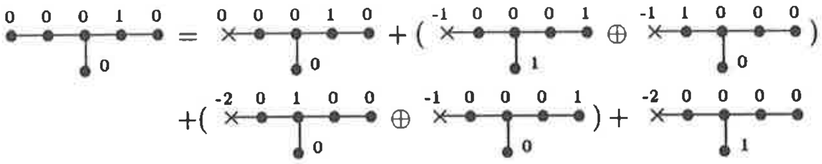




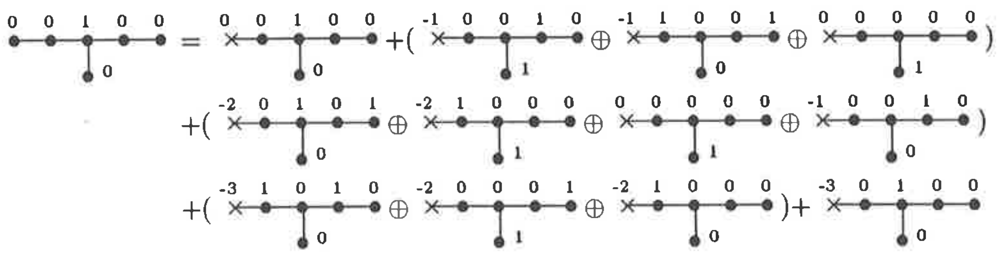


Length 3



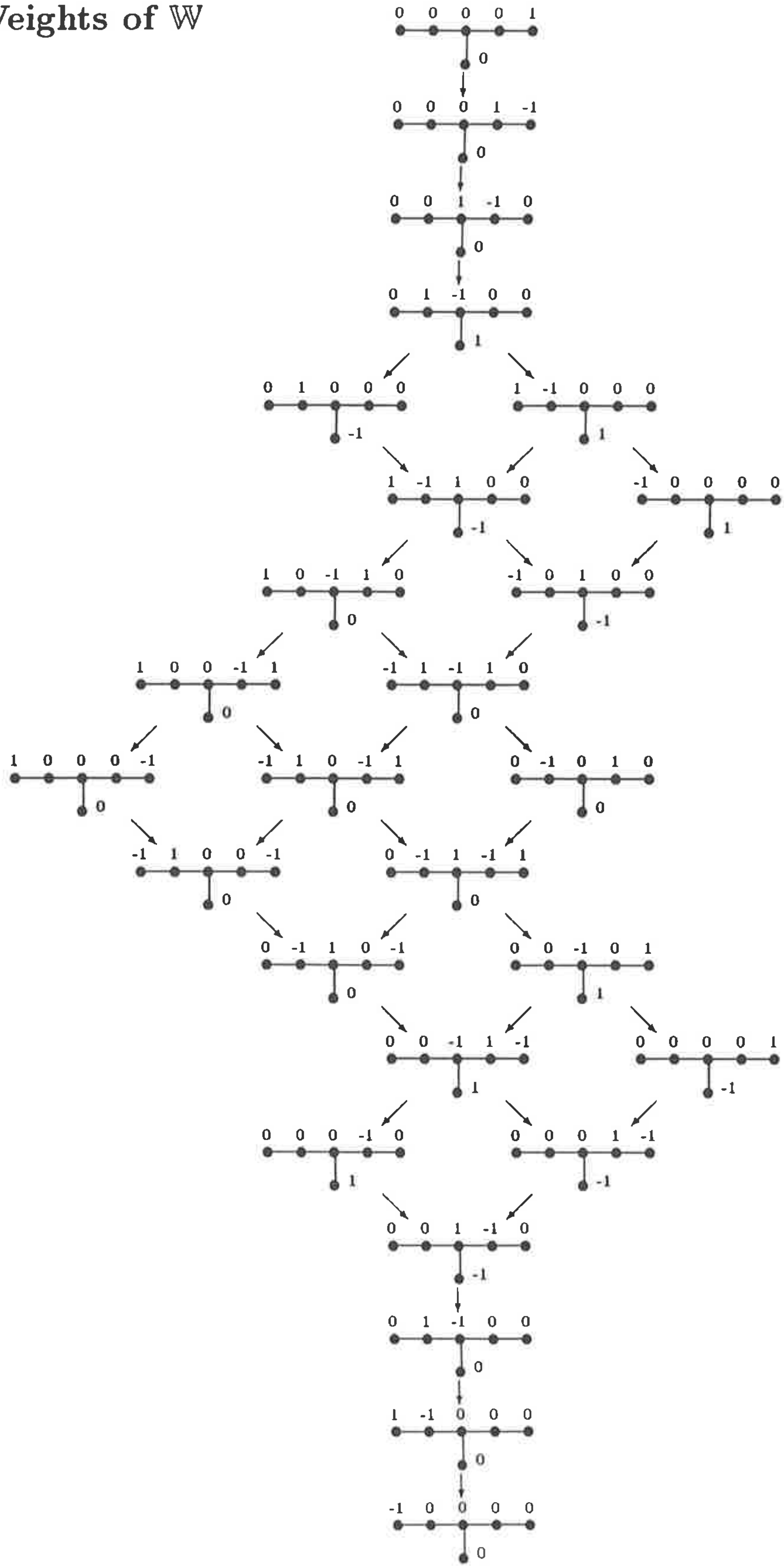


Length 4

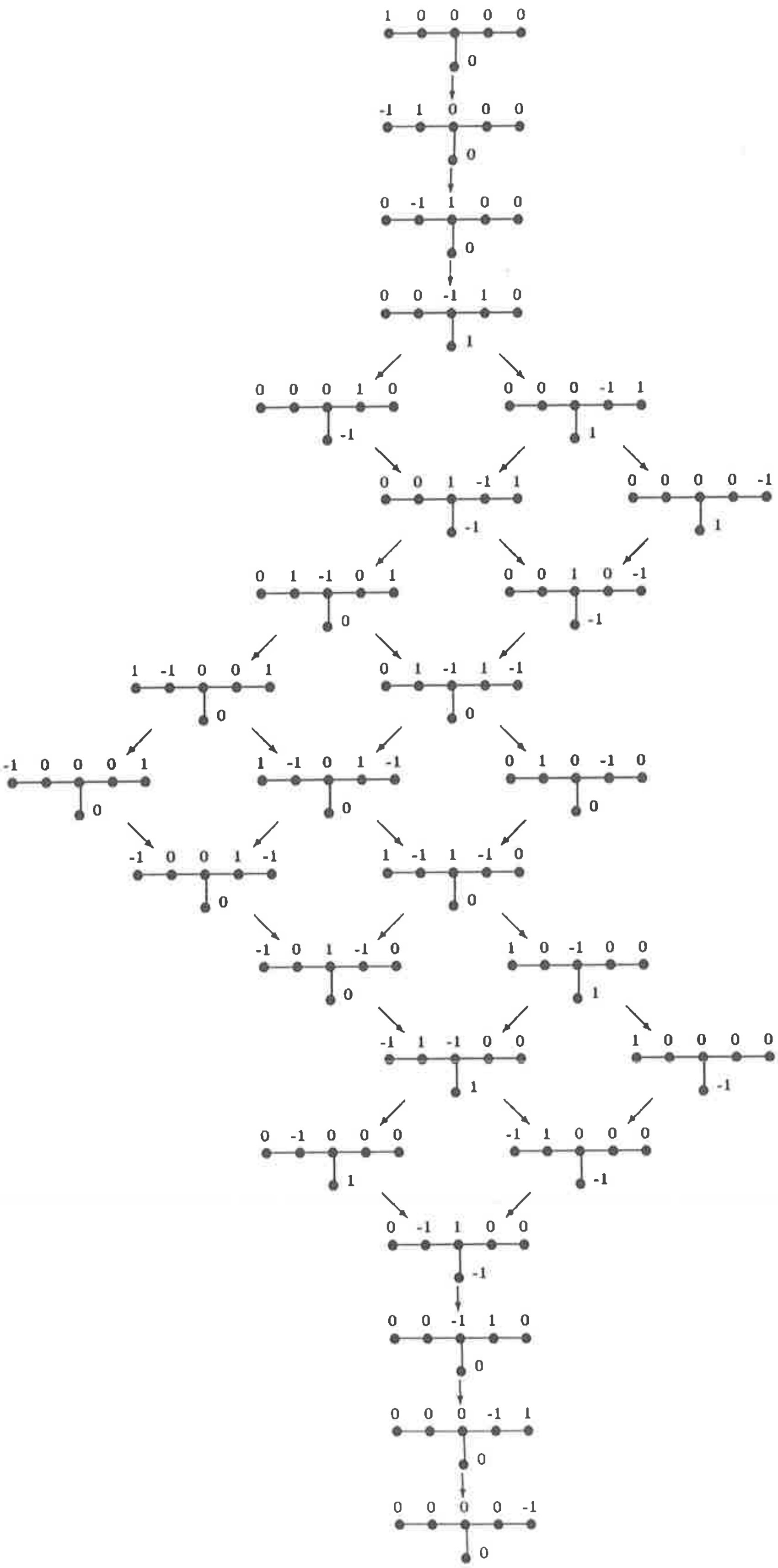


C Weyl Group Orbits of Highest Weights

C.1 Weights of 𝕎

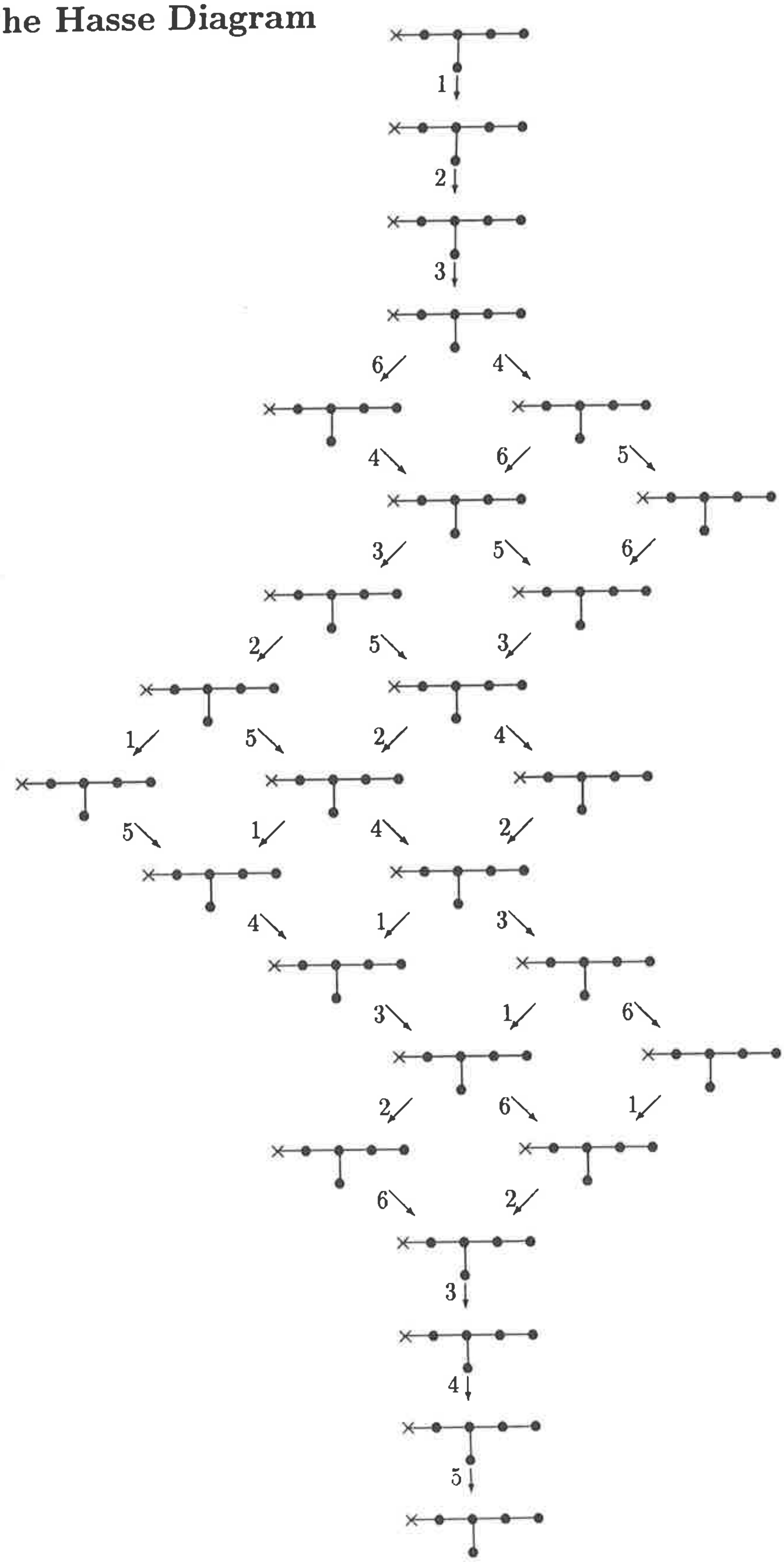


C.2 Weights of $\mathbb{W}^*$

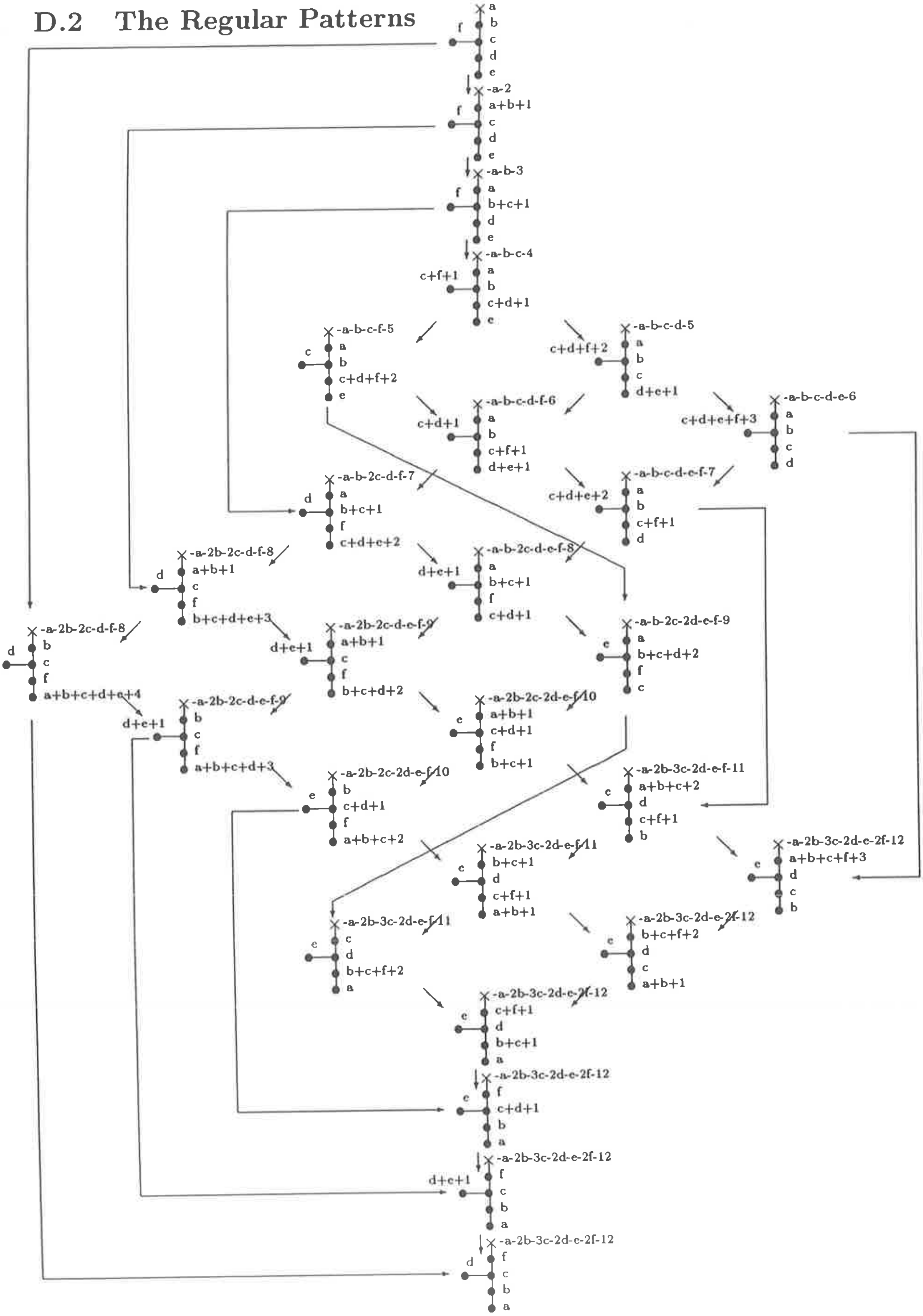


D Classifying Patterns

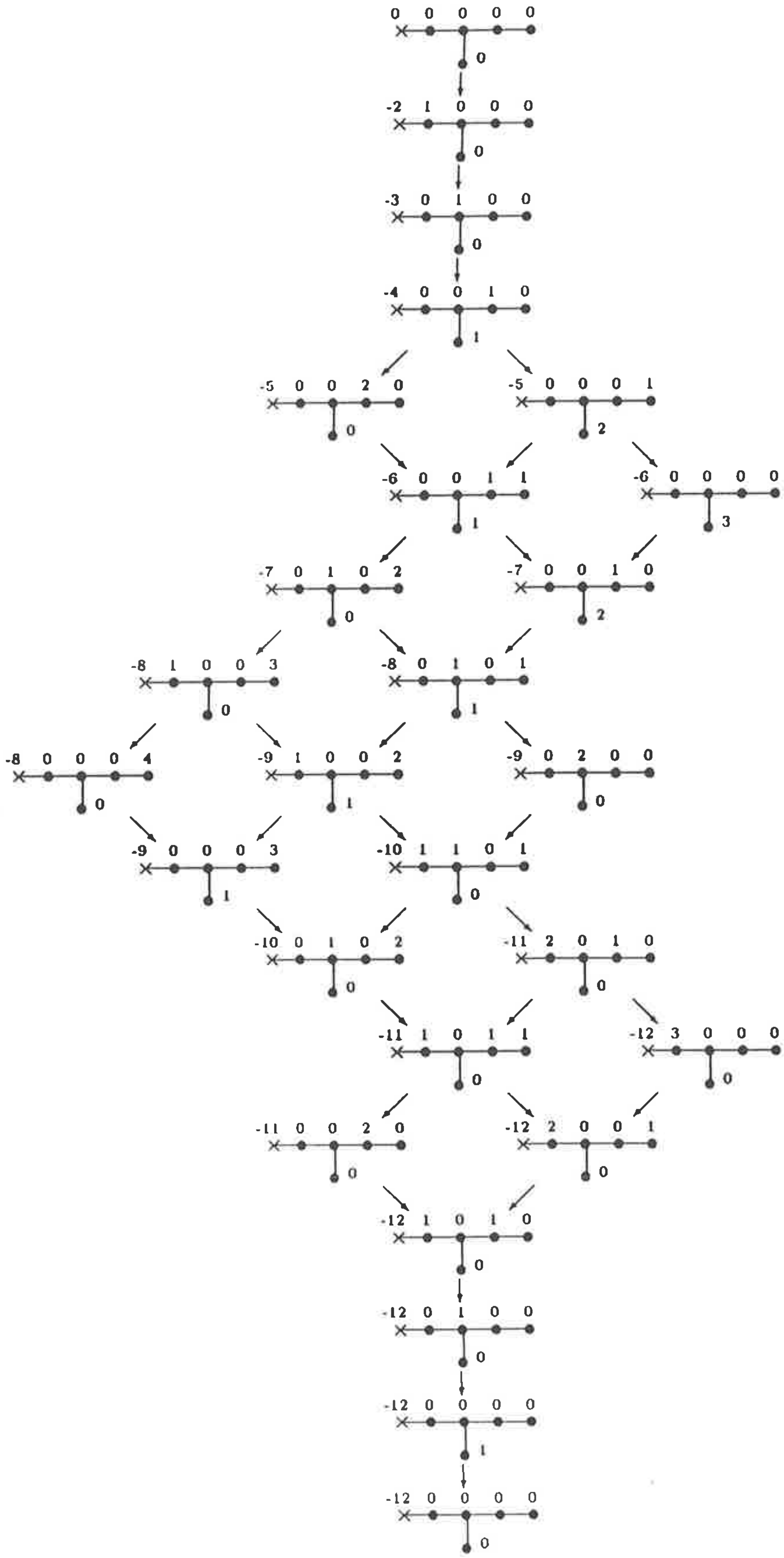
D.1 The Hasse Diagram



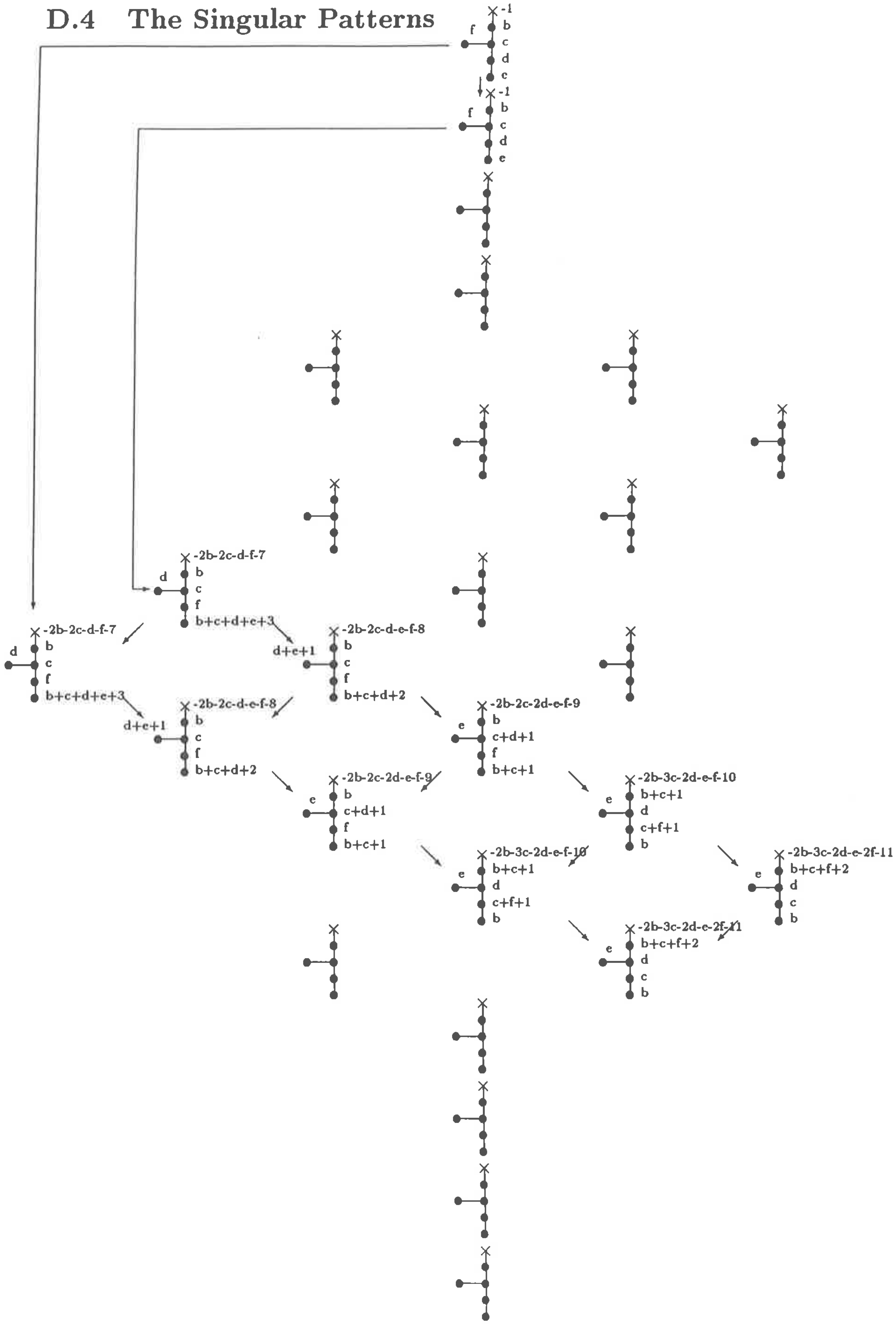
D.2 The Regular Patterns

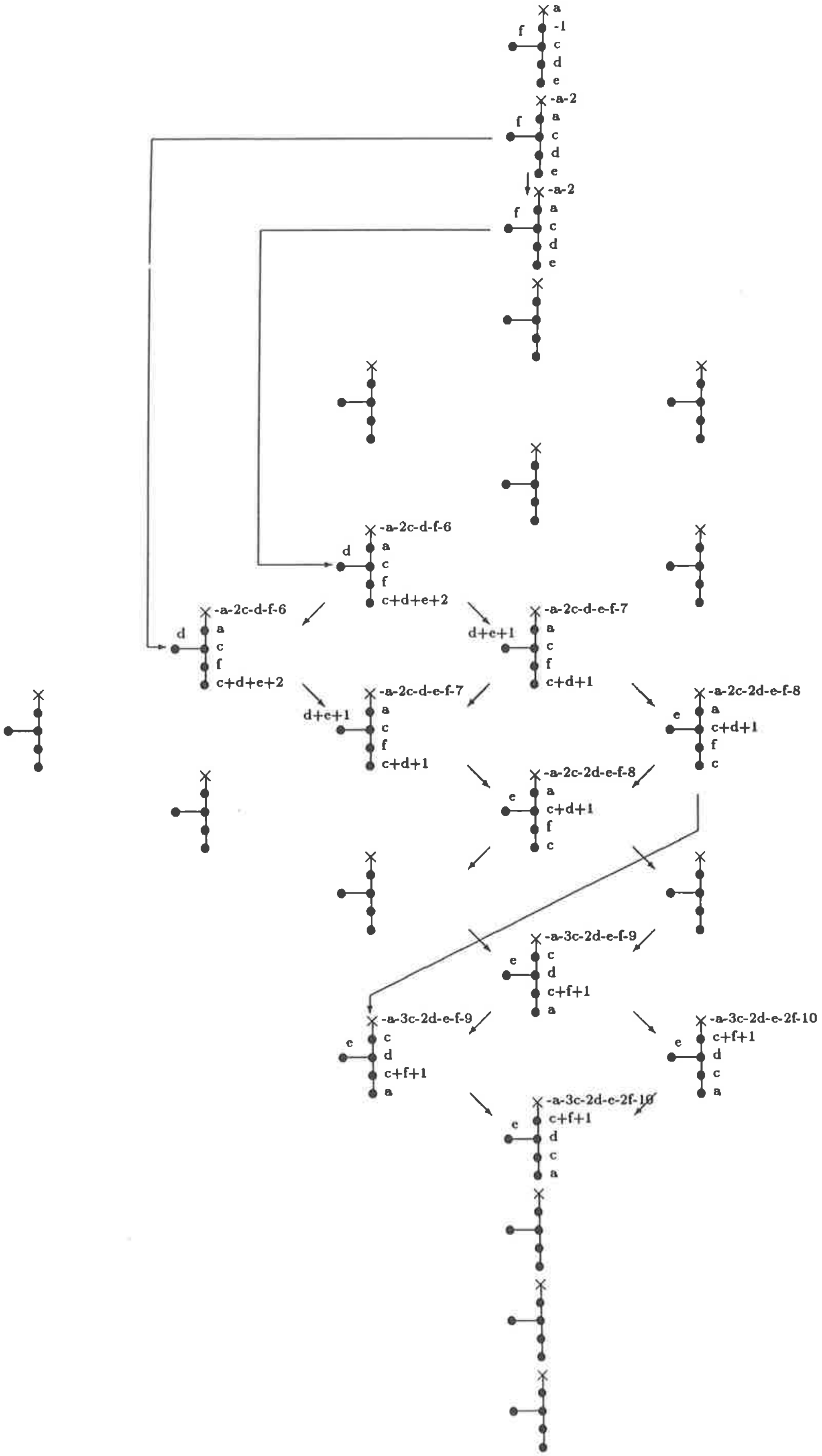


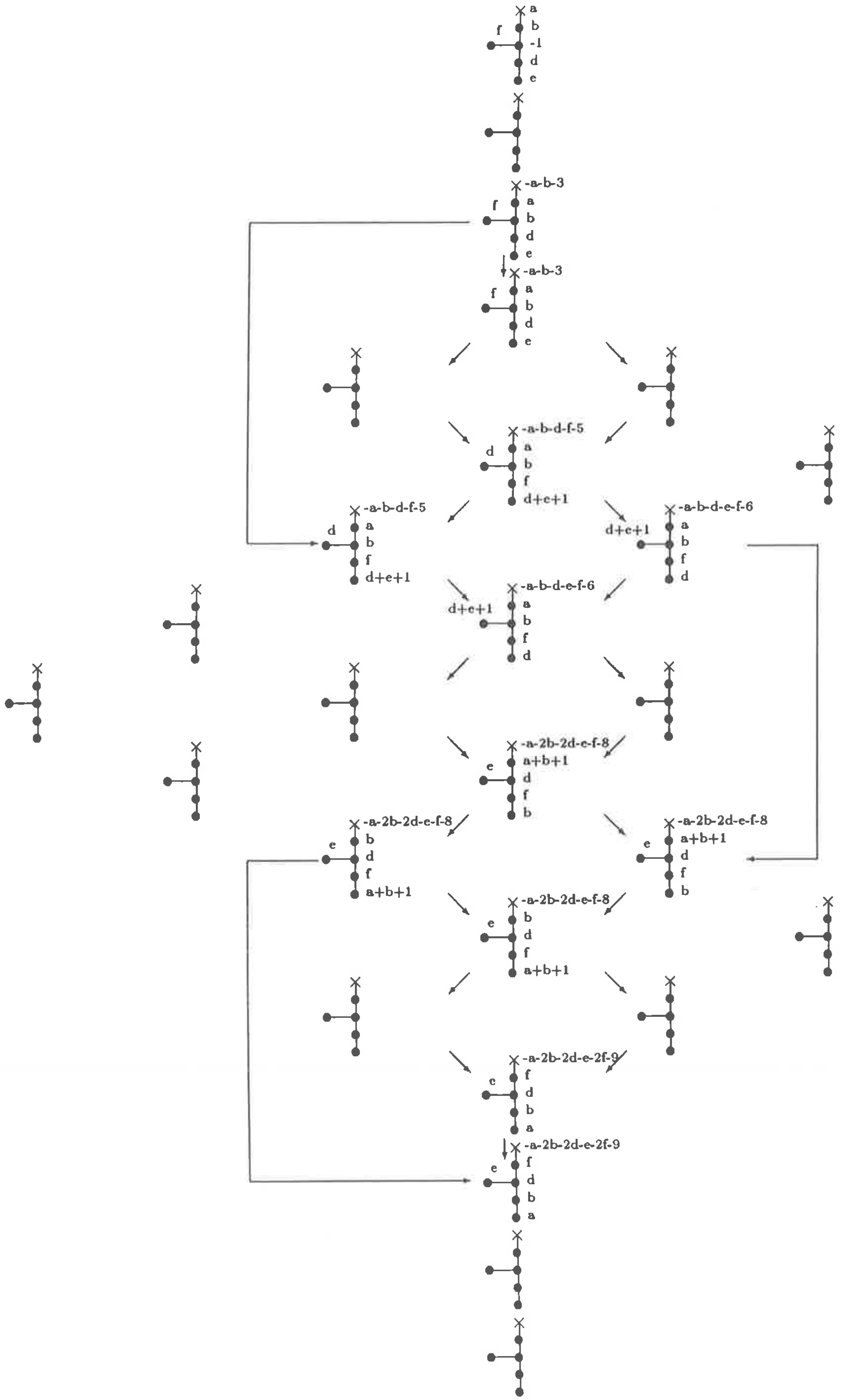
D.3 The De Rham Sequence



D.4 The Singular Patterns




a
-1
c
d
e
f
-a-2
-a-2c-d-f-6
c+d+e+2
-a-2c-d-e-f-7
d+e+1
c+d+1
-a-2c-2d-e-f-8
-a-3c-2d-e-f-9
c+f+1
-a-3c-2d-e-2f-10


a
b
-1
d
e
f
-a-b-3
a
b
d
e
f
-a-b-3
a
b
d
e
f
-a-b-d-f-5
a
b
f
d+e+1
d
-a-b-d-f-5
a
b
f
d+e+1
d
-a-b-d-e-f-6
a
b
f
d
d+e+1
-a-b-d-e-f-6
a
b
f
d
d+e+1
-a-2b-2d-e-f-8
a+b+1
d
f
b
e
-a-2b-2d-e-f-8
b
d
f
a+b+1
e
-a-2b-2d-e-f-8
a+b+1
d
f
b
e
-a-2b-2d-e-f-8
b
d
f
a+b+1
e
-a-2b-2d-e-2f-9
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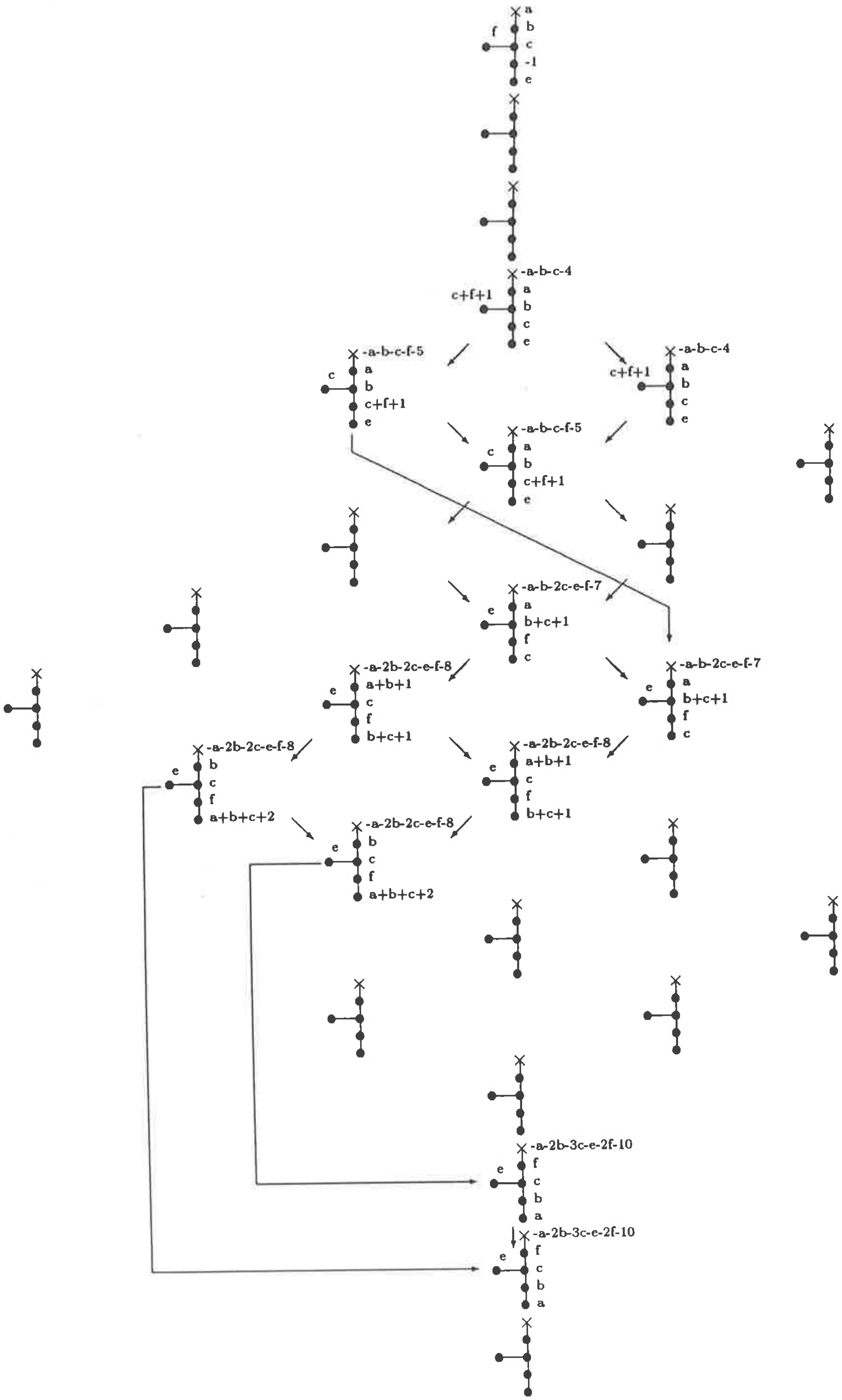
f
d
b
a
e


a
f
b
c
-1
e
-a-b-c-4
c+f+1
a
b
c
e
-a-b-c-f-5
c
a
b
c+f+1
e
-a-b-c-4
c+f+1
a
b
c
e
-a-b-c-f-5
c
a
b
c+f+1
e
-a-b-2c-e-f-7
e
a
b+c+1
f
c
-a-2b-2c-e-f-8
e
a+b+1
c
f
b+c+1
-a-b-2c-e-f-7
e
a
b+c+1
f
c
-a-2b-2c-e-f-8
e
b
c
f
a+b+c+2
-a-2b-2c-e-f-8
e
a+b+1
c
f
b+c+1
-a-2b-2c-e-f-8
e
b
c
f
a+b+c+2
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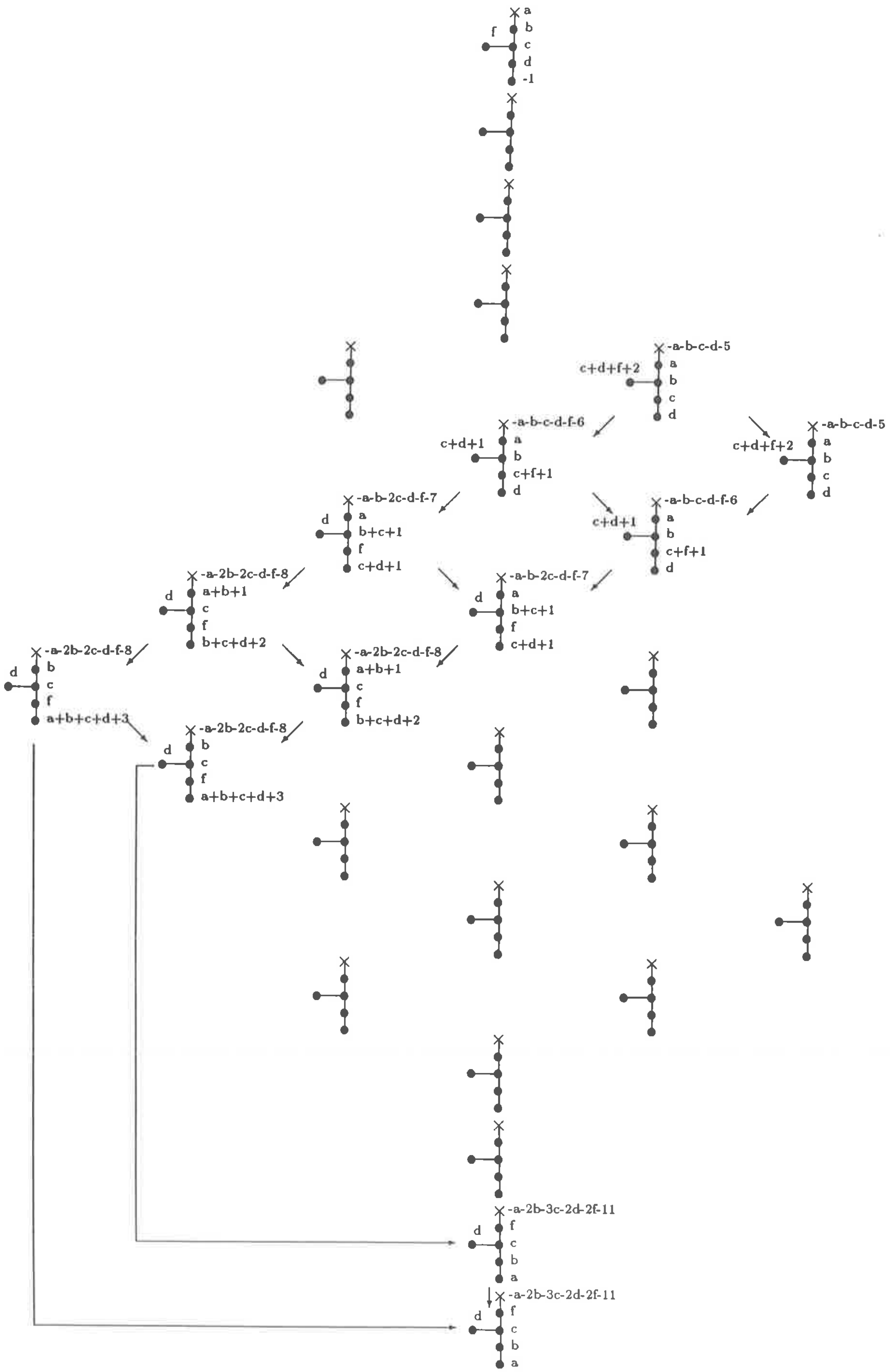
e
f
c
b
a


a
b
c
d
-1
f
-a-b-c-d-5
c+d+f+2
-a-b-c-d-f-6
c+d+1
c+f+1
-a-b-c-d-5
c+d+f+2
-a-b-2c-d-f-7
b+c+1
c+d+1
-a-b-c-d-f-6
c+d+1
c+f+1
-a-2b-2c-d-f-8
a+b+1
b+c+d+2
-a-b-2c-d-f-7
b+c+1
c+d+1
-a-2b-2c-d-f-8
a+b+c+d+3
-a-2b-2c-d-f-8
a+b+1
b+c+d+2
-a-2b-2c-d-f-8
a+b+c+d+3
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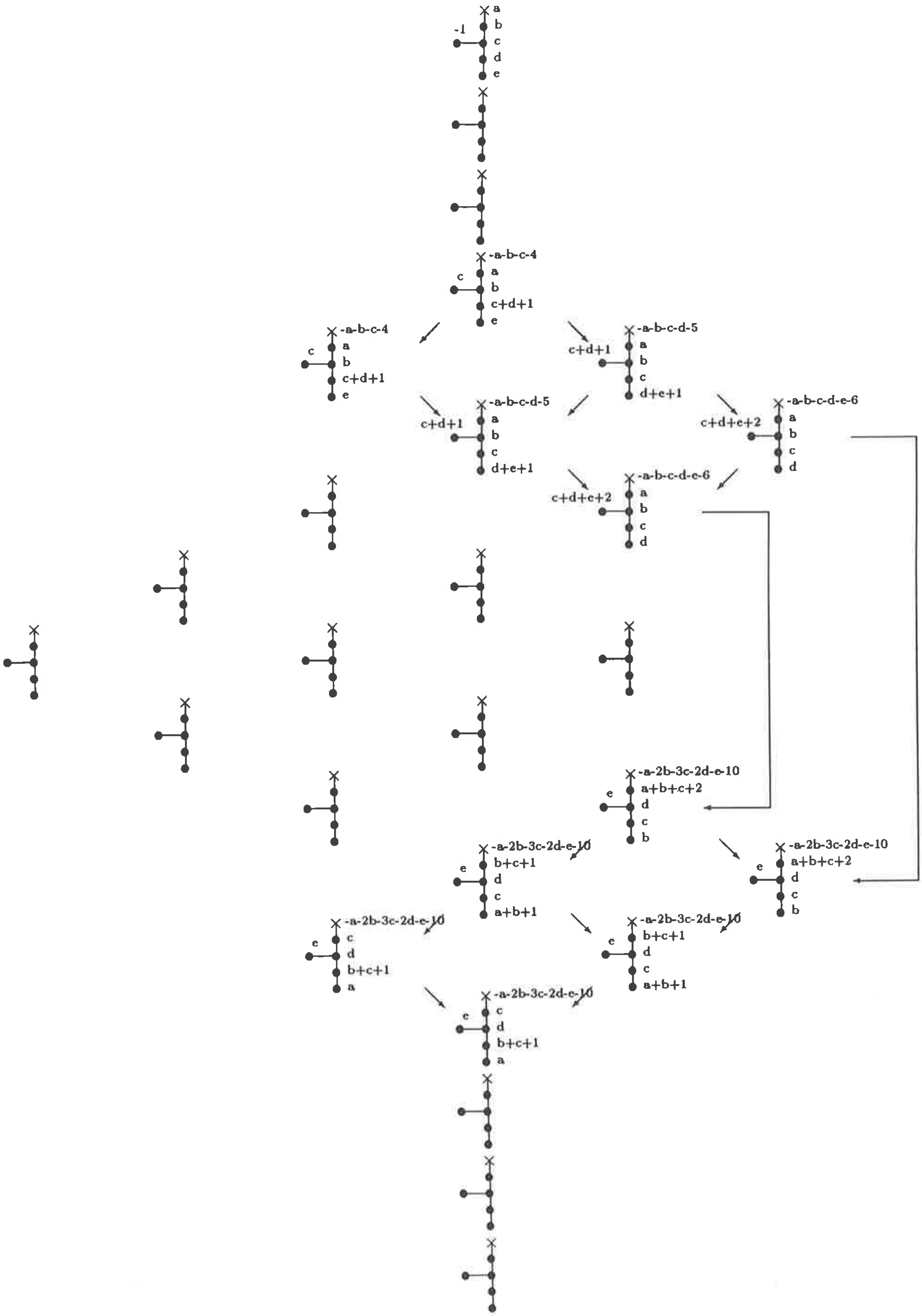
-a-2b-3c-2d-2f-11



a
-1
b
c
d
e
-a-b-c-4
c
a
b
c+d+1
e
-a-b-c-4
c
a
b
c+d+1
e
-a-b-c-d-5
c+d+1
a
b
c
d+e+1
-a-b-c-d-5
c+d+1
a
b
c
d+e+1
-a-b-c-d-e-6
c+d+e+2
a
b
c
d
-a-b-c-d-e-6
c+d+e+2
a
b
c
d
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e
a+b+c+2
d
c
b
-a-2b-3c-2d-e-10
e
b+c+1
d
c
a+b+1
-a-2b-3c-2d-e-10
e
a+b+c+2
d
c
b
-a-2b-3c-2d-e-10
e
c
d
b+c+1
a
-a-2b-3c-2d-e-10
e
b+c+1
d
c
a+b+1
-a-2b-3c-2d-e-10
e
c
d
b+c+1
a

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