Holomorphic flexibility properties of complements and mapping spaces

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Contents

Abstract

The classical Oka principle in complex analysis is a heuristic, supported by theorems of Oka, Grauert and others, to the effect that certain holomorphically defined problems involving Stein manifolds have only topological obstructions to their solution. Gromov's influential 1989 paper on the Oka principle introduced the class of so-called *elliptic manifolds*, and gave an Oka principle for maps from Stein manifolds into elliptic manifolds.

More recently, Forstnerič and Lárusson have introduced the category of *Oka manifolds* and *Oka maps*, which fit naturally into an abstract homotopy-theoretic framework; every elliptic manifold in Gromov's sense is also an Oka manifold. Examples of Oka manifolds are complex Lie groups and their homogeneous spaces (which are also elliptic); the complement in $\mathbb{P}^n$ of an algebraic subvariety of codimension at least 2; Hirzebruch surfaces; and more generally any fibre bundle whose base and fibre are Oka.

The Oka property can be thought of as a sort of anti-hyperbolicity; the notion of Kobayashi hyperbolicity expresses a type of holomorphic rigidity, and conversely, Oka manifolds are those that enjoy a high degree of holomorphic flexibility. Other flexibility properties enjoyed by Oka manifolds include strong dominability: for every Oka manifold X and every $p \in X$ there exists a holomorphic map $\mathbb{C}^n \to X$ which maps 0 to p and is a submersion at 0; and $\mathbb{C}$-connectedness: every pair of points can be joined by an entire curve.

The aim of this thesis is to provide new examples of Oka manifolds, and to shed light on the relationship between the Oka property and other types of holomorphic flexibility. Naturally occurring candidates for examples include complements of hypersurfaces in $\mathbb{P}^n$, especially low-degree or non-algebraic hypersurfaces (in contrast with Kobayashi's conjecture that the complement of a generic high-degree hypersurface should be hyperbolic), and spaces of holomorphic maps.

This thesis contains three chapters. The first chapter outlines the historical development of Oka theory, gives an overview of the remaining chapters, and suggests some directions for future research.

Chapter 2 is a paper entitled *Oka properties of some hypersurface complements*, to appear in Proceedings of the American Mathematical Society. There are two main results: a characterisation of when a

complement in $\mathbb{P}^n$ of hyperplanes is Oka, and the result that the complement of the affine graph of a meromorphic function is Oka, subject to some restrictions. The proof of the second result involves an extension to Gromov's technique of localisation of algebraic subellipticity.

Chapter 3 is a paper entitled *Holomorphic flexibility properties of the space of cubic rational maps*. Define R_d to be the space of rational functions of degree d on the Riemann sphere. Geometric invariant theory can be used to explore the structure of R_d: the Möbius group acts on R_d by precomposition and postcomposition. The two-sided action on R_2 is transitive, implying that R_2 is an Oka manifold. The action on R_3 has $\mathbb{C}$ as its categorical quotient; Section 3.4 gives an explicit formula for the quotient map and describes its structure in some detail. Furthermore, R_3 is strongly dominable and $\mathbb{C}$-connected.

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CHAPTER 1

Contextual statement

The broad topic of this thesis is *holomorphic flexibility*. This is best understood by contrast with its opposite.

Many results in classical complex analysis express a type of *rigidity*: that is, holomorphic maps are rare and special. For example, the theorems of Liouville and Picard describe strong restrictions on the existence of holomorphic functions $\mathbb{C} \to \mathbb{C}$. Kobayashi's theory of hyperbolic complex spaces extends these ideas to a more general setting: see Section 3 below.

On the other hand, the work of Oka, Grauert, Gromov and others throughout the 20th century shows that in some special settings, holomorphic maps are easier to come by than one might naively expect. Section 1 describes a collection of results broadly referred to as *the Oka principle*. During the last ten years this has crystallised into the notion of an *Oka manifold*, defined in 2009 by Franc Forstnerič [**17**]. Section 2 gives an overview of Oka manifolds and related flexibility properties.

Section 4 of this introduction outlines some ideas from geometric invariant theory which play a role in Chapter 3. The remaining sections give an overview of Chapters 2 and 3 and suggest some directions for future research in this area.

1. The Oka principle

The first example of the Oka principle concerns the so-called *Cousin problems*. Cousin's seminal paper of 1895 [**9**] was the first step in generalising the classical Mittag-Leffler and Weierstrass theorems to higher dimensions. The two Cousin problems were first explicitly stated by Cartan [**7**], along the following lines:

Let X be a complex manifold and $\{U_\alpha\}$ a family of open subsets covering X.

Problem 1. For each α, let g_α be a meromorphic function on U_α, and suppose that $g_\alpha - g_\beta$ is holomorphic on $U_\alpha \cap U_\beta$ for all α, β. Does there exist a meromorphic function g on X such that $g - g_\alpha$ is holomorphic on U_α for all α?

Problem 2. For each α, let g_α be a holomorphic function on U_α, and suppose that g_α/g_β is holomorphic and nowhere vanishing on $U_\alpha \cap U_\beta$

for all α, β. Does there exist a holomorphic function g on X such that g/g_α is holomorphic and nowhere vanishing on U_α for all α?

Cousin showed that if X is a polydisc in $\mathbb{C}^n$, then both problems can be solved. In fact, Cousin claimed a proof not only for polydiscs but for all product domains. However, Gronwall [**36**] pointed out an error in the proof, and gave a counterexample for the second problem in the case where $X \subset \mathbb{C}^2$ is a product of two annuli. (See [**8**, Section 3] for a detailed history of the Cousin problems; note that Chorlay uses the word *polycylinder* to refer to product domains.)

Over the next few decades, it became clear that Stein manifolds play a fundamental role in the theory of several complex variables. Consequently, the Cousin problems are of particular interest for the case where X is Stein. A Stein manifold is a complex manifold that is holomorphically separable and holomorphically convex:

Definition 1.1. A complex manifold X is *Stein* if

(1) for all $x, y \in X$ with $x \neq y$, there exists a holomorphic function $f \in \mathcal{O}(X)$ with $f(x) \neq f(y)$; and
(2) for every compact $K \subset X$, the holomorphically convex hull
$$\hat{K} = \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$
is compact.

A nontrivially equivalent definition is that a manifold if Stein if and only if every connected component is biholomorphic to a closed submanifold of $\mathbb{C}^n$ for some n.

The function theory of Stein manifolds is similar to that of domains in the plane. The most basic examples of Stein manifolds are the domains of holomorphy in $\mathbb{C}^n$. Further background on Stein manifolds and the theory of several complex variables can be found in many textbooks, including [**41, 50, 46, 28**].

If X is Stein, then the first Cousin problem always has a solution: this is an immediate consequence of Cartan's Theorem B. (See for example [**28**, Proposition V.1.6] or [**41**, Section 5.5].) However, there exist Stein manifolds on which the second Cousin problem is not always solvable: for instance, Gronwall's counterexample is Stein. Oka found a surprising criterion for when the second Cousin problem is solvable:

Theorem 1.2 ([**65**])**.** *Let $X \subset \mathbb{C}^n$ be a domain of holomorphy and $\{U_\alpha\}$ a family of open subsets covering X. For each α, let g_α be a holomorphic function on U_α, and suppose that g_α/g_β is holomorphic and nowhere vanishing on $U_\alpha \cap U_\beta$ for all α, β. Then the following are equivalent:*

(1) There exists a continuous function g on X such that g/g_α is continuous and nowhere vanishing on U_α for all α.

(2) *There exists a holomorphic function g on X such that g/g_α is holomorphic and nowhere vanishing on U_α for all α.*

In fact the result holds when X is any Stein manifold [**41**, Theorem 5.5.2]. Hörmander comments: *[This] is a case of the Oka principle: on a Stein manifold it is "usually" possible to do analytically what one can do continuously.* Grauert and Remmert give a more precise statement [**32**, page 145]: *On a reduced Stein space X, problems which can be cohomologically formulated have only topological obstructions. In other words, such problems are holomorphically solvable if and only if they are continuously solvable.*

There are many further developments of this theory, most notably by Grauert and Gromov. Oka's result can be viewed as a theorem about line bundles; Grauert generalised this to bundles of arbitrary rank over Stein spaces [**30, 31**]. The following theorem is a special case of what is sometimes called the *Oka–Grauert principle*:

Theorem 1.3. *Two holomorphic vector bundles over a Stein space that are isomorphic as topological complex vector bundles are also holomorphically isomorphic.*

See [**19**, Section 5.3] for a proof and further discussion of this result.

Gromov's wide-ranging paper [**35**] represents a shift of emphasis away from cohomology, highlighting the role of homotopy in the theory. He introduced the class of elliptic manifolds, greatly enlarging the context in which the Oka principle applies. An elliptic manifold is one which has a *dominating spray*, which is a generalisation of the exponential map of a Lie group:

Definition 1.4. Let X be a complex manifold. A *spray* over X is a holomorphic vector bundle $E \to X$ together with a holomorphic map $s\colon E \to X$ such that $s(0_x) = x$ for all $x \in X$. The spray is *dominating* if for all $x \in X$, the differential ds_{0_x} maps the vertical subspace E_x of $T_{0_x}E$ surjectively onto T_xX. The manifold X is *elliptic* if there exists a dominating spray over X.

Weaker notions of *subellipticity* were introduced by Forstnerič beginning with [**14**]; see also [**19**, pages 202–206].

Lie groups and their homogeneous spaces are elliptic. Other examples of elliptic manifolds will be described in Section 2 below.

Gromov's central result is what he calls the "main h-principle" [**35**, Section 0.6]:

Theorem 1.5. *Let S be a Stein manifold and X an elliptic manifold. Then every continuous map from S to X is homotopic to a holomorphic map. Furthermore, the inclusion $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}(S,X)$ of the space of holomorphic maps from S to X into the space of continuous maps is a weak homotopy equivalence in the compact-open topology.*

Section 3 of Gromov's paper describes some other "notions of ellipticity", sowing the seeds of modern Oka theory.

One notable application of the Oka principle is to Forster's conjecture that every Stein manifold of dimension n can be embedded into $\mathbb{C}^{\lfloor \frac{3n}{2} \rfloor + 1}$ [**13**]. This was proved for $n \geq 2$ by Eliashberg and Gromov [**12**] and Schürmann [**68, 69**]. The conjecture remains open for $n = 1$, but some partial results are given in [**26, 27, 61**].

Another significant application is the solution of the holomorphic Vaserstein problem by Ivarsson and Kutzschebauch [**42, 44, 43**]

2. Oka manifolds and holomorphic flexibility properties

An elliptic manifold enjoys a number of other Oka properties. These were explored in a series of papers in the period 2000–2009 by Forstnerič, some jointly with Prezelj, and by Lárusson; for a unified account of the subject, see the book [**19**] and the survey papers [**21, 20**]. The following properties are of fundamental importance.

Definition 2.1 ([**15, 16**]). A complex manifold X satisfies the *convex approximation property (CAP)* if every holomorphic map $K \to X$ from (a neighbourhood of) a convex compact subset K of $\mathbb{C}^n$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to X$.

Definition 2.2 ([**54**]). A complex manifold X satisfies the *convex interpolation property (CIP)* if whenever T is a contractible subvariety of $\mathbb{C}^m$ for some m, every holomorphic map $T \to X$ extends to a holomorphic map $\mathbb{C}^m \to X$. (Equivalently, we could take T to be any submanifold of $\mathbb{C}^m$ that is biholomorphic to a convex domain in $\mathbb{C}^n$ for some n; hence the use of the word *convex*.)

Definition 2.3 ([**16**]). Let X be a complex manifold, S a Stein manifold and K a holomorphically convex compact subset of S. Then X satisfies the *basic Oka property with approximation (BOPA)* if every continuous map $f\colon S \to X$ that is holomorphic on K can be deformed to a holomorphic map $S \to X$ keeping the intermediate maps holomorphic and arbitrarily close to f on K.

The CAP and the CIP can be viewed as generalisations of the classical theorems of Runge and Weierstrass respectively, while the BOPA is an extension of the conclusion of Theorem 1.5 above. Many variations on these themes are possible: as well as approximation, the BOPA can be supplemented with interpolation and jet interpolation conditions, and there exist parametric versions (replacing a single holomorphic map with a family of maps) of each Oka property. See [**19**, Section 5.15] for a list of fourteen Oka properties including the three given above.

In fact, all fourteen Oka properties are equivalent, motivating the following definition.

Definition 2.4 ([**17**]). An *Oka manifold* is a complex manifold satisfying the CAP.

The proofs of the equivalences are in some cases very involved. The papers of Forstnerič and Prezelj [**23, 24, 25**] began developing the necessary tools, giving rigorous proofs of many of Gromov's results. In particular, every elliptic manifold satisfies the BOPA, and is therefore Oka, and every Stein Oka manifold is elliptic. Lárusson in [**53**] proved the first nontrivial implication between Oka properties, and further results appeared in papers of Forstnerič [**15, 16, 17, 18**]. A complete account of the proofs is given in [**19**, Chapter 5].

Examples of Oka manifolds include complex Lie groups and their homogeneous spaces, in particular, complex affine space $\mathbb{C}^n$ and complex projective space $\mathbb{P}^n$ for all n and complex Grassmannians; complex tori; $\mathbb{C}^n \setminus A$ and $\mathbb{P}^n \setminus A$ where A is an algebraic subvariety of codimension at least 2; Hirzebruch surfaces and Hopf manifolds. A list of all known Oka manifolds as of September 2010 appears in [**21**, end of Section 3]. More recent examples are discussed in [**20**] and in Chapter 2 of this thesis; see also [**55, 56, 22**].

So far, most known examples of Oka manifolds are elliptic, and it is not known whether in fact every Oka manifold is elliptic. The class of Oka manifolds satisfies some functorial properties for which no analogues are known for elliptic manifolds. In particular, if $E \to B$ is a holomorphic fibre bundle with Oka fibres, then E is Oka if and only if B is Oka. This result fits into a larger context: Lárusson [**52, 53**] showed how to embed the category of complex manifolds into a model category in such a way that the Oka manifolds are exactly those manifolds that are fibrant objects, and the Stein manifolds are exactly those manifolds that are cofibrant objects.

Some elementary consequences of the Oka properties are of independent interest. Every Oka manifold enjoys the following two holomorphic flexibility properties:

Definition 2.5. Let X be a complex manifold. We say that X is *dominable* if for some n there exists a holomorphic map $\phi\colon \mathbb{C}^n \to X$ which is a submersion at some point. Furthermore, X is *strongly dominable* if for every $p \in X$ there exists $\phi\colon \mathbb{C}^n \to X$ such that $\phi(0) = p$ and $d\phi_0$ is surjective.

Definition 2.6. A complex manifold X is *strongly* $\mathbb{C}$*-connected* if every pair of points can be joined by an entire curve; that is, for every pair of points of X there is a holomorphic map $\mathbb{C} \to X$ whose image contains both points. The manifold X is *(weakly)* $\mathbb{C}$*-connected* if every pair of points can be joined by a finite chain of entire curves.

Buzzard and Lu [**5**] give a detailed account of dominable surfaces. The notion of $\mathbb{C}$-connectedness has received less attention in the literature so far, and is a promising subject for further research.

3. Hyperbolic manifolds

The Oka properties, and the notions of (sub)ellipticity, dominability and $\mathbb{C}$-connectedness, encapsulate in various ways the idea that there are "many" holomorphic maps $\mathbb{C} \to X$ into a complex manifold X. The various notions of hyperbolicity embody the opposite. For example:

Definition 3.1. A complex manifold X is *Brody hyperbolic* if there are no nonconstant holomorphic maps $\mathbb{C} \to X$.

Liouville's theorem can be expressed as the statement that every bounded domain in $\mathbb{C}$ is Brody hyperbolic. Similarly, Picard's little theorem is the statement that $\mathbb{C}$ with two points deleted is Brody hyperbolic.

The Schwarz lemma, giving restrictions on the possible self-maps of the complex unit disc, can be thought of as a quantitative version of the same idea. Developing this line of thought further leads eventually to the following definitions.

Definition 3.2. The *Kobayashi pseudo-distance* on a complex manifold X is the largest pseudo-distance such that every holomorphic map $\mathbb{D} \to X$ is distance-decreasing, where $\mathbb{D}$ denotes the complex unit disc with the Poincaré metric. We say that X is *hyperbolic* if the Kobayashi pseudo-distance is a distance.

The standard reference for the theory of hyperbolic complex manifolds (and, more generally, complex spaces) is Kobayashi's monograph [**49**]; his earlier book [**48**] gives a more elementary introduction to the main ideas.

It is immediate from the definitions that every hyperbolic manifold is also Brody hyperbolic. The converse is true for compact manifolds [**3**], but not in general; see [**48**, page 130] for a counterexample. On an Oka manifold, the Kobayashi pseudo-distance is identically zero; thus the Oka properties can be thought of as a sort of anti-hyperbolicity.

Examples of hyperbolic manifolds include bounded domains in $\mathbb{C}^n$; Hermitian manifolds whose holomorphic sectional curvature is bounded above by a negative constant; and the complement in $\mathbb{P}^n$ of at least $2n+1$ hyperplanes in general position. For further results on hyperplane complements, see [**33**] and [**49**, Section 3.10].

A major theme in the study of hyperbolicity has been the so-called *Kobayashi conjecture*: that the complement in $\mathbb{P}^n$ of a generic hypersurface of sufficiently high degree is hyperbolic [**48**, problem 3 on page 132]. Siu has recently announced a proof [**73**]; previous work on this question includes [**34, 77, 4, 74, 71**]

4. Geometric invariant theory

Chapter 3 of this thesis applies geometric invariant theory to the study of holomorphic flexibility properties. This is motivated by the fact, mentioned above, that the Oka property passes up and down in fibre bundles with Oka fibres. Therefore a promising way to study a complex manifold X from the point of view of Oka theory is to find fibre bundles $X \to Y$ for which Y is in some way simpler than X, and whose fibres are Oka.

This is in general a difficult problem: there are no general criteria for determining whether X can be decomposed in this way. However, we can form quotient spaces using group actions, and geometric invariant theory provides some criteria for determining whether the quotients are tractable for our purposes.

The modern approach to geometric invariant theory was introduced principally by Mumford [**63**]. The emphasis was on applications to moduli problems, and on developing a theory suitable for schemes in general. For the present study, a more concrete version, giving detailed information about the structure of various quotient maps, is desirable. The necessary results are provided for the algebraic category by Luna [**59**], and Snow [**75**] gives analogues of Luna's results for Stein spaces.

Let a complex Lie group G act holomorphically on a complex manifold X. Then the orbit space X/G exists as a topological space. Furthermore, the fibres of the quotient map are homogeneous spaces for G, and may therefore be viewed as Oka manifolds. There are two important questions to be answered: can X/G be given the structure of a complex manifold in such a way that the quotient map is holomorphic; and is the quotient map then a fibre bundle?

If G is reductive and X is Stein, then the results of Snow apply. If the action of G is free, or, more generally, if every orbit of G is closed, then the answer to both questions is affirmative [**75**, Corollary 5.5]. When non-closed orbits exist, it is necessary to replace the geometric quotient with the *categorical quotient*: this is the quotient of X by the equivalence relation defined by the ring of G-invariant holomorphic functions on X. See [**75**, Section 3] for details. The categorical quotient can also be defined in terms of a universal property: every G-invariant holomorphic map $X \to Z$, where X is a complex space, factors uniquely through the categorical quotient.

Snow's main theorem is that if G is reductive and X is Stein, then the categorical quotient is isomorphic to a Stein space (often not smooth) such that the quotient map is holomorphic. This takes us part of the way towards our goal, but the two questions above still need to be addressed. In the situation described in Chapter 3, the categorical quotient can be described explicitly, verifying that it is in fact smooth. However, the quotient map is not a even a fibration, let alone a fibre bundle. Nevertheless, we show how the quotient map is useful in proving $\mathbb{C}$-connectedness and strong dominability. Chapter 2, Section 4 gives an unrelated example where the Oka property can be shown to pass up through a map which is not a fibration; applying such techniques to quotient maps of group actions is a promising area for future research.

5. Research overview

Over the last decade, there has been substantial progress in the theory of Oka manifolds: as outlined in Section 2 above, sophisticated new techniques have been introduced for proving the equivalences between the various Oka properties, and the context of model categories provides compelling evidence for the importance of the class of Oka manifolds. However, the list of examples of known Oka manifolds is still short, and a variety of ad hoc methods have been used to prove the Oka property in specific cases. In general, the problem of determining whether or not a given manifold is Oka is difficult, and there is a need to develop new tools for studying specific examples.

Chapters 2 and 3 of this thesis are devoted to three classes of examples within the common theme of hypersurface complements. A simple heuristic seems to apply here: the "smaller" a manifold is, the more likely it is to be hyperbolic; a "bigger" manifold is more likely to be Oka. For example, $\mathbb{C}^n$ is Oka, and so is the complement in $\mathbb{C}^n$ of a subvariety of codimension at least 2; but bounded domains in $\mathbb{C}^n$ are hyperbolic. The complement of a hypersurface is an intermediate case which, depending on the choice of hypersurface, can be Oka or hyperbolic or neither.

For the case of hyperbolicity, we can give a precise result: it is easy to prove from the definitions that every open subset of a hyperbolic manifold is hyperbolic. Corresponding results for Oka manifolds are not known: for instance, one might hope that if X minus a point is Oka, then so is X, but a proof this or any similar fact remains elusive. However, all examples studied so far are in accord with the heuristic.

5.1. Hyperplane complements. As mentioned above, the complement in $\mathbb{P}^n$ of at least $2n+1$ hyperplanes is hyperbolic. This can be viewed as a degenerate case of the Kobayashi conjecture. Chapter 2, Section 3 addresses the opposite problem: for which arrangements of hyperplanes is the complement Oka? A complete answer is given: the complement is Oka if and only if the number of hyperplanes is at most $n+1$ and the hyperplanes are in general position. Furthermore, if the complement is not Oka then it is not even dominable.

5.2. Complements of graphs of meromorphic functions. Buzzard and Lu showed that the complement in $\mathbb{P}^2$ of a smooth cubic curve is dominable [**5**, Proposition 5.1]. This raises the question of whether the complement is Oka [**19**, Problem 5.16.15]; this is still an open problem. Buzzard and Lu's proof of dominability involves a branched covering map from the complement of the graph of a certain meromorphic function onto the cubic complement. Thus meromorphic graph complements appear as natural objects of study.

Chapter 2, Section 4 shows that the complement of the graph of a meromorphic function on $\mathbb{C}$ is always Oka. This result is generalised to meromorphic functions on Oka manifolds other than $\mathbb{C}$, subject to an additional hypothesis. Namely, if a meromorphic function $m\colon X \to \mathbb{P}^1$ can be written in the form $f + 1/g$ for holomorphic functions f and g, then the graph complement is Oka if and only if X is Oka. The condition on the form of m arises naturally from Buzzard and Lu's construction. This condition is satisfied for all meromorphic functions on $\mathbb{C}$, but not always on more general domains. Lemma 2.4.2 describes the meromorphic functions on $\mathbb{C}^n$ that satisfy this condition.

Chapter 3, Appendix B describes a special case of this phenomenon.

5.3. Spaces of rational maps. Let X and Y be complex manifolds with X compact. Then the space of holomorphic maps $\mathcal{O}(X, Y)$ can be given the structure of a complex manifold in a natural way (see Chapter 3, Section 2.1 for details). Such mapping spaces are natural candidates for Oka manifolds: the Oka property for $\mathcal{O}(X, Y)$ can be viewed as a holomorphic analogue of the parametric Oka property for Y.

The simplest interesting case, $X = Y = \mathbb{P}^1$, explored in Chapter 3, already leads into deep waters. Writing R_d for the set of degree d rational functions on the Riemann sphere, the connected components of $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ are exactly R_0, R_1, R_2,.... Each R_d can be realised as the complement in $\mathbb{P}^{2d+1}$ of a hypersurface. Spaces of rational maps occur naturally in a number of contexts. For example, R_d occurs as the moduli space of magnetic monopoles of charge d [**1, 64**]. Brockett's study of real rational maps in control theory [**2**] provided the initial motivation for Segal's influential work on the topology of R_d [**70**].

It is straightforward to prove that R_0, R_1 and R_2 are all Oka manifolds. However, it is not known whether R_d is Oka for any $d \geq 3$. Chapter 3 describes an action of the complex Lie group $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on R_3. The three main results are that the categorical quotient for this action is $\mathbb{C}$, and that R_3 is strongly dominable and $\mathbb{C}$-connected. Furthermore, Chapter 3, Section 4 gives an explicit description of the categorical quotient map in terms of cross-ratios of critical points and critical values of rational functions, and uses this construction to derive further information about the structure of the quotient map.

6. Conclusions

As well as contributing further examples to the class of known Oka manifolds, this thesis introduces several new techniques that are likely to be applicable in more general contexts. This suggests a number of promising directions for further research.

In the case of meromorphic graph complements, discussed in Chapter 2, Section 4, the projection map from the graph complement onto the first factor is a submersion. The main technical obstacle in this situation is that the map has mixed fibres: it is not a topological fibration. The solution is to "untwist" the fibres by passing to a suitable covering space. It may be possible to apply similar techniques to other complement spaces; natural choices would be meromorphic functions not of the form considered in Chapter 2, and complements of low degree hypersurfaces, in particular the complement of a cubic in $\mathbb{P}^2$.

Chapter 3 illustrates a situation where dominability and $\mathbb{C}$-connectedness of a quotient space pass up to the total space; the tools used here are geometric invariant theory and the "composition of dominability" lemma. It is likely that similar results apply in more general settings. In particular, these tools could be used to investigate dominability and $\mathbb{C}$-connectedness of R_d for $d \geq 4$.

The Oka property for R_d is another natural area for further work, although more challenging. For R_3, it is possible that a variation of the ideas of Chapter 2 for mixed fibres, combined with the categorical quotient map, could yield new results. For $d \geq 4$, methods based on cross-ratios would not easily generalise, and some new techniques may be needed.

The questions posed for $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ could also be asked for $\mathcal{O}(\mathbb{P}^n, \mathbb{P}^m)$ in general, and for many other mapping spaces.

CHAPTER 2

Oka properties of some hypersurface complements

Abstract. Oka manifolds can be viewed as the "opposite" of Kobayashi hyperbolic manifolds. Kobayashi asked whether the complement in projective space of a generic hypersurface of sufficiently high degree is hyperbolic. Therefore it is natural to investigate Oka properties of complements of low degree hypersurfaces. We determine which complements of hyperplane arrangements in projective space are Oka. A related question is which hypersurfaces in affine space have Oka complements. We give some results for graphs of meromorphic functions.

1. Introduction

A complex manifold X is *hyperbolic* (in the sense of Kobayashi) if, informally speaking, there are "few" maps $\mathbb{C} \to X$, and *Oka* if there are "many" maps $\mathbb{C} \to X$, in a sense to be made precise in Section 2 below. Hyperbolic manifolds have been extensively studied since the late 1960s. Oka theory is a more recent development, motivated by Gromov's paper [**35**] of 1989; the definition of an Oka manifold was only published in 2009, by Forstnerič [**17**].

Many interesting examples of hyperbolic manifolds arise from complements of projective hypersurfaces. In particular, Kobayashi asked [**48**, problem 3 on page 132] whether the complement in $\mathbb{P}^n$ of a generic hypersurface of sufficiently high degree should be hyperbolic. This has been proved for $n = 2$ by Siu and Yeung [**74**], but is still an open problem in higher dimensions. The degenerate case of the complement of a finite collection of hyperplanes is well understood. In particular, the complement in $\mathbb{P}^n$ of at least $2n + 1$ hyperplanes in general position is hyperbolic, and the complement of a collection of $2n$ or fewer hyperplanes is never hyperbolic. For hyperplanes not in general position, some necessary conditions for hyperbolicity are known. See Kobayashi's monograph [**49**, Section 3.10] for details.

Since the Oka property can be viewed as a sort of anti-hyperbolicity, it makes sense to ask which hypersurfaces have Oka complements. In Section 3 of this paper we give a complete answer to this question for complements of hyperplane arrangements in projective space. The main result of this section, Theorem 3.1, states that the complement of N hyperplanes in $\mathbb{P}^n$ is Oka if and only if the hyperplanes are in general position and $N \leq n+1$. We also investigate the weaker Oka-type properties of dominability by $\mathbb{C}^n$ and $\mathbb{C}$-connectedness: in this context we find that a non-Oka complement also fails to possess these weaker properties.

In Section 4 we give a sufficient condition for the complement of the graph of a meromorphic function to be Oka. Our Theorem 4.6 states that if $m : X \to \mathbb{P}^1$ can be written in the form $m = f + 1/g$ for holomorphic functions f and g, then the graph complement is Oka if and only if X is Oka. This is motivated by the open problem of whether the complement in $\mathbb{P}^2$ of a smooth cubic curve is Oka: given a cubic curve, there is an associated meromorphic function and a branched covering map from the graph complement of that function to the cubic complement. For details, see Buzzard and Lu [**5**, page 644–645]. We also explore the question of when the decomposition $m = f + 1/g$ exists (Lemma 4.2). For meromorphic functions that cannot be written in this form, further work is required to understand the Oka properties of the graph complements.

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2. Oka manifolds and hyperbolic manifolds

In this section we recall the definitions of Oka manifolds and hyperbolic manifolds, and collect some results that will be used later. For background, motivation and further details of Oka theory, see the survey article [**21**] of Forstnerič and Lárusson and the recently published book [**19**] of Forstnerič. For more on hyperbolicity, see the monograph [**49**] of Kobayashi.

Definition 2.1. A complex manifold X is an *Oka manifold* if every holomorphic map $K \to X$ from (a neighbourhood of) a convex compact subset K of $\mathbb{C}^n$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to X$.

This defining property is also referred to as the *convex approximation property (CAP).*

Definition 2.2. The *Kobayashi pseudo-distance* on a complex manifold X is the largest pseudo-distance such that every holomorphic map $\mathbb{D} \to X$ is distance-decreasing, where $\mathbb{D}$ denotes the complex unit disc with the Poincaré metric. We say that X is *hyperbolic* if the Kobayashi pseudo-distance is a distance.

If X is Oka then the Kobayashi pseudo-distance on X is identically zero; thus Oka manifolds can be viewed as "anti-hyperbolic". The most fundamental examples of Oka manifolds are complex Lie groups and their homogeneous spaces; in particular, $\mathbb{P}^n$ and $\mathbb{C}^n$ are Oka. Bounded domains in $\mathbb{C}^n$ are always hyperbolic. If X is a Riemann surface, then X is Oka if and only if it is one of $\mathbb{C}$, $\mathbb{C}^*$ (the punctured plane), $\mathbb{P}^1$ or a torus; otherwise it is hyperbolic.

Every Oka manifold X of dimension n is *dominable* by $\mathbb{C}^n$, in the sense that there exists a holomorphic map $\mathbb{C}^n \to X$ that has rank n at some point of $\mathbb{C}^n$.

Oka manifolds are also $\mathbb{C}$-connected: every pair of points can be joined by an entire curve, i.e. for any pair of points there exists a holomorphic map from $\mathbb{C}$ into the manifold whose image contains both points. This property is mentioned by Gromov [**35**, 3.4(B)], and follows easily from the "basic Oka property" described in [**21**, page 16]. (The definition of $\mathbb{C}$-*connected* is not standardised: the term can also refer to the weaker property that every pair of points can be joined by a finite chain of entire curves, by analogy with the case of rational connectedness.)

In general it is difficult to verify the condition of Definition 2.1 directly. Instead, sprays (in the sense of Gromov: see below) and fibre bundles are of fundamental importance. If $\pi : X \to Y$ is a holomorphic fibre bundle with Oka fibres, then X is Oka if and only if Y is Oka. (In fact there is a far more general notion of an *Oka map* which preserves the Oka property, but this will not be needed here.) In particular, products of Oka manifolds are Oka, and a manifold is Oka if it has a covering space that is Oka.

Definition 2.3. A *spray* over a complex manifold X consists of a holomorphic vector bundle $\pi : E \to X$ and a holomorphic map $s : E \to X$ such that $s(0_x) = x$ for all $x \in X$. We say that s is *dominating* at the point $x \in X$ if the differential ds_{0_x} maps the vertical subspace E_x of $T_{0_x}E$ surjectively onto T_xX. A family of sprays (E_j, π_j, s_j), $j = 1, \dots, m$, is *dominating* at x if

$$(ds_1)_{0_x}(E_{1,x}) + \cdots + (ds_m)_{0_x}(E_{m,x}) = T_xX.$$

The manifold X is *elliptic* if there exists a spray that is dominating at every point of X, and *weakly subelliptic* if for every compact set $K \subset X$ there exists a finite family of sprays over X that is dominating at every point of K.

The concept of a spray can be viewed as a generalisation of the exponential map for a complex Lie group: for example, see [**21**, Examples 5.3] or [**19**, Proposition 5.5.1].

Every elliptic or weakly subelliptic manifold is Oka.

The following property is equivalent to the CAP.

Definition 2.4. A complex manifold X satisfies the *convex interpolation property (CIP)* if whenever T is a contractible subvariety of $\mathbb{C}^m$

for some m, every holomorphic map $T \to X$ extends to a holomorphic map $\mathbb{C}^m \to X$.

(Equivalently, we could take T to be any subvariety of $\mathbb{C}^m$ that is biholomorphic to a convex domain in $\mathbb{C}^n$; hence the use of the word *convex*.)

A useful tool for proving hyperbolicity is Borel's generalisation of Picard's little theorem. Kobayashi gives three equivalent formulations (see [**49**, Theorem 3.10.2 on page 134]), of which we only need the following.

Theorem 2.5 (Picard–Borel)**.** *Let $g_0, \ldots, g_N$ be nowhere vanishing holomorphic functions on $\mathbb{C}$, and suppose*

$$g_0 + \cdots + g_N = 0.$$

Partition the index set $\{0, 1, \ldots, N\}$ into subsets, putting two indices j and k into the same subset if and only if g_j/g_k is constant. Then for each subset J,

$$\sum_{j \in J} g_j = 0.$$

Remark 2.6. Since the g_j are nowhere vanishing, it follows that each subset must have size greater than 1. In particular, if $N = 2$ then the partition has only one part, hence g_0, g_1 and g_2 are constant multiples of each other.

3. Hyperplane complements

Let $F_1, \ldots, F_N$ be nonzero homogeneous linear forms in $n+1$ variables. We say that the hyperplanes in $\mathbb{P}^n$ defined by the equations $F_j = 0$, $j = 1, \ldots, N$, are in *general position* if every subset of $\{F_1, \ldots, F_N\}$ of size at most $n+1$ is linearly independent. If $N \leq n+1$, then a set of N hyperplanes is in general position if and only if coordinates can be chosen on $\mathbb{P}^n$ so that the given hyperplanes are the coordinate hyperplanes $x_j = 0$, $j = 0, \ldots, N-1$.

Theorem 3.1. *Let $H_1, \ldots, H_N$ be distinct hyperplanes in $\mathbb{P}^n$. Then the complement $X = \mathbb{P}^n \setminus \bigcup_{j=1}^N H_j$ is Oka if and only if the hyperplanes are in general position and $N \leq n+1$. Furthermore, if X is not Oka then it is not dominable by $\mathbb{C}^n$ and not $\mathbb{C}$-connected.*

Before proving this, we state and prove a sharper form of Theorem 3.10.15 of Kobayashi's book [**49**, page 142]. To state the theorem, it is convenient to introduce the following terminology.

Definition 3.2. Let $H_1, \ldots, H_k$ be distinct hyperplanes in $\mathbb{P}^n$ defined by linear forms $F_1, \ldots, F_k$, and suppose the forms satisfy a minimal linear relation of the form

$$c_1 F_1 + \cdots + c_k F_k = 0$$

where $c_j \neq 0$ for all j. (By "minimal" we mean that $\sum_{j\in J} c_j F_j \neq 0$ for every proper nonempty subset J of $\{1,\dots,k\}$.) Then the *diagonal hyperplanes* of the linear relation are the hyperplanes defined by the linear forms $\sum_{j\in J} c_j F_j$ where J is a subset of $\{1,\dots,k\}$ with $2 \leq |J| \leq k-2$. (If $k \leq 3$, there are no diagonal hyperplanes.) The *associated subspaces* of $\{H_1,\dots,H_k\}$ are the linear subspaces of $\mathbb{P}^n$ which contain $\bigcap_{j=1}^k H_j$ with codimension 1. (If $\bigcap H_j = \varnothing$, the associated subspaces are exactly the points of $\mathbb{P}^n$.)

Remark 3.3. If $p \in \mathbb{P}^n \setminus \bigcap H_j$, then p is contained in exactly one associated subspace for each minimal linear relation.

Example 3.4. On $\mathbb{P}^2$ consider the linear forms

$$\begin{aligned} F_1 &= x_1, \\ F_2 &= x_2, \\ F_3 &= x_1 - x_0, \\ F_4 &= x_2 - x_0. \end{aligned}$$

If we consider $x_0 = 0$ to be the line at infinity, then the lines $F_j = 0$, $j = 1,2,3,4$, are the sides of a "unit square" in the affine plane. The linear relation $F_1 - F_2 - F_3 + F_4 = 0$ has three diagonal lines (noting that $J = \{1,2\}$ and $J = \{3,4\}$ give the same line, and so on). They are the two diagonals of the square ($x_1 = x_2$ and $x_1 + x_2 = x_0$), and the line at infinity ($x_0 = 0$).

Example 3.5. Let P be any point of $\mathbb{P}^2$ and let F_1, F_2 and F_3 be linear forms defining three distinct lines through P. Then there exists a linear relation among F_1, F_2 and F_3, and the associated subspaces are the lines through P.

Theorem 3.6. *Let $H_1,\dots,H_N$ be distinct hyperplanes in $\mathbb{P}^n$ defined by linear forms $F_1,\dots,F_N$, and let $f : \mathbb{C} \to \mathbb{P}^n \setminus \bigcup H_j$ be a holomorphic map. Suppose that $F_1,\dots,F_N$ are linearly dependent. Then for each subset of $F_1,\dots,F_N$ satisfying a minimal linear relation, there is a diagonal hyperplane or an associated subspace containing the image of f.*

Remark 3.7. In the case where $\bigcap H_j = \varnothing$, the associated subspaces are points, so the conclusion is that either f is constant or the image is contained in a diagonal hyperplane.

Proof of Theorem 3.6. First we note that f can be lifted to a holomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{C}^{n+1} \setminus \{0\}$. To see this, observe that the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ can be regarded as the universal line bundle over $\mathbb{P}^n$ with the zero section removed. Thus lifting f is equivalent to finding a nowhere vanishing section of the pullback by f of the universal bundle. But the vanishing of the cohomology group $H^1(\mathbb{C}, \mathcal{O}^*)$ guarantees that line bundles over $\mathbb{C}$ are trivial, and therefore a nowhere vanishing section always exists.

By reordering and rescaling the defining forms, we can put a minimal linear relation in the form

$$F_1 + \cdots + F_k = 0.$$

Define entire functions $h_1, \ldots, h_k$ by $h_j = F_j \circ \tilde{f}$. Then the h_j satisfy the hypotheses of the Picard–Borel theorem (Theorem 2.5): each h_j vanishes nowhere (because the image of f misses all the hyperplanes), and the h_j sum to the zero function (because of the linear relation between the F_j). Theorem 2.5 tells us that there is a subset $J \subset \{1, \ldots, k\}$ with

$$\sum_{j \in J} h_j = 0$$

and such that all the ratios h_μ / h_ν are constant for $\mu, \nu \in J$. There are two possibilities.

First, if J is a proper subset of $\{1, \ldots, k\}$ then J must have size at least 2 and at most $k - 2$. (If J either consisted of or omitted only a singleton j, then the corresponding h_j would be identically zero.) In this case the linear form

$$F = \sum_{j \in J} F_j$$

defines a diagonal hyperplane in $\mathbb{P}^n$. (The minimality of the linear relation implies that F is nonzero.) The image of f lies in this hyperplane.

The second case is that $J = \{1, \ldots, k\}$. Then there exist nonzero constants $c_1 \ldots, c_{k-1}$ such that

$$h_j = c_j h_k$$

for $j = 1, \ldots, k - 1$. This means that the image of $\tilde{f}$ lies in each of the hyperplanes $F_j = c_j F_k$. Let A and B be the linear subspaces of $\mathbb{C}^{n+1}$ given by

$$A = \bigcap_{j=1}^{k-1} \{F_j - c_j F_k = 0\},$$

$$B = \bigcap_{j=1}^{k} \{F_j = 0\}.$$

Clearly $B \subset A$. It remains to show that B has codimension at most 1 in A. Equivalently, we wish to show that given $x, y \in A \setminus B$, some nontrivial linear combination of x and y lies in B. The numbers $\alpha = F_k(x)$ and $\beta = F_k(y)$ are both nonzero. Then $F_k(\beta x - \alpha y) = 0$, so $F_j(\beta x - \alpha y) = 0$ for all j, hence $\beta x - \alpha y \in B$. □

Remark 3.8. In the above proof, the subspaces A and B can never be equal (because the image of f misses all of the H_j). A naive dimension argument might suggest that A and B have the same dimension. However, the fact that the h_j sum to zero implies the relation $c_1 + \cdots + c_{k-1} = -1$, so the forms $F_j - c_j F_k$ are linearly dependent: their sum is zero.

Remark 3.9. In the case $J = \{1, \ldots, k\}$, Kobayashi states that f is constant ([**49**, page 142, proof of Theorem 3.10.15, last paragraph]). In fact there exist nonconstant maps whose images lie in associated subspaces, but this does not invalidate the conclusion of Kobayashi's Theorem 3.10.15. For an example of such a map, take $n = 4$ with linear forms

$$F_1 = x_0 - x_2, \quad F_2 = x_2 - x_1, \quad F_3 = x_1 - x_3, \quad F_4 = x_3 - x_0, \quad F_5 = x_4$$

so that $F_1 + F_2 + F_3 + F_4 = 0$. Let $f : \mathbb{C} \to X$ be a function that lifts to

$$\tilde{f}(t) = (t, t+1, t+2, t+3, 1).$$

Then $h_1, \ldots, h_4$ are the constant functions $-2, 1, -2, 3$ respectively, but f is not constant.

Corollary 3.10. *Let $H_1, \ldots, H_N$ be distinct hyperplanes in $\mathbb{P}^n$, not in general position. Let p be any point of $X = \mathbb{P}^n \setminus \bigcup H_j$. Then there is a finite collection of proper subspaces of T_pX with the property that for every map $f : \mathbb{C} \to X$ with $f(0) = p$, the derivative $df(0)$ lies in one of those subspaces.*

Proof. By Theorem 3.6, the image of f is restricted to a proper linear subspace of X. Thus there is a corresponding subspace of T_pX containing $df(0)$. We just need to verify that there are only finitely many possible subspaces. But each point of X is contained in exactly one associated subspace for each minimal linear relation among the F_j, and there are only finitely many diagonal hyperplanes. □

Corollary 3.11. *If the distinct hyperplanes $H_1, \ldots, H_N$ in $\mathbb{P}^n$ are not in general position, then $\mathbb{P}^n \setminus \bigcup H_j$ is not dominable by $\mathbb{C}^n$.*

Proof. Let f be a map from $\mathbb{C}^n$ into $\mathbb{P}^n \setminus \bigcup H_j$, with $f(0, \ldots, 0) = p$. The image of $df(0)$ is spanned by the n vectors

$$d(t \mapsto f(te_j))|_{t=0} \quad (j = 1, \ldots, n)$$

where $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{C}^n$. If $df(0)$ is surjective, those vectors are linearly independent, so there will be no finite set of proper subspaces containing

$$d(t \mapsto f(tv))|_{t=0}$$

for all $v \in \mathbb{C}^n$, contradicting the previous corollary. □

Proof of Theorem 3.1. Write X for the space $\mathbb{P}^n \setminus \bigcup_{j=1}^{N} H_j$.

Case 1: hyperplanes in general position and $N > n+1$. In this case, Kobayashi's Theorem 3.6.10 [**49**, page 136] tells us that the image of a nonconstant holomorphic map $\mathbb{C} \to X$ must lie in one of a finite collection of hyperplanes. Therefore X is not dominable by $\mathbb{C}^n$. Also, X is not $\mathbb{C}$-connected: distinct points outside the finite collection of hyperplanes cannot be joined by an entire curve.

Case 2: hyperplanes in general position and $N \leq n+1$. If $N = 0$, then $X = \mathbb{P}^n$ is Oka. For $N > 0$, the fact that the hyperplanes are in general position means that we can choose coordinates so that H_j is the hyperplane $x_{j-1} = 0$ for $j = 1, \ldots, N$. Then we see that $X \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^* \times \mathbb{C} \times \cdots \times \mathbb{C}$ with $N-1$ factors $\mathbb{C}^*$ and $n+1-N$ factors $\mathbb{C}$. This is a product of Oka manifolds, hence Oka.

Case 3: hyperplanes not in general position. The fact that X is not dominable, and therefore not Oka, is just Corollary 3.11 above. To see that X is not $\mathbb{C}$-connected, choose any point $p \in X$. Then Theorem 3.6 gives a finite collection of hyperplanes containing every entire curve through p. If q is a point of X outside that finite collection, then p and q cannot be joined by an entire curve. $\square$

Remark 3.12. In fact if X is not Oka, then it does not satisfy the weaker version of $\mathbb{C}$-connectedness referred to above: there exist pairs of points that cannot be connected even by a finite chain of entire curves. In case 1 of the above proof this is immediate. For case 3, some further work is needed, since the finite collection of hyperplanes referred to can vary with the choice of the point p. The key ingredients are the fact that there are only finitely many diagonal hyperplanes in total, and that given an associated subspace A and a diagonal hyperplane D of the same configuration, either $A \subset D$ or $A \cap D \subset \bigcup H_j$. In other words, points inside a diagonal hyperplane cannot be joined to points outside via associated subspaces.

4. Complements of graphs of meromorphic functions

Buzzard and Lu [**5**, Proposition 5.1] showed that the complement in $\mathbb{P}^2$ of a smooth cubic curve is dominable by $\mathbb{C}^2$. Their method of proof was to construct a meromorphic function associated with the cubic, and a branched covering map from the complement of the graph of that function to the complement of the cubic, and then show that the graph complement is dominable. We will show that the graph complement is in fact Oka; this result can be generalised to meromorphic functions on Oka manifolds other than $\mathbb{C}$, subject to an additional hypothesis. (Note that our result is not enough to settle the question of whether the cubic complement is Oka. We know that the Oka property passes down through *unbranched* covering maps, but no similar result is known for branched coverings.)

For a holomorphic map $m : X \to \mathbb{P}^1$ on a complex manifold X, that is to say either a meromorphic function with no indeterminacy or else the constant function ∞, we will write Γ_m for the affine graph

$$\Gamma_m = \{(x, m(x)) \in X \times \mathbb{C} : m(x) \neq \infty\}.$$

This is a closed subset of $X \times \mathbb{C}$, so the set $(X \times \mathbb{C}) \setminus \Gamma_m$ is a complex manifold. (If m is identically ∞, then Γ_m is the empty set.)

Buzzard and Lu's result relies on the fact that meromorphic functions on $\mathbb{C}$ can be written in the following form.

Lemma 4.1. *For every meromorphic function $m : \mathbb{C} \to \mathbb{P}^1$ there exist holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ such that*

$$m = f + \frac{1}{g}.$$

In other words, the projection map from $\mathbb{C}^2 \setminus \Gamma_m$ onto the first coordinate has a holomorphic section given by $x \mapsto (x, f(x))$.

The result follows from the classical theorems of Mittag-Leffler and Weierstrass; see [**5**, page 645] for details.

The analogous result in higher dimensions is not true. We have the following topological criterion.

Lemma 4.2. *Let m be a holomorphic map from $\mathbb{C}^n$ to $\mathbb{P}^1$, and write $m = h/k$ for holomorphic functions $h, k : \mathbb{C}^n \to \mathbb{C}$ with no common zeros. Then the following statements are equivalent.*

(1) There exist holomorphic functions $f, g : \mathbb{C}^n \to \mathbb{C}$ such that

$$m = f + \frac{1}{g}.$$

(2) The function h has a logarithm on the zero set $Z(k)$ of k.

(3) The function h has a logarithm on a neighbourhood of $Z(k)$.

Proof. $(1 \Rightarrow 2)$: The function

$$\frac{1}{g} = m - f = \frac{h - kf}{k}$$

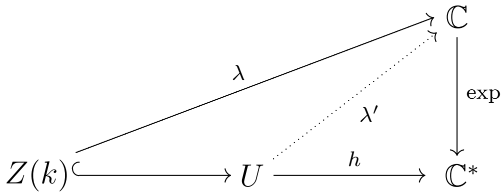
has no zeros. Therefore $h - kf$ is a nowhere vanishing entire function, so

$$h - kf = e^{\mu}$$

for some holomorphic $\mu : \mathbb{C}^n \to \mathbb{C}$. Then $\mu|_{Z(k)}$ is the desired logarithm of h.

$(2 \Rightarrow 3)$: Suppose there is a holomorphic function λ on $Z(k)$ such that $e^{\lambda} = h|_{Z(k)}$. We wish to find a neighbourhood U of $Z(k)$, small enough that h vanishes nowhere on U, such that the inclusion map

$Z(k) \hookrightarrow U$ induces an epimorphism of fundamental groups. Given such a neighbourhood, we have the following situation.



Then the existence of λ, together with the epimorphism $\pi_1(Z(k)) \to \pi_1(U)$, tells us that $h|_U$ satisfies the lifting criterion for the covering map $\exp : \mathbb{C}^* \to \mathbb{C}$. Therefore there exists a holomorphic function λ' such that $e^{\lambda'} = h$ on U.

To find a suitable neighbourhood U, we start by realising $\mathbb{C}^n$ as a simplicial complex with $Z(k)$ as a subcomplex. (The existence of such a simplicial complex is guaranteed by standard results on the topology of subanalytic varieties: see for example [**58**, Theorem 1].) Then we can find a basis of neighbourhoods of $Z(k)$ such that $Z(k)$ is a strong deformation retract of each basis set (this is a general fact about CW-complexes: see [**39**, Prop. A.5, p. 523]). Finally, we choose a basis set U small enough that h does not vanish on U.

($3 \Rightarrow 1$): First consider the situation of hypothesis (2): suppose $\lambda : Z(k) \to \mathbb{C}$ is a logarithm for h. We wish to find a suitable holomorphic function $\mu : \mathbb{C}^n \to \mathbb{C}$ which extends λ. Then we can define

$$f = \frac{h - e^{\mu}}{k} \text{ and } g = \frac{k}{e^{\mu}}.$$

In order for such f to be a well defined holomorphic function, we require that $h - e^{\mu}$ should vanish on $Z(k)$ to order at least the order of vanishing of k, in other words that the divisors should satisfy

$$(h - e^{\mu}) \geq (k).$$

By hypothesis, we in fact have a neighbourhood U of $Z(k)$ and a logarithm $\lambda' : U \to \mathbb{C}$ for h on U. Then the above condition is equivalent to requiring that μ agrees with λ' on $Z(k)$ up to the order of vanishing of k.

We write $\mathcal{O}$ for the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. Consider the exact sequence of sheaves

$$0 \to k\mathcal{O} \to \mathcal{O} \to \mathcal{O}/k\mathcal{O} \to 0.$$

The sheaf $k\mathcal{O}$ is coherent, so by Cartan's Theorem B, $H^1(\mathbb{C}^n, k\mathcal{O}) = 0$, and therefore the map $\mathcal{O}(\mathbb{C}^n) \to (\mathcal{O}/k\mathcal{O})(\mathbb{C}^n)$ is surjective. Noting that the stalk of $\mathcal{O}/k\mathcal{O}$ at any point outside $Z(k)$ is zero, we see that the function λ' represents an element of $(\mathcal{O}/k\mathcal{O})(\mathbb{C}^n)$. Then we can choose μ to be any preimage of that element. □

Remark 4.3. When $n = 1$ in the above lemma, the zero set of k is discrete, and so a logarithm of h always exists on $Z(k)$. Thus Lemma 4.1 is a special case of Lemma 4.2.

Remark 4.4. The only properties of $\mathbb{C}^n$ used in the above proof are that it is Stein and simply connected and that all meromorphic functions on $\mathbb{C}^n$ can be written as a quotient. Thus we can generalise the result: if X is a simply connected Stein manifold, $h, k \in \mathcal{O}(X)$ have no common zeros and $m = h/k$, then the three statements given in the lemma are equivalent.

Example 4.5. For positive integers ν, the functions $m_\nu : \mathbb{C}^2 \to \mathbb{P}^1$ given by

$$m_\nu(x, y) = \frac{x}{xy^\nu - 1}$$

cannot be written in the form $f + 1/g$. At present it is not known whether any of the spaces $\mathbb{C}^3 \setminus \Gamma_{m_\nu}$ for $\nu \geq 2$ are Oka. In the case $\nu = 1$, the Oka property for $\mathbb{C}^3 \setminus \Gamma_{m_1}$ follows from work of Ivarsson and Kutzschebauch [**43**, Lemmas 5.2 and 5.3]. Specifically, let p be the polynomial $p(x, y, z) = xyz - x - z$. Then Γ_{m_1} is the level set $p^{-1}(0)$, and the complement is the union of all the other level sets. The complement is isomorphic to the product of $\mathbb{C}^*$ with the level set $p^{-1}(1)$ via the map $\mathbb{C}^* \times p^{-1}(1) \to \mathbb{C}^3 \setminus \Gamma_{m_1}$ given by

$$(\lambda, x, y, z) \mapsto (\lambda x, \lambda^{-1} y, \lambda z).$$

Now $p^{-1}(1)$ is smooth, and by the results of Ivarsson and Kutzschebauch, its tangent bundle is spanned by finitely many complete holomorphic vector fields. This implies that the set is Oka (see for example [**19**, Example 5.5.13(B)]), so $\mathbb{C}^* \times p^{-1}(1)$ is Oka.

With these considerations in mind, we are ready to state the main result of this section.

Theorem 4.6. *Let X be a complex manifold, and let $m : X \to \mathbb{P}^1$ be a holomorphic map. Suppose m can be written in the form $m = f + 1/g$ for holomorphic functions f and g. Then $(X \times \mathbb{C}) \setminus \Gamma_m$ is Oka if and only if X is Oka.*

Remark 4.7. The existence of the decomposition $m = f + 1/g$ is a geometric condition that is of some independent interest: it is equivalent to the condition that the projection map from $(X \times \mathbb{C}) \setminus \Gamma_m$ onto the first factor has a holomorphic section. The projection map is an elliptic submersion in the sense defined in [**21**, page 24]. (It is easy to see that it is a *stratified* elliptic submersion, as defined in [**21**, page 25]. For a sketch of why it is an (unstratified) elliptic submersion, see Remark 4.12 below.) However, unless either m has no poles or $m = \infty$, the projection is not an Oka map, because it is not a topological fibration.

In the case where X is Stein, it follows from [**21**, Theorem 5.4 (iii)] that the existence of a continuous section of the elliptic submersion implies the existence of a holomorphic section. For general X, one might expect that this ellipticity property could be applied to yield a simpler proof than the one presented below. So far, such a proof has been elusive.

We will first prove Theorem 4.6 for the special case $X = \mathbb{C}^n$, and then show how the convex interpolation property (Definition 2.4) for general X reduces to the special case. The proof for $X = \mathbb{C}^n$ involves a variation of Gromov's technique of localisation of algebraic subellipticity (see [**35**, Lemma 3.5B] and [**19**, Proposition 6.4.2]). This relies on the following lemma.

Lemma 4.8. *Let $g : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, not identically zero, and suppose $x_0 \in \mathbb{C}^n$ satisfies $g(x_0) = 0$. Then for all $s \in \mathbb{C}^n$,*

$$\lim_{\substack{x \to x_0 \\ g(x) \neq 0}} \left(\frac{1}{g(x)} - \frac{1}{g(x + g(x)^2 s)} \right) = g'(x_0)(s).$$

Proof. First, in order for the limit to make sense, we need to verify that x_0 has a neighbourhood on which $g(x+g(x)^2 s)$ vanishes only when $g(x)$ vanishes. We use the approximation

$$g(x + h) = g(x) + g'(x)(h) + O(|h|^2). \tag{4.1}$$

With $h = g(x)^2 s$, this gives

$$g(x + g(x)^2 s) = g(x) + g'(x)(g(x)^2 s) + O(|g(x)^4|).$$

When $x \neq x_0$ and $g(x) \neq 0$, if x is close to x_0, then the second and third terms of the right hand side are much smaller than the first, so $g(x + g(x)^2 s) \neq 0$, as required.

Now, using (4.1) again, we obtain

$$\begin{aligned} \frac{1}{g(x)} - \frac{1}{g(x+h)} &= \frac{g(x+h) - g(x)}{g(x)g(x+h)} \\ &= \frac{g'(x)(h) + O(|h|^2)}{g(x)(g(x) + g'(x)(h) + O(|h|^2))}. \end{aligned}$$

(In the event that $g(x+h)$ vanishes, we interpret the fractions as meromorphic functions.) Replacing h with $g(x)^2 s$, and using the fact that $g'(x)$ is a linear map, gives

$$\begin{aligned} \frac{1}{g(x)} - \frac{1}{g(x + g(x)^2 s)} &= \frac{g'(x)(g(x)^2 s) + O(|g(x)|^4)}{g(x)(g(x) + g'(x)(g(x)^2 s) + O(|g(x)|^4))} \\ &= \frac{g'(x)(s) + O(|g(x)|^2)}{1 + g(x)g'(x)(s) + O(|g(x)|^3)}. \end{aligned}$$

As $x \to x_0$ this expression tends to $g'(x_0)(s)$. $\square$

Remark 4.9. The exponent 2 in the lemma corresponds to a doubly twisted line bundle in the proof of Proposition 4.10 below. A single twist would not be sufficient: for example, if we take $n = 1$ and $g(x) = x$ then for $s \neq 0$ the expression $\dfrac{1}{g(x)} - \dfrac{1}{g(x + g(x)s)}$ does not have a finite limit as $x \to 0$.

Proposition 4.10. *Let $m : \mathbb{C}^n \to \mathbb{P}^1$ be a holomorphic map, and suppose m can be written in the form $m = f + 1/g$ for holomorphic functions f and g. Then $\mathbb{C}^{n+1} \setminus \Gamma_m$ is Oka.*

Proof. If $g = 0$, so that $m = \infty$, then $\Gamma_m = \varnothing$, so $\mathbb{C}^{n+1} \setminus \Gamma_m = \mathbb{C}^{n+1}$ is Oka. For the rest of the proof, assume that $g \neq 0$.

We will write points of $\mathbb{C}^{n+1}$ as (x, y) or (s, t) where $x, s \in \mathbb{C}^n$ and $y, t \in \mathbb{C}$. Let X denote the complement of the graph of $1/g$; i.e.

$$X = \{(x, y) \in \mathbb{C}^{n+1} : g(x)y \neq 1\}.$$

The map $X \to \mathbb{C}^{n+1} \setminus \Gamma_m$ given by $(x, y) \mapsto (x, y + f(x))$ is a biholomorphism. Hence it suffices to prove that X is Oka.

We begin by describing a covering space Y of X. Then we shall exhibit sprays on trivial bundles over certain subsets of the covering space. Finally, these sprays will be extended to sprays on twisted bundles over Y, using the above lemma. (This is the localisation step referred to above.) This will be sufficient to establish that Y is weakly subelliptic, hence Oka. Therefore X is Oka.

The covering space Y is constructed as follows. Define an equivalence relation $\sim$ on $\mathbb{C}^{n+1} \times \mathbb{Z}$ by

$$(x, y, k) \sim (x', y', k') \text{ if } x = x',\ g(x) \neq 0 \text{ and } \quad g(x)(y - y') = (k - k')2\pi i. \tag{4.2}$$

Then Y is the quotient space $(\mathbb{C}^{n+1} \times \mathbb{Z})/\sim$. From now on we will write $[x, y, k]$ as shorthand for the equivalence class in Y of (x, y, k), and Y_k for the kth "layer" $\{[x, y, k] : (x, y) \in \mathbb{C}^{n+1}\}$.

Note that Y can be described in concrete terms as a hypersurface in $\mathbb{C}^{n+2}$: see Remark 4.11. The description of Y used here is chosen to emphasise the simple form of the sprays σ_k described below.

It is straightforward to verify that Y is a Hausdorff space. We can map each Y_k bijectively to $\mathbb{C}^{n+1}$ by sending $[x, y, k]$ to (x, y). Thus Y has the structure of an $(n+1)$-dimensional complex manifold.

By way of motivation for this construction, observe that if $x_0 \in \mathbb{C}^n$ with $g(x_0) \neq 0$, then the set $\{[x, y, k] \in Y : x = x_0\}$ is a copy of $\mathbb{C}$, whereas if $g(x_0) = 0$, then $\{[x, y, k] \in Y : x = x_0\}$ is a countable union of disjoint copies of $\mathbb{C}$. The covering map described below looks like an exponential map when $g \neq 0$, but the identity map when $g = 0$. The construction involves a holomorphically varying family of holomorphic maps which include both exponentials and the identity.

We follow Buzzard and Lu [**5**, page 645] in defining a function ϕ on $\mathbb{C}^2$ by

$$\phi(x,y) = \begin{cases} \dfrac{e^{xy}-1}{x} & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases} \tag{4.3}$$
$$= y + \frac{xy^2}{2} + \frac{x^2y^3}{3!} + \cdots .$$

From the series expansion we see that ϕ is holomorphic. Then we define $\pi : Y \to X$ by

$$\pi[x,y,k] = (x, -\phi(g(x), y)).$$

If (x,y) is a point of X with $g(x) = 0$, then the fibre over (x,y) is the set

$$\pi^{-1}(x,y) = \{[x,-y,k] : k \in \mathbb{Z}\}.$$

If $g(x) \neq 0$, then a set of unique representatives for $\pi^{-1}(x,y)$ is given by

$$\{[x, \frac{\log(1-g(x)y)}{g(x)}, 0]\}$$

for all possible branches of the logarithm. It follows that all fibres of π are isomorphic to $\mathbb{Z}$. It can be verified that every point of X has a neighbourhood that is evenly covered by π, and so π is a covering map.

For each layer Y_k of Y there is a dominating spray σ_k on the trivial bundle $Y_k \times \mathbb{C}^{n+1}$, given by

$$\sigma_k([x,y,k]; s,t) = [x+s, y+t, k].$$

We wish to construct a bundle E_k over Y and a spray $\tilde{\sigma}_k : E_k \to Y$ such that $\tilde{\sigma}_k$ agrees with σ_k with respect to a trivialisation of $E_k|_{Y_k}$. Since every compact subset of Y is covered by finitely many Y_k, this will establish that Y is weakly subelliptic (Definition 2.3).

To simplify the notation, we will only describe E_0 and $\tilde{\sigma}_0$; the construction for $k \neq 0$ is similar. Define open subsets U_1 and U_2 of Y by

$$U_1 = \{[x,y,k] : k \neq 0\},$$
$$U_2 = \{[x,y,k] : k = 0\} = Y_0.$$

As each $[x,y,k]$ is an equivalence class, these sets are not in fact disjoint. (This is the only part of the proof where the assumption $g \neq 0$ is required.) The intersection $U_1 \cap U_2$ is the set of points $[x,y,0]$ with $g(x) \neq 0$. The bundle E_0 is described by local trivialisations $E_0|_{U_\alpha} \to U_\alpha \times \mathbb{C}^{n+1}$, $\alpha = 1,2$, with transition map $\theta_{12} : (U_1 \cap U_2) \times \mathbb{C}^{n+1} \to (U_1 \cap U_2) \times \mathbb{C}^{n+1}$ given by

$$\theta_{12}([x,y,0]; s,t) = ([x,y,0]; g(x)^2 s, t). \tag{4.4}$$

Define $\tilde{\sigma}_0$ by

$$\tilde{\sigma}_0|_{U_1}([x,y,k];s,t) = \begin{cases} [x+g(x)^2 s, y - k2\pi i/g(x) + t, 0] & \text{if } g(x) \neq 0, \\ [x, y - k2\pi i g'(x)(s) + t, k] & \text{if } g(x) = 0, \end{cases}$$

$$\tilde{\sigma}_0|_{U_2}([x,y,0];s,t) = \sigma_0([x,y,0];s,t) = [x+s, y+t, 0].$$

The fact that $\tilde{\sigma}_0|_{U_1}$ is continuous follows from Lemma 4.8 together with equation (4.2). It is easy to verify from equations (4.2) and (4.4) that $\tilde{\sigma}_0|_{U_1}$ and $\tilde{\sigma}_0|_{U_2}$ agree on $U_1 \cap U_2$. Thus $\tilde{\sigma}_0$ is a well defined holomorphic map from E_0 to Y extending σ_0. Finally, $\tilde{\sigma}_0([x,y,k];0,0) = [x,y,k]$, so $\tilde{\sigma}_0$ is a spray. This completes the proof. □

Remark 4.11. The covering space Y from the above proof can be embedded into $\mathbb{C}^{n+2}$ by the map $[x,y,k] \mapsto (x, -\phi(g(x),y), g(x)y + 2\pi i k)$. The image of this map is the set

$$Z = \{(x,y,z) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : 1 - g(x)y = e^z\}$$

and a covering map $Z \to X$ is given by $(x,y,z) \mapsto (x,y)$.

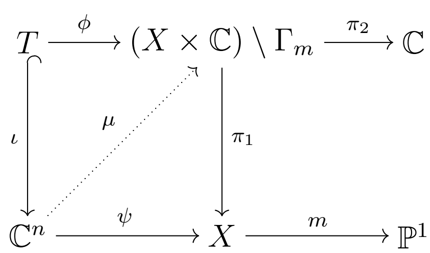
Proof of Theorem 4.6. We will write π_1 and π_2 for the projections of the complement $(X \times \mathbb{C}) \setminus \Gamma_m$ onto X and $\mathbb{C}$ respectively. The map $\sigma : X \to (X \times \mathbb{C}) \setminus \Gamma_m$ given by $\sigma(x) = (x, f(x))$ is a holomorphic section of π_1.

First suppose $(X \times \mathbb{C}) \setminus \Gamma_m$ is Oka. The convex interpolation property for X can easily be verified as follows. Let $\phi : T \to X$ be a holomorphic map from a contractible subvariety T of some $\mathbb{C}^n$. Then by the CIP of $(X \times \mathbb{C}) \setminus \Gamma_m$, the composite map $\sigma \circ \phi : T \to (X \times \mathbb{C}) \setminus \Gamma_m$ has a holomorphic extension $\psi : \mathbb{C}^n \to (X \times \mathbb{C}) \setminus \Gamma_m$. The composition $\pi_1 \circ \psi$ is a map $\mathbb{C}^n \to X$ extending ϕ. Therefore X is Oka.

Conversely, suppose X is an Oka manifold, and $m : X \to \mathbb{P}^1$ a holomorphic map with $m = f + 1/g$ as in the statement of the theorem. We will verify the CIP for $(X \times \mathbb{C}) \setminus \Gamma_m$.

Suppose T is a contractible subvariety of $\mathbb{C}^n$ for some n, and let $\phi : T \to (X \times \mathbb{C}) \setminus \Gamma_m$ be a holomorphic map. We want to find a holomorphic map $\mu : \mathbb{C}^n \to (X \times \mathbb{C}) \setminus \Gamma_m$ which extends ϕ.

First of all we can use the CIP for X to extend the composite map $\pi_1 \circ \phi$ to a holomorphic map $\psi : \mathbb{C}^n \to X$. This is indicated in the following diagram.



Now we have a holomorphic map $m \circ \psi : \mathbb{C}^n \to \mathbb{P}^1$. In fact $m \circ \psi = f \circ \psi + 1/(g \circ \psi)$, so we know by Proposition 4.10 that $\mathbb{C}^{n+1} \setminus \Gamma_{m \circ \psi}$ is

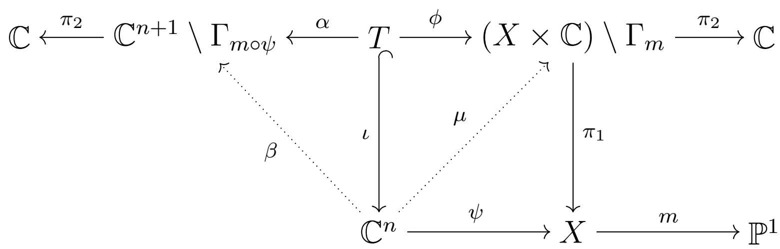
Oka. We want to map T into $\mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi}$, then use the CIP to extend this map.

Define $\alpha : T \to \mathbb{C}^{n+1}$ by

$$\alpha(x) = (\iota(x), \pi_2(\phi(x))).$$

Since $\phi(x)$ is an element of $(X \times \mathbb{C}) \setminus \Gamma_m$, it follows that $\pi_2(\phi(x))$ is never equal to $m(\pi_1(\phi(x)))$. By the definition of ψ, this means that $\pi_2(\phi(x)) \neq m(\psi(\iota(x)))$ for all $x \in T$. Therefore the image of α is contained in $\mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi}$.

The CIP for $\mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi}$ tells us that α extends to a map $\beta : \mathbb{C}^n \to \mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi}$, as in the following diagram. (The map $\pi_2 : \mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi} \to \mathbb{C}$ is the restriction to $\mathbb{C}^{n+1} \setminus \Gamma_{m\circ\psi}$ of the projection of $\mathbb{C}^n \times \mathbb{C}$ onto the last coordinate. The use of π_2 for two different projection maps should not cause any confusion, as the domain is always clear from the context.)



Finally, we can define $\mu : \mathbb{C}^n \to (X \times \mathbb{C}) \setminus \Gamma_m$ by

$$\mu(x) = (\psi(x), \pi_2(\beta(x))).$$

Since $\pi_2(\beta(x))$ can never equal $m(\psi(x))$, we see that the image of μ is indeed contained in $(X \times \mathbb{C}) \setminus \Gamma_m$. And from the definitions, the fact that β is an extension of α implies that μ is an extension of ϕ. □

Remark 4.12. As mentioned in Remark 4.7 above, the projection map from $(X \times \mathbb{C}) \setminus \Gamma_m$ onto the first factor is an elliptic submersion, in the case where m can be written as $f + 1/g$. To prove this, it is necessary to construct a dominating fibre spray ([**21**, page 24]). This can be done using the function ϕ defined by equation (4.3) in the proof of Proposition 4.10 above. As previously, there is no loss of generality in assuming $f = 0$. Denoting points of $X \times \mathbb{C}$ by (x, y) with $x \in X$ and $y \in \mathbb{C}$, define a map $s : ((X \times \mathbb{C}) \setminus \Gamma_m) \times \mathbb{C} \to (X \times \mathbb{C}) \setminus \Gamma_m$ by

$$s(x, y, t) = (x, ye^{tg(x)} - \phi(g(x), t)).$$

The verification that the image of s is indeed contained in $(X \times \mathbb{C}) \setminus \Gamma_m$, and that s is a dominating fibre spray, is routine.

CHAPTER 3

Holomorphic flexibility properties of the space of cubic rational maps

Abstract. For each natural number d, the space R_d of rational maps of degree d on the Riemann sphere has the structure of a complex manifold. The topology of these manifolds has been extensively studied. The recent development of Oka theory raises some new and interesting questions about their complex structure. We apply geometric invariant theory to the cases of degree 2 and 3, studying a double action of the Möbius group on R_d. The action on R_2 is transitive, implying that R_2 is an Oka manifold. The action on R_3 has $\mathbb{C}$ as a categorical quotient; we give an explicit formula for the quotient map and describe its structure in some detail. We also show that R_3 enjoys the holomorphic flexibility properties of strong dominability and $\mathbb{C}$-connectedness.

1. Introduction and statement of results

The space of rational maps on the Riemann sphere can be given the structure of a complex manifold. The topology of this manifold (the compact-open topology) has been studied extensively, beginning with the work of Segal [**70**]. In this paper we study rational maps from a geometric point of view, motivated by the recent development of Oka theory. In particular, we are interested in the holomorphic flexibility properties of dominability and $\mathbb{C}$-connectedness (defined below), which can be viewed as opposite to Kobayashi hyperbolicity.

We write R_d for the set of rational maps of degree d. Each such map can be written as a quotient of two relatively prime polynomials whose maximum degree is d. The space $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ of all rational maps is the union of R_d for $d = 0, 1, 2, \ldots$; the R_d are exactly the connected components of this space. Section 2 describes the complex structure on R_d and gives a brief overview of the relevant concepts from Oka theory.

Basic question: *Is $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ an Oka manifold?*

The question can be approached one degree at a time: is each component R_d an Oka manifold? We apply geometric invariant theory, using the results of Snow [**75**]. In particular, the Möbius group $\mathrm{PSL}_2(\mathbb{C})$ acts on R_d in two ways, by precomposition and postcomposition (see for example Ono and Yamaguchi [**66**]). We combine these two actions into a two-sided action of $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$, described in Section 3.

For low degree, we have $R_0 = \mathbb{P}^1$ and $R_1 = \mathrm{PSL}_2(\mathbb{C})$, both of which are known examples of Oka manifolds. For $d = 2$, the two-sided group action is transitive:

Theorem 1.1 ([**37**, Proposition 2.1])**.** *The space R_2 of rational maps of degree 2 is a complex homogeneous manifold.*

At the end of Section 3 we give an alternative proof of this result, as an introduction to the methods used later in this paper. Complex homogeneous manifolds are always Oka (see for example [**19**, Proposition 5.5.1]), so we have the following consequence:

Corollary 1.2. *R_2 is an Oka manifold.*

For $d \geq 3$ it is presently unknown whether R_d is an Oka manifold. The ideal situation would be to express R_d as a holomorphic fibre bundle whose base and fibre are Oka. This would be sufficient to show that R_d is Oka (see for example [**19**, Theorem 5.5.4]). The quotient map of a group action would be a natural candidate for such a bundle.

The nearest we can get at present is to exhibit a group action whose *categorical* (rather than geometric) quotient is Oka, and to prove the following two weaker properties for R_3.

Definition 1.3. Let X be a complex manifold, $p \in X$ and $\phi\colon \mathbb{C}^n \to X$ a holomorphic map with $\phi(0) = p$. (The number n is not necessarily equal to the dimension of X.) We say that ϕ *dominates* X *at* p if $d\phi_0$ is surjective. If such a ϕ exists, then X is *dominable at* p, and ϕ is a *dominating map.* If X is dominable at every $p \in X$, then X is *strongly dominable.*

Definition 1.4. A manifold X is *strongly* $\mathbb{C}$*-connected* if every pair of points can be joined by an entire curve; that is, for every pair of points of X there is a holomorphic map $\mathbb{C} \to X$ whose image contains both points.

Remark 1.5. Every Oka manifold is strongly $\mathbb{C}$-connected: this follows from the basic Oka property described in [**21**, page 16]. The definition of "$\mathbb{C}$-connected" is not standardised: Gromov in [**35**, 3.4(B)] uses the term to refer to strong $\mathbb{C}$-connectedness as described here, while other authors use it to refer to the weaker property that every pair of points can be joined by a finite chain of entire curves.

The three main results of this paper are Theorems 1.6 through 1.8 below.

Theorem 1.6. *The categorical quotient for the action of* $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ *on* R_3 *by pre- and postcomposition is* $\mathbb{C}$.

In fact we can give an explicit formula for the quotient map, as well as a detailed description of the orbits of the group action.

Overview of proof. The quotient map $\pi\colon R_3 \to \mathbb{C}$ is constructed in Section 4. Given $f \in R_3$, the value of $\pi(f)$ is expressed as a rational function of cross-ratios of critical points and critical values of f. Section 4.1 introduces the role of symmetric polynomials in describing this function; Section 4.3 describes a standard form for elements of R_3 as an aid to computation, and Section 4.4 gives an explicit description of π on an open subset of R_3. The extension of π to the rest of R_3 is given in Section 4.5.

To show that π is the desired quotient map, we need to describe the orbits of the group action and determine which orbits are closed. This is the content of Section 4.2. Then Lemma 4.4.7 tells us that π distinguishes the closed orbits, Lemma 4.4.8 and Section 4.5 tell us that that the image of π is all of $\mathbb{C}$, and Corollary 4.5.2 tells us that π is holomorphic. Remark 4.5.3 explains why these properties imply that π is the quotient map. □

We can say a little more about the group action: Section 4.6 describes the stabilisers of the orbits. Also, it is interesting to notice that all the constructions of Section 4 can be carried out within the algebraic category, whereas the proof of the next two theorems involves exponential maps.

Theorem 1.7. *The space R_3 of rational maps of degree 3 is strongly dominable.*

Overview of proof. Proposition 5.1.1 describes a method of constructing dominating maps, and the rest of Section 5.1 gives explicit maps from $\mathbb{C}^8$ to R_3. The building blocks of this construction are a map η_0 from a subset of $\mathbb{C}^2$ to R_3 that is transverse to the orbits of the group action, a dominating map from $\mathbb{C}^6$ to the group, and an embedding of R_3 into $\mathbb{P}^7$. We also need the fact that the domain of η_0 is dominable: this is Proposition 5.2.8.

Section 5.2 shows that the image of η_0 intersects every orbit of the group action. This fact allows us to use translates of η_0 to obtain dominating maps at each point of R_3.

Section 5.3 contains the proof that η_0 is transverse to the orbits. □

Theorem 1.8. *The space R_3 is strongly $\mathbb{C}$-connected.*

Proof. The dominating maps of the previous theorem are in fact surjective: this follows from Proposition 5.2.6. Thus there exists a surjective map $\mathbb{C}^8 \to R_3$. Given f and g in R_3, we can choose any preimages of f and g, join the preimages by an entire curve in $\mathbb{C}^8$, and compose the entire curve with the surjective map to obtain an entire curve in R_3 joining f and g. □

I thank Finnur Lárusson for many helpful discussions during the preparation of this paper.

2. Context: rational maps and the parametric Oka property

In Section 2.1 we describe the complex structure for the spaces R_d of rational maps of degree d. It is most convenient to use the coefficients of rational maps as coordinates. In fact the complex structure thus obtained has an important universal property, telling us that it is the right complex structure for our purposes.

In Section 2.2 we introduce some relevant concepts from Oka theory. These serve as motivation for the questions addressed in this paper; however, the definition of an Oka manifold is not directly used. Consequently, only a brief sketch is given here. The interested reader can refer to the survey paper [**21**] of Forstnerič and Lárusson for definitions and examples, or to the book [**19**] of Forstnerič for a more detailed exposition.

2.1. The space of rational maps.

For each $d = 0, 1, 2 \ldots$, we can embed R_d (as a set) into $\mathbb{P}^{2d+1}$ by sending a rational function

$$\frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0}$$

to the point with homogeneous coordinates

$$(a_d\colon a_{d-1} \cdots : a_0 \colon b_d \colon b_{d-1} \colon \cdots : b_0).$$

We introduce a complex structure on R_d as the pullback of the complex structure on $\mathbb{P}^{2d+1}$. The image of R_d under this embedding is an open subset of $\mathbb{P}^{2d+1}$. Specifically, the condition for a rational function p/q to belong to R_d, where p and q are polynomials of maximum degree d, is that p and q should have no common factors. This is equivalent to the non-vanishing of the resultant of p and q, and so the image of R_d is the complement of the resultant locus in $\mathbb{P}^{2d+1}$.

The topology induced on R_d by this complex structure coincides with the compact-open topology on $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1) = \bigcup_{d=0}^{\infty} R_d$. In this topology, each R_d is connected, and the map $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1) \to \mathbb{Z}$ sending a rational function to its degree is continuous. Therefore the connected components of $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ are precisely the R_d.

Proposition 2.1.1. *With the complex structure described above, the space $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ is an internal hom-object in the category of reduced complex spaces and holomorphic maps.*

Proof. We wish to show that $\mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ is a representing object for the functor $\mathcal{O}(- \times \mathbb{P}^1, \mathbb{P}^1)$.

Given a reduced complex space T and a holomorphic map $\phi\colon T \times \mathbb{P}^1 \to \mathbb{P}^1$, define $\tilde{\phi}\colon T \to \mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ by $\tilde{\phi}(t)(x) = \phi(t, x)$. For each t, the map $\tilde{\phi}(t)$ is a member of some R_d; by continuity of the degree map, the degree d must be constant on each connected component of T. Thus we can use the embedding $R_d \to \mathbb{P}^{2d+1}$ to write $\tilde{\phi}$ in local coordinates. From this, it is easy to see that $\tilde{\phi}$ is holomorphic.

Let $\eta_T \colon \mathcal{O}(T \times \mathbb{P}^1, \mathbb{P}^1) \to \mathcal{O}(T, \mathcal{O}(\mathbb{P}^1, \mathbb{P}^1))$ be the map sending ϕ to $\tilde{\phi}$. Then η is natural transformation from the functor $\mathcal{O}(- \times X, Y)$ to the functor $\mathcal{O}(-, \mathcal{O}(X, Y))$.

If $\psi \colon T \to \mathcal{O}(\mathbb{P}^1, \mathbb{P}^1)$ is holomorphic, then $\psi = \tilde{\phi}$ where $\phi(t, x) = \psi(t)(x)$, and it is immediate that ϕ is holomorphic. Therefore η_T is a bijection, and η is a natural isomorphism. □

This result is implicit in the work of Kaup [**47**], but is not stated explicitly there. See also Douady [**11**, Section 10.2] regarding universal properties of mapping spaces.

2.2. A brief outline of Oka theory. Informally speaking, Oka manifolds can be viewed as the opposite of Kobayashi hyperbolic manifolds. Hyperbolicity is a type of holomorphic rigidity property; conversely, Oka manifolds enjoy a variety of holomorphic flexibility properties.

The most concrete expression of flexibility for a manifold X is that there should be "many" holomorphic maps $\mathbb{C} \to X$. This is formalised in Gromov's notion of ellipticity, introduced in [**35**]. Dominability and $\mathbb{C}$-connectedness (Definitions 1.3 and 1.4) express weaker versions of the same idea. There is a chain of implications

$$\text{elliptic} \Rightarrow \text{Oka} \Rightarrow \text{strongly dominable and strongly } \mathbb{C}\text{-connected};$$

at present, it is unknown whether the reverse implications hold in general. (Campana and Winkelmann [**6**, Example 8.3] give an example of a manifold that is $\mathbb{C}$-connected but not Oka. However, there are no known examples of manifolds that are strongly dominable but not Oka.)[1]

Oka manifolds also enjoy a number of homotopy properties. The simplest is the so-called basic Oka property (BOP): every continuous map from a Stein manifold to an Oka manifold is homotopic to a holomorphic map. Oka manifolds satisfy a stronger version of the BOP with added approximation and interpolation conditions (see [**21**, page 16] for details).

Forstnerič and Lárusson have identified a number of equivalent properties which characterise Oka manifolds. The following is of particular interest in the context of mapping spaces.

Definition 2.2.1 (Parametric Oka property (POP), simple version)**.** A manifold X satisfies the *parametric Oka property* if for every Stein manifold S and every compact subset $P \subset \mathbb{R}^m$, every continuous map

[1] Note added in proof: the recent preprint of Andrist and Wold, 'The complement of the closed unit ball in $\mathbb{C}^3$ is not subelliptic', `arXiv:1303.1804`, describes a manifold that is strongly dominable but not subelliptic.

$f\colon P \times S \to X$ is homotopic to a map $f_1\colon P \times S \to X$ such that $f_1(\cdot, x)\colon S \to X$ is holomorphic for every $x \in P$.

In other words, a family of continuous maps can be deformed to a family of holomorphic maps with continuous dependence on the parameter.

This is apparently a stronger condition than the BOP. However, it turns out that the POP with approximation and interpolation is equivalent to the BOP with approximation and interpolation; either condition can be taken as the definition of an Oka manifold (see [**17**, Section 1]).

It is natural to ask whether continuous dependence on a parameter can be replaced by holomorphic dependence. To put it another way, if P is a compact complex manifold, then is the mapping space $\mathcal{O}(P, X)$ an Oka manifold or similar? (The results of [**11**] guarantee that if P is compact, then $\mathcal{O}(P, X)$ carries a universal complex structure.) In this paper we begin with the simplest interesting case, that of $P = X = \mathbb{P}^1$.

3. Group actions and the degree 2 case

The action of the Möbius group $\mathrm{PSL}_2(\mathbb{C})$ on the Riemann sphere $\mathbb{P}^1$ is sharply 3-transitive: given any two triples of distinct points of $\mathbb{P}^1$, there is a unique Möbius transformation taking the first triple to the second. Because of this transitivity, the Möbius group is a valuable tool for simplifying the study of R_d when d is small. Specifically, rational maps can be composed with Möbius transformations, giving rise to a number of interesting group actions on R_d. Since $\mathrm{PSL}_2(\mathbb{C})$ is reductive (being the complexification of the compact subgroup PSU_2), we can study its actions using geometric invariant theory as described in [**75**] (which gives analytic analogues of the results of [**59**]).

First there are the pre- and postcomposition actions. Precomposition is the action $R_d \times \mathrm{PSL}_2(\mathbb{C}) \to R_d$ defined by $f \cdot g = f \circ g$ for $f \in R_d$ and $g \in \mathrm{PSL}_2(\mathbb{C})$. Postcomposition is the action $\mathrm{PSL}_2(\mathbb{C}) \times R_d \to R_d$ defined by $g \cdot f = g \circ f$. An interesting asymmetry appears here. It is easy to verify that postcomposition is a free action. Therefore there is a well defined geometric quotient: the set of orbits has the structure of a complex manifold, and the quotient map is a fibre bundle [**75**, Corollary 5.5]. This quotient space is a useful tool in studying the topology of R_d: see for example [**40**] and [**66**]. On the other hand, the precomposition action is not free. (For example, consider $f(x) = x^d$ and let g be multiplication by a dth root of unity.) This group action has received much less attention in the literature.

The two group actions can be combined to give the conjugation action: $f \cdot g = g^{-1} \circ f \circ g$. This action is of interest in the study of holomorphic dynamics: see for example [**62**, Section 3]. In particular, the quotient space of R_2 under this action is $\mathbb{C}^2$ (see also [**72**, Section 5]). For $d > 2$, the quotient of R_d is a rational variety [**57**, Section 4]. However, the behaviour of holomorphic flexibility properties under birational maps is not well understood, so it is not obvious how to apply this group action to our present investigations.

The actions can also be considered jointly via the two-sided action of $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on R_d: if $g = (g_1, g_2) \in G$ and $f \in R_d$, then define f^g by

$$f^g = g_1^{-1} \circ f \circ g_2.$$

The two-sided action, like the precomposition action, is not free. Furthermore, for $d \geq 3$ there exist non-closed orbits (see Section 4.2 below), so the orbit space is not Hausdorff; the geometric quotient does not exist as a manifold. However, Snow's main theorem tells us that the categorical quotient for this action exists as a reduced complex space. Since R_3 is 7-dimensional and G is 6-dimensional, we expect to find a 1-dimensional quotient space. Study of this quotient reveals a great deal about the structure of R_3. This will be explored further in Section 4 below.

For the case $d = 2$, the action is transitive. This can be proved by elementary means, as in [**37**]. An alternative and more intuitive proof can be obtained by considering rational maps in terms of critical values.

A rational map of degree d can be viewed as a branched d-sheeted covering map $\mathbb{P}^1 \to \mathbb{P}^1$. By the Riemann–Hurwitz formula, there are $2d-2$ critical values when counted with multiplicity. Each critical value has multiplicity at most $d - 1$, so there must be at least two distinct critical values.

The Riemann existence theorem (see for example [**10**, page 49]) plays a key role in understanding the orbits. We need only the following special case.

Theorem 3.1 (Riemann existence theorem, special case)**.** *Let Δ be a finite subset of $\mathbb{P}^1$, and $\phi\colon \pi_1(\mathbb{P}^1 \setminus \Delta) \to \mathfrak{S}_d$ a group homomorphism (where $\mathfrak{S}_d$ denotes the symmetric group on d symbols). Suppose the image of ϕ is transitive. Then there exists a compact connected Riemann surface X and a d-fold branched holomorphic covering map $f\colon X \to \mathbb{P}^1$ with critical values Δ and monodromy given by ϕ. If $f_1\colon X_1 \to \mathbb{P}^1$ and $f_2\colon X_2 \to \mathbb{P}^1$ are two such coverings, then there exists a biholomorphic map $g\colon X_1 \to X_2$ such that $g \circ f_1 = f_2$.*

The genus of the surface X is given by the Riemann–Hurwitz formula; the multiplicities of the critical values can be calculated from the cycle structure of the permutations as described in Chapter 1 of [**51**].

If the multiplicities sum to $2d-2$, then the resulting covering map is a rational function $\mathbb{P}^1 \to \mathbb{P}^1$, and the orbit of this rational function under the precomposition action described above is uniquely determined by the critical values and monodromy.

In the case of $d = 2$, there are exactly two distinct critical values. For a two-sheeted covering, there is only one possible monodromy permutation. Triple transitivity of the group implies that pairs of critical values are all equivalent under the postcomposition action. It follows that the two-sided action on R_2 has only one orbit. This proves Theorem 1.1.

4. Degree 3: the categorical quotient

In this section we study the two-sided group action of $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on R_3 from the point of view of geometric invariant theory. The group is reductive, and R_3, being the complement of a hypersurface in $\mathbb{P}^7$, is Stein. Therefore the main theorem of Snow's paper [**75**] tells us that the categorical quotient exists and is a reduced Stein space. We will show that the quotient is in fact $\mathbb{C}$, proving Theorem 1.6 above. The quotient map is explicitly described in Section 4.4, and the proof is completed in Section 4.5.

A rational function of degree 3 has four critical points and four critical values, counted with multiplicity. The multiplicity of each critical value is either one or two. Therefore there are only three possible cases (all of which occur):

- four distinct simple critical values;
- one double and two simple critical values;
- two double critical values.

The set of rational functions with four distinct critical values is an open subset of R_3, and will be referred to as the *open stratum*, denoted R_3^O. The open stratum is the set of $f \in R_3$ such that the zeros of f' are distinct; in other words, in local coordinates it is the complement of the zero locus of the discriminant of the numerator of f'. Therefore it is an open subset of R_3, in both the compact-open topology and the Zariski topology.

The complement of R_3^O will be called the *null fibre*, for reasons that will become clear later. Most of this section will be concerned with understanding the orbits in the open stratum. For each of the other two cases there is a single orbit; this is proved in Section 4.2 below.

Remark 4.1. Since R_3 is 7-dimensional and G is 6-dimensional, it is reasonable to expect that the generic orbit will have codimension 1. In fact, if $g = (g_1, g_2) \in G$ fixes $f \in R_3$, then g_1 must permute the critical values of f, and g_2 permutes the critical points. Since an element of the Möbius group is uniquely determined by the image of three points, it follows that if f has at least three critical values (and therefore at least three critical points), then the stabiliser of f is finite, so the orbit is 6-dimensional.

In the case where f has only two critical values, the stabiliser can contain a one-parameter subgroup. For example, $f = x^3$ is fixed by the group element (a^3x, ax) for all $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Hence the orbit of such f is at most 5-dimensional.

More details about the stabilisers are given in Section 4.6.

Remark 4.2. The null fibre, being locally the zero locus of a discriminant polynomial, is a proper analytic subvariety of R_3. Since it contains at least one 6-dimensional orbit (the non-closed orbit of Proposition 4.2.1), it has codimension 1.

4.1. Cross-ratio and symmetrised cross-ratio. Given four distinct points (a *quartet*) $z_1, z_2, z_3, z_4 \in \mathbb{C}$, their cross-ratio is the number

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

The definition is extended to quartets in $\mathbb{P}^1$ by adopting the convention that $\infty/\infty = 1$. For example, $(0, \infty; 1, \lambda) = (-1 \cdot \infty)/(\infty \cdot (-\lambda)) = \lambda$. For quartets of distinct points, the cross-ratio can take on any value except 0, 1 or ∞. The Möbius group $\mathrm{PSL}_2(\mathbb{C})$ preserves cross-ratio. Therefore we aim to use the cross-ratio to construct invariant functions for the action of $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on R_3. In particular, we are interested in the cross-ratios of the critical points and of the critical values of an element of the open stratum.

There is a technical issue that needs to be addressed: there is no canonical way of ordering the four critical points or values. Therefore the "cross-ratio of the critical points" is not well defined. We will address this by symmetrising the cross-ratio. (This is analogous to the relationship between the elliptic modular function λ and Klein's j-invariant given by $j(\tau) = 256(1-\lambda+\lambda^2)^2/\lambda^2(1-\lambda^2)$; the j-invariant is a symmetrised version of the modular function.)

Generically, the 24 possible orders of four points give rise to six cross-ratios. If one ratio is λ, then the six ratios are

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}. \tag{4.1.1}$$

For most values of λ these six numbers are distinct. If λ is one of -1, $\frac{1}{2}$ or 2, then there are only three distinct cross-ratios, namely $\{-1, \frac{1}{2}, 2\} = \{\lambda, 1/\lambda, 1-\lambda\}$. If λ is a primitive sixth root of unity, i.e. $\lambda = e^{\pm\pi i/3}$, then there are only two distinct cross-ratios, namely λ and $\bar{\lambda}$.

We will write $\sigma_1, \dots, \sigma_6$ for the elementary symmetric functions of six variables. Thus $\sigma_1(x_1, \dots, x_6) = x_1 + \dots + x_6$, $\sigma_2(x_1, \dots, x_6) = x_1x_2 + \dots + x_5x_6$ (fifteen terms), and so on up to $\sigma_6(x_1, \dots, x_6) = x_1x_2x_3x_4x_5x_6$.

For $k = 1, \dots, 6$, define functions $s_k \colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ by

$$s_k(\lambda) = \sigma_k\left(\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\right).$$

It follows that for a quartet (z_1, z_2, z_3, z_4) of distinct points of $\mathbb{P}^1$, the quantity $s_k((z_1, z_2; z_3, z_4))$ depends only on the set $\{z_1, z_2, z_3, z_4\}$. Thus we can regard these quantities as cross-ratios of an unordered set.

Routine calculations (easily verified using a computer algebra system: see Appendix C) show that s_k is constant when $k = 1$, 5 or 6: we have $s_1 = s_5 = 3$ and $s_6 = 1$. Also $s_4 = s_2$ and $s_3 = 2s_2 - 5$. We will only use s_2 in the sequel.

It is also worth noting that the function $s_2 + 3/4$ factorises nicely. Thus we define $s : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ by

$$s(\lambda) = s_2(\lambda) + \frac{3}{4} = -\frac{(\lambda+1)^2(2\lambda-1)^2(\lambda-2)^2}{4\lambda^2(\lambda-1)^2}, \tag{4.1.2}$$

and we say that the *symmetrised cross-ratio* of a set of four distinct points $\{z_1, z_2, z_3, z_4\}$ of $\mathbb{P}^1$ is the number $s((z_1, z_2; z_3, z_4))$.

Proposition 4.1.1. *The symmetrised cross-ratio is a complete invariant for the action of the Möbius group on unordered sets of four distinct points of* $\mathbb{P}^1$.

Proof. The (usual) cross-ratio is a complete invariant for the action on ordered sets of four points. Changing the order of the points transforms the cross-ratio as described above. Therefore we simply need to show that if $s(\mu) = s(\lambda)$, then μ is one of the six quantities listed above at (4.1.1). This follows from the following identity, easily verified by mechanical calculation:

$$\mu^2(\mu-1)^2(s_2(\lambda) - s_2(\mu)) = (\mu - \lambda)(\mu - \frac{1}{\lambda}) \cdots (\mu - \frac{\lambda - 1}{\lambda}). \quad \square$$

Using this symmetrised cross-ratio, we define a map $R_3^O \to \mathbb{C}$, also called s, as follows. For $f \in R_3^O$ with critical points z_1, z_2, z_3, z_4,

$$s(f) = s((z_1, z_2; z_3, z_4)). \tag{4.1.3}$$

The dependence of the critical points on f is continuous by Hurwitz's theorem, and so s is continuous.

4.2. Closed and non-closed orbits. We can think of the categorical quotient as parametrising the closed orbits of the group action. Therefore we need to determine which orbits are closed.

First we will deal with the null fibre, i.e. the set of functions in R_3 whose critical values are not distinct. There are two cases.

Suppose f has exactly three critical values: one double and two simple. By the transitivity of the postcomposition action, we see that f lies in the same orbit as a function g with ∞ as a double critical value, i.e. a polynomial. We can also assume that 0 is a simple critical value; and by transitivity of the precomposition action we can require $g(0) = 0$ and $g(\infty) = \infty$. Such g must be of the form $ax^3 + bx^2$ for some $a, b \in \mathbb{C}^*$. Conversely, every such function has exactly three critical values: ∞ is a double critical value of every cubic polynomial, and the finite critical values are the two roots 0 and $-2b/3a$ of g'.

If $g(x) = ax^3 + bx^2$, then

$$\frac{a^2}{b^3} g\left(\frac{b}{a}x\right) = \frac{a^2}{b^3}\left(a\frac{b^3}{a^3}x^3 + b\frac{b^2}{a^2}x^2\right) = x^3 + x^2,$$

so g, and therefore f, is in the same orbit as $x^3 + x^2$. Thus the set of functions with exactly three critical values is a single orbit.

It is easy to see that this orbit is not closed: the sequence $(x^3 + \frac{1}{n}x^2)_{n=1}^{\infty}$ converges to x^3, which is outside the orbit because it has only two critical values. This fact has a geometrical interpretation: travelling along the sequence, the two simple critical values of $(x^3 + \frac{1}{n}x^2)$ get closer and eventually coalesce.

The second case is that of functions with two double critical values. This is the smallest possible number of critical values for an element of R_3. Bearing in mind the above geometrical interpretation, it is immediate that having two critical values is a closed condition: it is not possible for the critical values to coalesce within R_3. Thus the set of such functions is closed. Similar arguments to those presented above show that all such functions lie in the same orbit as x^3.

To summarise: the null fibre consists of exactly two orbits, namely a non-closed orbit consisting of the functions with exactly three critical values, and a closed orbit consisting of the functions with exactly two critical values.

Recall (Remark 4.1) that orbits of functions with at least three critical values are 6-dimensional, and that any other orbits are of strictly smaller dimension. It follows that the closed orbit in the null fibre is the unique orbit in R_3 of minimal dimension. We will see in Theorem 4.6.3 that this orbit is in fact 5-dimensional.

Now we turn our attention to the open stratum R_3^O, i.e. the set of functions in R_3 with four distinct critical values. Here we use the map $s\colon R_3^O \to \mathbb{C}$ defined in the previous section. Since the group action preserves cross-ratio, each fibre of s is a union of orbits.

Suppose the orbit f^G of $f \in R_3^O$ is not closed. Since s is continuous, the closure of f^G is contained in $s^{-1}(s(f))$. Proposition 2.3 of [**75**] tells us that the closure of f^G contains an orbit of strictly smaller dimension, and therefore $s^{-1}(s(f))$ contains orbits of at least two different dimensions. But since all orbits in the open stratum are 6-dimensional, this is impossible.

The following proposition collects together the results obtained so far.

Proposition 4.2.1. *The orbits for the action of* $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ *on* R_3 *are of three types.*

- *The points with two distinct critical values form a single orbit, which is closed and* 5*-dimensional.*

- *The points with three distinct critical values form a single orbit, which is non-closed and* 6*-dimensional. The closure of this orbit is its union with the* 5*-dimensional orbit.*
- *The orbit of a function with four distinct critical values is closed and* 6*-dimensional.*

4.3. Standard form for cubic rational functions. The following will be useful as an aid to calculation.

Definition 4.3.1. Let $a \in \mathbb{C}$. Then f_a will denote the rational function

$$f_a(x) = \frac{x^2(x+a)}{(2a+3)x-(a+2)}.$$

Remark 4.3.2. The function f_a fixes the points 0, 1 and ∞. If a is -1 or -2, then f_a equals x^2 or $x(2-x)$ respectively. Otherwise $f_a \in R_3$, and 0, 1 and ∞ are critical points and critical values of f_a.

Lemma 4.3.3. *Suppose $f \in R_3^O$ has critical points* 0, 1, ∞ *and* μ *and critical values* 0, 1, ∞ *and* λ, *with f sending* 0, 1, ∞ *and* μ *to* 0, 1, ∞ *and* λ *respectively. Then there exists unique $a \in \mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$ such that $f = f_a$. The values of a, μ and λ are related by the equations*

$$\mu = -\frac{a(a+2)}{2a+3}, \quad \lambda = \frac{\mu^3}{(a+2)^2}, \quad a = \frac{\mu^3 + 3\mu\lambda - 4\lambda}{2\lambda(1-\mu)}. \qquad (4.3.1)$$

Conversely, $f_a \in R_3^O$ for all $a \in \mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$.

Remark 4.3.4. Given an arbitrary element of R_3^O, we can pre- and postcompose with Möbius transformations to send three of the critical points and values to $\{0, 1, \infty\}$. Thus every orbit in R_3^O contains at least one f_a.

Proof of lemma. For 0 to be a double zero of f but not a triple zero, the numerator of f must take the form $cx^2(x+a)$ for some $a, c \in \mathbb{C}^*$. Since $f(1) \neq 0$, we have the condition $a \neq -1$. For ∞ to map to ∞ with multiplicity exactly 2, the denominator must be a linear polynomial, say $x+b$ for some $b \in \mathbb{C}^* \setminus \{a\}$. (We can take the leading coefficient to be 1 because we have the coefficient c in the numerator. We require $b \notin \{0, a\}$ in order for f to have degree 3.) Thus f is of the form

$$f(x) = \frac{cx^2(x+a)}{x+b}.$$

The condition $f(1) = 1$ gives

$$\frac{c(1+a)}{1+b} = 1,$$

so $b = c(a+1) - 1$, and therefore

$$f(x) = \frac{cx^2(x+a)}{x+(a+1)c-1}.$$

The finite critical points are exactly the zeros of f'. These zeros must be 0, 1 and μ. By the quotient rule, the numerator of f' is

$$cx(3x + 2a)(x + (a + 1)c - 1) - cx^2(x + a). \tag{4.3.2}$$

Evaluating this at $x = 1$ gives

$$c^2(3 + 2a)(a + 1) - c(1 + a) = (a + 1)c((2a + 3)c - 1),$$

which vanishes when $a = -1$ or $c = 0$ or $c = 1/(2a + 3)$. The first two cases are impossible. Substituting the third value of c into the above expression for f, and dividing the numerator and denominator by c, gives

$$f(x) = \frac{x^2(x + a)}{(2a + 3)x + a + 1 - (2a + 3)} = \frac{x^2(x + a)}{(2a + 3)x - (a + 2)},$$

as required.

Now we wish to calculate the values of μ and λ. We have already ensured that 0, 1 and ∞ are critical points of f; the fourth critical point is μ. Substituting the value of c into (4.3.2) and dividing by the common factor cx gives

$$\begin{aligned}
&(3x + 2a)(x + (a + 1)c - 1) - x(x + a)\\
&= (3x + 2a)(x + \tfrac{a+1}{2a+3} - 1) - x(x + a)\\
&= \tfrac{1}{2a+3}((3x + 2a)((2a + 3)x + a + 1 - (2a + 3)) - (2a + 3)x(x + a))\\
&= \tfrac{1}{2a+3}((4a + 6)x^2 + (2a^2 - 6)x - 2a(a + 2))\\
&= \tfrac{2}{2a+3}((2a + 3)x^2 + (a^2 - 3)x - a(a + 2))\\
&= \tfrac{2}{2a+3}(x - 1)((2a + 3)x + a(a + 2)).
\end{aligned}$$

This vanishes at $x = 1$ (which we already know to be a critical point) and at $x = \mu = -a(a+2)/(2a+3)$. Then λ is given by $f(\mu)$; multiplying numerator and denominator by $(2a + 3)^3$ we obtain

$$\begin{aligned}
f(\mu) &= \frac{a^2(a + 2)^2(-a(a + 2) + a(2a + 3))}{-a(a + 2)(2a + 3)^2 - (a + 2)(2a + 3)^3}\\
&= \frac{a^2(a + 2)^2(a^2 + a)}{-(2a + 3)^2(a(a + 2) + (a + 2))}\\
&= \frac{-a^3(a + 2)^2(a + 1)}{(2a + 3)^2(a + 1)(a + 2)}\\
&= \frac{-a^3(a + 2)}{(2a + 3)^3}\\
&= \frac{\mu^3}{(a + 2)^2},
\end{aligned}$$

as required.

Conversely, a can be calculated from μ and λ as follows. The second equation of the lemma can be rearranged to give

$$(2a+3)\mu = -a(a+2),$$

and so

$$a^2 + 2(\mu+1)a + 3\mu = 0. \tag{4.3.3}$$

The third equation from Lemma 4.3.3 gives

$$a^2 + 4a + 4 = \mu^3/\lambda. \tag{4.3.4}$$

Subtracting (4.3.3) from (4.3.4):

$$2(1-\mu)a + 4 - 3\mu = \mu^3/\lambda,$$

which gives the required expression for a.

Next, we need to identify the "forbidden" values of a. These come from the constraints $\mu, \lambda \notin \{0, 1, \infty\}$. We have $\mu = 0$ exactly when $a = 0$ or $a = -2$, and $\mu = \infty$ when $a = -3/2$. The equation $\mu = 1$ gives $a(a+2)+2a+3 = 0$, which factorises as $a^2+4a+3 = (a+1)(a+3) = 0$, eliminating the values $a = -1, -3$. Looking at $\lambda = 0$ and $\lambda = \infty$ gives nothing new. The equation $\lambda = 1$ gives

$$\begin{aligned}
0 &= \lambda - 1 \\
&= \mu^3 - (a+2)^2 \qquad (\text{if } a \neq -2) \\
&= (2a+3)^3(\mu^3 - (a+2)^2) \qquad (\text{if } a \neq -3/2) \\
&= -a^3(a+2)^3 - (a+2)^2(2a+3)^3 \\
&= -(a+2)^2(a^3(a+2) + (2a+3)^3) \\
&= -(a+2)^2(a^4 + 2a^3 + 8a^3 + 36a^2 + 54a + 27) \\
&= -(a+2)(a+1)(a^3 + 9a^2 + 27a + 27) \\
&= -(a+1)(a+2)(a+3)^3,
\end{aligned}$$

so again no new forbidden values are obtained.

Finally, if a is not one of the forbidden values, then it is clear that we can form the function f_a of Definition 4.3.1, that it is an element of R_3, and that the corresponding values of μ and λ are not in $\{0, 1, \infty\}$, so that $f_a \in R_3^O$. Thus every value of a in $\mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$ can be realised. □

Remark 4.3.5. Given μ, equation (4.3.3) in general gives two possible values of a, and therefore two possible values of λ. The discriminant of (4.3.3) is

$$\Delta = (2(\mu+1))^2 - 12\mu = 4(\mu^2 - \mu + 1).$$

Therefore there is a unique value of a exactly when $\mu = e^{\pm\pi i/3}$; as mentioned in Section 4.1, these are the cross-ratio values for which different orderings of the critical points give only two distinct cross-ratios rather than the usual six.

4.4. Cross-ratio and invariant functions. Our goal is to find a complete set of invariants for the action of G. The function s of (4.1.3) is invariant, but we will see in Example 4.4.4 that s is not sufficient to distinguish the closed orbits.

In this section we define a new function π, described in (4.4.1) below, using the results of the previous section. We will see that this function is in fact the categorical quotient map. The definition parallels that of s: the quantity a of Lemma 4.3.3 plays the role of the cross-ratio, Lemma 4.4.3 plays the role of (4.1.1), and the elementary symmetric function σ_2 is again used.

Let $f \in R_3^O$. Choose an ordering σ of the critical values, and let λ be the cross-ratio of the critical values in that order. Each critical value has two preimages, one of which is a critical point, so there is an induced ordering of the critical points. Let μ be the cross-ratio of the critical points in this order.

Definition 4.4.1. The *signature* of f with respect to σ is the pair (μ, λ).

Lemma 4.4.2. *Two elements of R_3^O are in the same orbit if and only if there exist orderings for which they have the same signature.*

Proof. Let $f \in R_3^O$ and choose an ordering σ of its critical values. If we precompose f with a Möbius transformation, then the critical points move but the critical values are unchanged. Similarly, postcomposition will move the critical values but leave the critical points unchanged. Given σ, there is a unique Möbius transformation α_1 moving the first three critical points to 0, ∞ and 1 in order. Since cross-ratio is preserved, the fourth critical point will be moved to μ. Similarly, there is a unique Möbius transformation α_2 moving the critical values to 0, ∞, 1 and λ in order. Write $f^{(\sigma)}$ for the function $\alpha_2 \circ f \circ \alpha_1^{-1}$. Note that $f^{(\sigma)}$ has critical points 0, 1, ∞ and μ, critical values 0, 1, ∞ and λ, and fixes the points 0, 1 and ∞.

It follows that $f^{(\sigma)}$ is in fact the function f_a of Definition 4.3.1, for the value of a given by (4.3.1). Hence $f^{(\sigma)}$ is uniquely determined by the signature.

If f and g have the same signature with respect to orderings σ, ρ, then $f^{(\sigma)} = g^{(\rho)}$, and hence f and g are in the same orbit. Conversely, suppose f and g are in the same orbit, and choose an ordering σ for f.

Then there exist Möbius transformations β_1 and β_2 such that $\beta_2 \circ g \circ \beta_1^{-1} = f^{(\sigma)}$. Taking the critical values of g in the ordering ρ given by $\beta^{-1}(0)$, $\beta^{-1}(1)$, $\beta^{-1}(\infty)$, $\beta^{-1}(\mu)$, we see that $g^{(\rho)} = f^{(\sigma)}$. □

We would like to use the quantity a of Lemma 4.3.3 to parametrise the orbits. However, an orbit can contain more than one f_a. The situation is analogous to that of a quartet of points having more than one cross-ratio, and we resolve it in the same way, by symmetrising with respect to the set of values that can occur.

Lemma 4.4.3. *Let $a, b \in \mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$. Then f_a and f_b are in the same orbit if and only if b is in the set*

$$\left\{a,\ -\frac{2a+3}{a+2},\ -(a+3),\ -\frac{a}{a+1},\ -\frac{2a+3}{a+1},\ -\frac{a+3}{a+2}\right\}.$$

Proof. By Remark 4.3.2, f_a has critical points 0, 1, ∞ and μ, and critical values 0, 1, ∞ and λ, for some $\mu, \lambda \in \mathbb{P}^1$. Therefore f_a has signature (μ, λ). Since $a \notin \{0, -1, -3/2, -2, -3\}$, Lemma 4.3.3 implies that $f_a \in R_3^O$, so $\mu, \lambda \in \mathbb{C} \setminus \{0, 1\}$, and

$$a = \frac{\mu^3 + 3\mu\lambda - 4\lambda}{2\lambda(1-\mu)}.$$

Similarly, let μ' and λ' be the fourth critical point and critical value respectively of f_b, so that f_b has signature (μ', λ') and

$$b = \frac{\mu'^3 + 3\mu'\lambda' - 4\lambda'}{2\lambda'(1-\mu')}.$$

If the critical points and critical values of f_a are taken in a different order, then the cross-ratios change as described in (4.1.1). Thus the signatures of f_a with respect to the various orderings are

$$(\mu, \lambda), \left(\tfrac{1}{\mu}, \tfrac{1}{\lambda}\right), (1-\mu, 1-\lambda), \left(\tfrac{1}{1-\mu}, \tfrac{1}{1-\lambda}\right), \left(\tfrac{\mu}{\mu-1}, \tfrac{\lambda}{\lambda-1}\right), \left(\tfrac{\mu-1}{\mu}, \tfrac{\lambda-1}{\lambda}\right).$$

It follows from Lemma 4.4.2 that f_a and f_b are in the same orbit if and only if (μ', λ') equals one of the six pairs listed above. We will show that $(\mu', \lambda') = (1/\mu, 1/\lambda)$ if and only if $b = -(2a+3)/(a+2)$. The other cases are handled similarly.

First suppose that $\mu' = 1/\mu$ and $\lambda' = 1/\lambda$. Using (4.3.1), we have

$$\begin{aligned}
b &= \frac{\mu^{-3} + 3\mu^{-1}\lambda^{-1} - 4\lambda^{-1}}{2\lambda^{-1}(1-\mu^{-1})} \\
&= \frac{\mu^{-3}\lambda + 3\mu^{-1} - 4}{2(1-\mu^{-1})} \\
&= \frac{(a+2)^{-2} - 3(2a+3)a^{-1}(a+2)^{-1} - 4}{2(1+(2a+3)a^{-1}(a+1)^{-1})} \\
&= \frac{a - 3(2a+3)(a+2) - 4a(a+2)^2}{2(a(a+2)^2 + (2a+3)(a+2))} \\
&= \frac{a - 3(2a^2+7a+6) - 4a(a^2+4a+4)}{2(a+2)(a^2+2a+2a+3)} \\
&= \frac{-4a^3 - 22a^2 - 36a - 18}{2(a+2)(a^2+4a+3)} \\
&= -\frac{2a^3 + 11a^2 + 18a + 9}{(a+2)(a+1)(a+3)} \\
&= -\frac{(a+1)(2a+3)(a+3)}{(a+1)(a+2)(a+3)} \\
&= -\frac{2a+3}{a+2}.
\end{aligned}$$

Conversely, if $b = -(2a+3)/(a+2)$, then

$$\begin{aligned}
\mu' &= -\frac{b(b+2)}{2b+3} \\
&= \frac{2a+3}{a+2} \cdot \frac{1}{a+2} \cdot \frac{-(a+2)}{a} \\
&= -\frac{2a+3}{a(a+2)} \\
&= 1/\mu,
\end{aligned}$$

and similarly $\lambda' = 1/\lambda$. □

Example 4.4.4. Recall that for most choices of μ there are two values of a (Remark 4.3.5). The two corresponding f_a may or may not belong to the same orbit. For example, if we take $\mu = 2$, then we find that $a = -3 \pm \sqrt{3}$. But if $a = -3 + \sqrt{3}$, then $-a/(a+1) = -3 - \sqrt{3}$, and so it follows from the lemma that there is only one orbit corresponding to $\mu = 2$.

On the other hand, $\mu = 5$ yields $a = -6 \pm \sqrt{21}$. By the lemma, these two values of a correspond to distinct orbits. This justifies our earlier claim that the function s of Section 4.1 is not sufficient to distinguish the closed orbits.

By analogy with the symmetrised cross-ratio of (4.1.2), we define a function $\pi\colon \mathbb{C}\setminus\{0,-1,-3/2,-2,-3\}\to\mathbb{C}$ by

$$\begin{aligned}\pi(a) &= \sigma_2(a, -\tfrac{2a+3}{a+2}, -(a+3), -\tfrac{a}{a+1}, -\tfrac{2a+3}{a+1}, -\tfrac{a+3}{a+2}) - \tfrac{117}{4}\\ &= \frac{a^2(3a+2)^2(a+3)^2}{(a+1)^2(a+2)^2}.\end{aligned} \tag{4.4.1}$$

The term $-\frac{117}{4}$ is again chosen to enable a nice factorisation, and also ensures that $\pi(a)\to 0$ as $a\to 0$.

Define a map from R_3^O to $\mathbb{C}$, also called π, by

$$\pi(f) = \pi(a) \text{ where } f \text{ is in the same orbit as } f_a. \tag{4.4.2}$$

It follows from Lemma 4.4.3 and the use of the symmetric polynomial σ_2 in (4.4.1) that $\pi(f)$ is well defined.

Lemma 4.4.5. *The map π is holomorphic on the open stratum.*

Proof. Given $f\in R_3^O$ let λ be the cross-ratio of the critical values of f taken in some order, and let μ be the cross-ratio of the critical points in the corresponding order. Then the value of a is given by (4.3.1). It follows that $\pi(f)$ is a holomorphic function of λ and μ.

A straightforward application of the argument principle shows that, locally, as f varies holomorphically then so does each critical point, and therefore so does each critical value. Hence the dependence of μ and λ on f is holomorphic, and so π is holomorphic. □

Remark 4.4.6. The dependence of f on μ and λ can be described explicitly: substituting the formulae of (4.3.1) into (4.4.1) yields

$$\pi(\mu,\lambda) = \frac{p(\mu,\lambda)}{4\lambda^2(\lambda-1)^2\mu^4(\mu-1)^4}$$

where

$$\begin{aligned}p(\mu,\lambda) = &-\lambda^2\mu^{12} + 6\lambda^2\mu^{11} + \lambda\mu^{12} + 160\lambda^4\mu^8 + 54\lambda^3\mu^9 - 45\lambda^2\mu^{10}\\ &- 4\lambda\mu^{11} - 640\lambda^4\mu^7 - 563\lambda^3\mu^8 + 89\lambda^2\mu^9 + 34\lambda\mu^{10} - \mu^{11}\\ &- 44\lambda^5\mu^5 + 1044\lambda^4\mu^6 + 1676\lambda^3\mu^7 + 173\lambda^2\mu^8 - 98\lambda\mu^9 + \mu^{10}\\ &+ 110\lambda^5\mu^4 - 782\lambda^4\mu^5 - 2340\lambda^3\mu^6 - 782\lambda^2\mu^7 + 110\lambda\mu^8\\ &+ \lambda^6\mu^2 - 98\lambda^5\mu^3 + 173\lambda^4\mu^4 + 1676\lambda^3\mu^5 + 1044\lambda^2\mu^6 - 44\lambda\mu^7\\ &- \lambda^6\mu + 34\lambda^5\mu^2 + 89\lambda^4\mu^3 - 563\lambda^3\mu^4 - 640\lambda^2\mu^5 - 4\lambda^5\mu\\ &- 45\lambda^4\mu^2 + 54\lambda^3\mu^3 + 160\lambda^2\mu^4 + \lambda^5 + 6\lambda^4\mu - \lambda^4.\end{aligned}$$

Lemma 4.4.7. *Two elements f_1 and f_2 of R_3^O are in the same orbit if and only if $\pi(f_1)=\pi(f_2)$.*

Proof. The forward implication is immediate from the definition.

Suppose f_1 is in the same orbit as f_{a_1}, and f_2 in the same orbit as f_{a_2}. Then a mechanical calculation gives

$$\pi(f_2) - \pi(f_1) = \frac{(a_1-a_2)(a_1+a_2+3)(a_1a_2+a_1+a_2)(a_1a_2+a_1+2a_2+3)(a_1a_2+2a_1+a_2+3)(a_1a_2+2a_1+2a_2+3)}{(a_1+1)^2(a_1+2)^2(a_2+1)^2(a_2+2)^2}.$$

If the right hand side vanishes, then one of the factors of the numerator must vanish. Each factor corresponds to one of the expressions of Lemma 4.4.3. For example, $a_2 = -(2a_1+3)/(a_1+1)$ is equivalent to $a_1a_2 + 2a_1 + a_2 + 3 = 0$.

Hence if $\pi(f_1) = \pi(f_2)$, then Lemma 4.4.3 implies that f_{a_1} and f_{a_2} are in the same orbit, and so f_1 and f_2 are in the same orbit. □

Lemma 4.4.8. *The image of the open stratum under π is $\mathbb{C}^*$.*

Proof. We need to determine the values of $c \in \mathbb{C}$ for which the equation

$$\frac{a^2(3a+2)^2(a+3)^2}{(a+1)^2(a+2)^2} = c$$

has a solution $a \in \mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$. We can multiply through to obtain

$$p(a,c) = a^2(3a+2)^2(a+3)^2 - c(a+1)^2(a+2)^2 = 0. \qquad (4.4.3)$$

For fixed c, this is polynomial in a, and will always have a solution in $\mathbb{C}$. For which c does there exist a solution in $\mathbb{C} \setminus \{0, -1, -3/2, -2, -3\}$?

If $a = -1$ or -2, then the second term of p vanishes: the value of p is independent of c, and is nonzero. If $a = 0$ or $-3/2$ or -3, then the first term of p vanishes: hence we find $c = 0$. Thus for nonzero c, the solutions a of (4.4.3) never lie in the set $\{0, -1, -3/2, -2, -3\}$, and so π maps R_3^O onto $\mathbb{C}^*$. □

4.5. The quotient map. In the previous section we defined a holomorphic map $\pi\colon R_3^O \to \mathbb{C}^*$. We extend π to all of R_3 by defining $\pi(f) = 0$ whenever f is in the null fibre.

Lemma 4.5.1. *The map $\pi\colon R_3 \to \mathbb{C}$ is continuous.*

Proof. We only need to prove continuity at the null fibre. That is, we want to show that $\pi(f) \to 0$ as f approaches the null fibre. First we give an intuitive picture of the situation; a precise calculation follows.

We start by examining the limiting cases for the formulae given in Lemma 4.3.3 as the parameter a approaches a "forbidden" value or ∞. Recall that the restrictions on a arise from requiring the critical points and critical values to be distinct. As a tends towards a forbidden value, μ, λ and f_a behave as in the following table.

a	μ	λ	f tends to
0	0	0	$x^3/(3x-2)$
-2	0	0	$x(2-x)$
-3	1	1	$x^2(x-3)/(1-3x)$
-1	1	1	x^2
$-3/2$	∞	∞	$x^2(3-2x)$
∞	∞	∞	$x^2/(2x-1)$

As a approaches -1, -2 or ∞, we see that the degree of f_a drops, so f_a "falls out of R_3" as the critical points coalesce. However, for the other cases, f_a approaches an element of the null fibre.

The idea of the proof, informally, is that as we approach the null fibre in R_3, the value of a must approach 0, $-3/2$ or -3. These are exactly the values for which (4.4.1) takes on the value 0.

Let us make this more precise. Suppose $(f_n)_{n=1}^{\infty}$ is a sequence in R_3^O tending to an element of the null fibre. For each n, choose an ordering σ_n of the critical points of f_n, and Möbius transformations taking the critical points in order to $\{0, 1, \infty, \mu_n\}$ and the corresponding critical values to $\{0, 1, \infty, \lambda_n\}$. Furthermore, choose σ_n so that μ_n is at least as close to 0 (in the spherical metric, say) as it is to 1 or ∞.

With respect to this choice of σ_n, we must have $\mu_n \to 0$ as $n \to \infty$.

Using (4.3.1), if we let

$$a_n = \frac{\mu_n^3 + 3\mu_n\lambda_n - 4\lambda_n}{2\lambda_n(1-\mu)_n},$$

then we have

$$\pi(f_n) = \frac{a_n^2(3a_n+2)^2(a_n+3)^2}{(a_n+1)^2(a_n+2)^2} = O(a_n^2).$$

Also

$$\mu_n = -\frac{a_n(a_n+2)}{2a_n+3} = O(a_n).$$

So as $n \to \infty$ and $\mu \to 0$ we have $a_n \to 0$ and therefore $\pi(f_n) \to 0$. □

Corollary 4.5.2. *The map $\pi\colon R_3 \to \mathbb{C}$ is holomorphic.*

Proof. We already know that π is holomorphic outside the null fibre (Lemma 4.4.5). Recall (Remark 4.2) that the null fibre has codimension 1. Since π is continuous at the null fibre, it follows from Riemann's removable singularity theorem that π is holomorphic everywhere. □

Remark 4.5.3. Lemma 4.4.7 tells us that π is constant on the orbits and distinguishes the closed orbits, and Lemma 4.4.8 tells us that π is surjective. Informally, this means that π does the best possible job of distinguishing the orbits—it is not possible for a holomorphic function (or even a continuous function) to distinguish the two orbits of the null fibre, since one is inside the closure of the other—and so π by itself forms a complete set of invariant functions for the group action. Therefore π is the categorical quotient map, proving Theorem 1.6.

This argument can be expressed more rigorously. We know that the categorical quotient map $\pi'\colon R_3 \to Y$ exists [**75**, page 70]. Furthermore, Y is a reduced complex space and inherits from R_3 the properties of being connected, irreducible and normal [**75**, page 84]. By the universal property of the quotient [**75**, Lemma 3.1], there exists a unique holomorphic map $\alpha\colon Y \to \mathbb{C}$ such that $\pi = \alpha \circ \pi'$. From the above mentioned properties of π, it follows that α is a bijection. The difficulty is that Y may have singularities, so we cannot assume immediately that α^{-1} is holomorphic.

In our case, however, Y is irreducible and reduced, and therefore pure-dimensional. Since there exists a holomorphic bijection $Y \to \mathbb{C}$, the dimension must be 1. But a 1-dimensional normal space is necessarily smooth. It follows that α is a biholomorphism, and so $\pi\colon R_3 \to \mathbb{C}$ is also a categorical quotient map. This completes the proof of Theorem 1.6.

4.6. Further structure: stabilisers and the exceptional orbit. From the point of view of Oka theory, we would like to know whether π is an Oka map, in the sense defined in [**21**, Definition 6.3]. A necessary condition is that π should be a topological fibration. In fact it is not. This can be seen by studying the stabilisers of elements of R_3 and applying the results of [**67**, Section 2.3].

Recall that if the cross-ratio of four points in some order is $e^{\pm\pi i/3}$, then the six cross-ratios listed in (4.1.1) take on only the two distinct values $e^{\pm\pi i/3}$.

Definition 4.6.1. The *exceptional orbit* is the set of elements of the open stratum of R_3 whose critical points, taken in some order, have a cross-ratio of $e^{\pm\pi i/3}$.

Remark 4.6.2. The exceptional orbit is in fact a single orbit of the group action. Straightforward calculations show that this orbit is $\pi^{-1}(-27/4)$: we can use (4.3.3) to find $a = -e^{\pm\pi i/3} - 1$, and for both choices of sign, (4.4.1) gives the value $-27/4$. Note also that $e^{\pm\pi i/3}$ are the special cross-ratio values referred to in Remark 4.3.5.

Theorem 4.6.3. *The stabilisers of elements of R_3 are of four types:*

(1) An element of the closed orbit in the null fibre has stabiliser of dimension 1. Hence this orbit is 5-dimensional. The stabiliser is the nontrivial semidirect product of $\mathbb{C}^*$ and $\mathbb{Z}_2$.
(2) An element of the non-closed orbit has finite stabiliser of size 2.
(3) An element of the exceptional orbit has finite stabiliser of size 12, isomorphic to the alternating group on four symbols.

(4) An element of the open stratum outside the exceptional orbit has finite stabiliser of size 4, isomorphic to the Klein 4-group. Furthermore, all such stabilisers are conjugate.

Remark 4.6.4. By [**75**, Corollary 5.5], the restriction of the quotient map to the open stratum minus the exceptional orbit is a holomorphic fibre bundle. The fibres are homogeneous spaces for G, therefore Oka manifolds, and so π restricted to this domain is an Oka map. This is the maximal open subset of R_3 over which π is a fibration.

Proof of theorem. Let $f \in R_3$ and $g = (\alpha, \beta) \in G$ such that $f^g = f$. Then $\alpha^{-1} \circ f \circ \beta = f$. In particular, f and $\alpha^{-1} \circ f \circ \beta$ have the same critical points and the same critical values. It follows that α permutes the critical values of f and β permutes the critical points of f. Furthermore, both permutations must preserve multiplicities.

Case (1): We can choose $f = x^3$ as a representative of the 5-dimensional orbit. The critical points are 0 and ∞, each with multiplicity 2, and the critical values are the same. The only Möbius transformations fixing the set $\{0, \infty\}$ are $x \mapsto cx$ and $x \mapsto c/x$ for $x \in \mathbb{C}^*$. Choosing α and β to be of this form, and adding the restriction that $\alpha^{-1} \circ f \circ \beta = f$, we obtain an explicit realisation of the stabiliser as

$$\{(c^3 x, cx) : c \in \mathbb{C}^*\} \cup \{(c^3/x, c/x) : c \in \mathbb{C}^*\}.$$

Case (2): Similarly, take $f = x^3 + x^2$ as a representative of the non-closed orbit. This has a double critical value of ∞ with preimage ∞, and finite critical points and values $0 \mapsto 0$ and $-2/3 \mapsto 4/27$. (The finite critical points are simply the zeros of the derivative $3x^2 + 2x$.)

Suppose $\alpha^{-1} \circ f \circ \beta = f$. Then α and β must both fix ∞. This means that they are both of the form $x \mapsto ax + b$ for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Also, α must either fix or interchange the points 0 and $4/27$. Thus α is either the identity or the map $x \mapsto 4/27 - x$. Similarly, β is either the identity or $x \mapsto -2/3 - x$.

If we set $\alpha(x) = 4/27 - x$ and $\beta(x) = -2/3 - x$, noting that $\alpha^{-1} = \alpha$, then we can calculate:

$$\begin{aligned} \alpha \circ f &= 4/27 - f \neq f, \\ f \circ \beta &= (-2/3 - x)^2(-2/3 - x + 1) \\ &= (4/9 + 4x/3 + x^2)(1/3 - x) \\ &= 4/27 - x^2 - x^3 \neq f, \\ \alpha \circ f \circ \beta &= 4/27 - (4/27 - x^2 - x^3) = f. \end{aligned}$$

and so the stabiliser is $\{(1, 1), (\alpha, \beta)\}$ which has size 2 as stated above.

Cases (3) and (4), descriptions of the stabilisers: For orbits of R_3^O, we can choose a representative f as described in Lemma 4.3.3, with distinct critical points 0, 1, ∞ and μ and distinct critical values 0, 1, ∞ and λ.

Step 1: If $\alpha^{-1} \circ f \circ \beta = f$, then α must permute the four critical points of f, and β must permute the four critical values; furthermore, β must induce the same permutation as α. Since a Möbius transformation is determined by the images of three points, this greatly restricts the possibilities for (α, β). To be specific, we can choose an ordering (z_1, z_2, z_3, z_4) of $(0, 1, \infty, t)$ (where t can stand for λ or μ), find the unique Möbius transformation g sending $(0, 1, \infty)$ to (z_1, z_2, z_3), and check whether $g(t) = z_4$. The 24 possibilities are listed in Appendix A.

For generic values of t, there are only four permutations, given by the rows of the table with "any" in the fourth column. For each permutation we can calculate $\alpha^{-1} \circ f \circ \beta = f$ explicitly and verify that (α, β) does indeed stabilise f. The nontrivial permutations are all pairs of transpositions, giving the Klein 4-group. This proves case (4).

We obtain additional permutations only when λ and μ are both special cross-ratio values, i.e. one of -1, $\frac{1}{2}$, 2 or $e^{\pm\pi i/3}$.

Step 2: If either μ or λ is not one of the above special values, then the only possible elements of the stabiliser are those identified in Step 1 above. So we need to check whether μ and λ can be simultaneously special. This is straightforward: for each value of μ we solve (4.3.3) above to find the corresponding values of a, and then calculate λ. The result is that if μ is one of -1, $\frac{1}{2}$ or 2, then λ is real and irrational, therefore not special, but if μ is $e^{\pm\pi i/3}$, then $\lambda = \bar{\mu}$ is special. In this case there are an additional eight candidate elements in the stabiliser.

The calculations for the special values of λ and μ are summarised in the following table.

μ	equation	a	$\lambda = \mu^3/(a+2)^2$
-1	$a^2 - 3 = 0$	$\pm\sqrt{3}$	real and irrational
$1/2$	$a^2 + 3a + 3/2 = 0$	$(-3 \pm \sqrt{3})/2$	real and irrational
2	$a^2 + 6a + 6 = 0$	$-3 \pm \sqrt{3}$	real and irrational
$e^{\pi i/3}$	$a^2 + 2(e^{\pi i/3} + 1) + 3e^{\pi i/3} = 0$	$-e^{\pi i/3} - 1$	$1 - e^{\pi i/3} = e^{-\pi i/3}$
$e^{-\pi i/3}$	$a^2 + 2(e^{-\pi i/3} + 1) + 3e^{-\pi i/3} = 0$	$-e^{-\pi i/3} - 1$	$1 - e^{-\pi i/3} = e^{\pi i/3}$

Step 3: For each of the candidate elements identified above, calculate $\alpha^{-1} \circ f \circ \beta$ and verify that it equals f. (In fact, knowing that the stabiliser is a group, we only need to verify this for one element outside the generic stabiliser. It is easiest to work with the permutation (01∞), for which $\alpha(x) = \beta(x) = 1/(1-x)$ and $\alpha^{-1}(x) = (x-1)/x$.) Hence the stabiliser of the exceptional orbit has size 12. From the table in Appendix A we see that all elements of this stabiliser induce even permutations on the set $\{0, 1, \infty, e^{\pi i/3}\}$, and so the stabiliser is isomorphic to the alternating group on four symbols.

Case (4), conjugacy of stabilisers: Every finite subgroup of the Möbius group is conjugate to a subgroup of the group $\mathrm{PSU}_2(\mathbb{C})$, which can be viewed as the group of rigid motions of the Riemann sphere with respect to the usual embedding into $\mathbb{R}^3$. See for example [**60**] or [**45**, Section 2.13].

Two finite subgroups of the Möbius group are conjugate if and only if they are isomorphic as abstract groups ([**45**, remarks after Corollary 2.13.7]). However, a slightly stronger result is needed for our purposes.

For a Klein 4-subgroup of $\mathrm{PSU}_2(\mathbb{C})$, viewed as a group of rigid motions of the sphere, each non-identity element is a rotation by an angle of π about some axis. It is clear that two rotations commute if and only if their axes are orthogonal. Thus Klein 4-groups correspond to sets of three mutually orthogonal axes. For any two such sets of axes, there is an orientation-preserving rigid motion of the sphere taking one to the other. This gives a group element conjugating one Klein 4-group to the other. Therefore any two such subgroups are conjugate.

We can go a little further. For a set of three mutually orthogonal axes, and for any permutation of those axes, there exists a rotation realising that permutation. Conjugating by this rotation will yield an automorphism of the corresponding Klein 4-group which permutes the non-identity elements in the same way.

Hence we can conclude that given Klein 4-subgroups $\{1, \alpha_1, \alpha_2, \alpha_3\}$ and $\{1, \beta_1, \beta_2, \beta_3\}$ of the Möbius group, there exists a group element g with $g\alpha_j g^{-1} = \beta_j$ for $j = 1, 2, 3$. This is the stronger result referred to above.

Now we apply this to stabilisers in $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$. If f is in the open stratum but not in the exceptional orbit, then the stabiliser is of the form $\{(1,1), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$, where each α_j permutes the critical values of f via a pair of disjoint transpositions, and each β_j carries out the same permutation on the corresponding critical points. Given another stabiliser of the form $\{(1,1), (\alpha_1', \beta_1'), (\alpha_2', \beta_2'), (\alpha_3', \beta_3')\}$, we seek (g, h) such that g conjugates $\{1, \alpha_1, \alpha_2, \alpha_3\}$ to $\{1, \alpha_1', \alpha_2', \alpha_3'\}$ in some order, and h conjugates $\{1, \beta_1, \beta_2, \beta_3\}$ to $\{1, \beta_1', \beta_2', \beta_3'\}$ in the same order. The fact that any two isomorphic finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$ are conjugate tells us that a suitable g exists. Then the existence of h is guaranteed by the stronger result that we can conjugate the elements of one Klein 4-subgroup to another in any desired order. □

5. Degree 3: dominability and $\mathbb{C}$-connectedness

5.1. Composition of dominability. When exploring the Oka property or related flexibility properties of a manifold X, a logical first step is to investigate holomorphic maps $\mathbb{C}^n \to X$, and in particular to look for dominating maps (Definition 1.3).

In the case of R_d, the principal difficulty in constructing explicit dominating maps is cancellation. The easiest way to write down a map $\mathbb{C}^n \to R_d$ is in the form $p(t)/q(t)$ where p and q are families of polynomials parametrised by $t \in \mathbb{C}^n$. However, it is necessary to ensure that p and q do not have common factors as the parameter t varies. We can achieve this by embedding R_d in a larger space and then applying the following result.

Proposition 5.1.1 (Composition of dominability). *Let X be an open subset of a complex manifold Z, and $p \in X$. Suppose $\phi\colon \mathbb{C}^n \to Z$ dominates Z at p. If $\phi^{-1}(X)$ is dominable at 0, then X is dominable at p.*

Proof. If $\psi\colon \mathbb{C}^m \to \phi^{-1}(X)$ dominates $\phi^{-1}(X)$ at 0, then $\phi \circ \psi$ dominates X at p. □

We construct a map $\mathbb{C}^8 \to R_3$ as follows.

We can view a point of $\mathbb{P}^7$ as a formal rational function:

$$(a_0\colon \cdots \colon a_7) \longleftrightarrow \frac{a_0x^3 + a_1x^2 + a_2x + a_3}{a_4x^3 + a_5x^2 + a_6x + a_7}.$$

We also have the group $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ acting on R_3 by pre- and post-composition. This extends to an action of G on $\mathbb{P}^7$: we can compose a formal rational function with a Möbius transformation to get a well-defined result.

Recall that we can embed R_3 into $\mathbb{P}^7$ by using the coefficients of a rational function as the homogeneous coordinates of a point of $\mathbb{P}^7$. This embedding is G-equivariant; in the following discussion we will identify R_3 with its image in $\mathbb{P}^7$.

Now choose $f \in R_3$ and suppose we have a map $\eta\colon \mathbb{C}^2 \to \mathbb{P}^7$ sending 0 to f. Let $\exp\colon \mathbb{C}^6 \to G$ be the exponential map. This map dominates G at the identity; for this particular group, it is also surjective [**29**, page 47]. Define $\phi\colon \mathbb{C}^8 \to \mathbb{P}^7$ by

$$\phi(s,t) = \eta(s)^{\exp(t)}, \qquad s \in \mathbb{C}^2,\ t \in \mathbb{C}^6. \tag{5.1.1}$$

Our strategy is to choose η so that ϕ dominates $\mathbb{P}^7$ at f, and find some $\psi\colon \mathbb{C}^m \to \phi^{-1}(R_3)$ which is dominating at 0. The proposition then tells us that $\phi \circ \psi$ dominates R_3 at f. This proves Theorem 1.7.

Furthermore, ψ can be chosen so that $\phi \circ \psi$ is surjective (Corollary 5.2.9). Since $\mathbb{C}^8$ is $\mathbb{C}$-connected (Definition 1.4), it follows that R_3 is $\mathbb{C}$-connected, proving Theorem 1.8.

To construct a suitable η, first we define $\eta_0\colon \mathbb{C}^2 \to \mathbb{P}^7$ by

$$\eta_0(a,b) = \frac{x^3 - ax}{-bx^2 + 1} = (1\colon 0\colon -a\colon 0\colon 0\colon -b\colon 0\colon 1). \tag{5.1.2}$$

We will see in Proposition 5.2.6 that the image of η_0 intersects every orbit of G on R_3. Thus given $f \in R_3$ there exist $a_0, b_0 \in \mathbb{C}$ and $g \in G$ such that g takes $\eta_0(a_0,b_0)$ to f. Define η by

$$\eta(a,b) = \eta_0(a + a_0, b + b_0)^g.$$

Remark 5.1.2. The form of η is not uniquely determined by the choice of f. For the purpose of proving strong dominability, this does not matter: all we need is that given f there exists at least one suitable η. If we could in fact find a canonical η for each f, in such a way that the map $f \mapsto \eta$ were holomorphic, then we could join the resulting dominating maps to make a spray ([**21**, Definition 5.1]). This would imply that R_3 is Oka.

The numerator of η_0 has roots 0 and $\pm\sqrt{a}$, and the denominator has roots $\pm 1/\sqrt{b}$. Therefore $\eta_0(a,b)$ fails to be in R_3 exactly when $ab = 1$. Similarly, given a_0, b_0 and g, the set of (a,b) such that $\eta(a,b) \notin R_3$ is a translate of $\mathbb{C}^2 \setminus \{(a,b) : ab = 1\}$. Therefore

$$\phi^{-1}(R_3) \cong (\mathbb{C}^2 \setminus \{(a,b) : ab = 1\}) \times \mathbb{C}^6.$$

Now $\mathbb{C}^2 \setminus \{ab = 1\}$ is Oka: this is a consequence of [**38**, Proposition 4.10], or see Appendix B for an elementary proof. In particular, $\mathbb{C}^2 \setminus \{ab = 1\}$, and hence $\phi^{-1}(R_3)$, is dominable at 0. Thus $\phi^{-1}(R_3)$ is dominable at 0. We will show in Section 5.3 that ϕ dominates $\mathbb{P}^7$ at f. Therefore $\phi \circ \psi$ dominates R_3 at f.

5.2. Proof of surjectivity. The goal of this section is to show that the map $\psi \colon \mathbb{C}^8 \to \phi^{-1}(R_3)$ can be chosen so that $\phi \circ \psi$ is surjective. First we prove that the image of the map η_0 defined by (5.1.2) intersects every orbit, and therefore ϕ is surjective. Then we will describe the choice of ψ.

For the first part, we exploit the fact that the critical values of $\eta_0(a,b)$ have a certain kind of symmetry.

Definition 5.2.1. Let $z_1, \ldots, z_4 \in \mathbb{C}$. We say that $(z_1, \ldots, z_4)$ is *balanced* if $z_1 + z_2 = z_3 + z_4 = 0$.

Lemma 5.2.2. *Let $z_1, \ldots, z_4$ be distinct points of $\mathbb{C}$. There exists a Möbius transformation α such that $(\alpha(z_1), \ldots, \alpha(z_4))$ is balanced.*

Proof. By transitivity of the Möbius group, we can assume that $z_3 = 1$ and $z_4 = -1$. We will find a Möbius transformation α fixing 1 and -1, and such that $\alpha(z_1) + \alpha(z_2) = 0$. Suppose

$$\alpha(x) = \frac{ax + b}{cx + d}.$$

Then $\alpha(1) = 1$ tells us that $a + b = c + d$, and $\alpha(-1) = -1$ implies $a - b = c - d$. Hence $a = d$ and $b = c$, so α is of the form

$$\alpha(x) = \frac{ax + b}{bx + a}$$

for some $a, b \in \mathbb{C}$. For α to be invertible we also need $a \neq \pm b$.

We wish to find a and b such that

$$\frac{az_1 + b}{bz_1 + a} + \frac{az_2 + b}{bz_2 + a} = 0.$$

This gives

$$(z_1 + z_2)a^2 + 2(1 + z_1 z_2)ab + (z_1 + z_2)b^2 = 0,$$

or, setting $A = a/b$ or $A = b/a$ (by symmetry, both are possible),

$$(z_1 + z_2)A^2 + 2(1 + z_1 z_2)A + (z_1 + z_2) = 0.$$

This always has a solution for A. The condition $a \neq \pm b$ means that we require $A \neq \pm 1$. But if $A = 1$, then the left hand side of the equation is

$$2(z_1 + z_2 + 1 + z_1 z_2) = 2(z_1 + 1)(z_2 + 1),$$

which is always nonzero when $z_1, z_2, \pm 1$ are distinct. Similarly, $A = -1$ will also give a nonzero left hand side. Hence it is always possible to find a and b satisfying the required conditions. □

The proof of the following elementary result is left as an exercise for the reader.

Lemma 5.2.3. *Let $U \subset \mathbb{C}$ be a connected open set containing 0 and such that $-x \in U$ for every $x \in U$. Let $f : U \to \mathbb{C}$ be a holomorphic function such that f' is even and $f(0) = 0$. Then f is odd.*

Lemma 5.2.4. *Let $f = p/q \in R_d$ where p and q are polynomials with no common factors. Suppose f is an odd function. Then either p is odd and q is even, or p is even and q is odd.*

Proof. Since p and q have no common factors, the zeros of p are precisely the zeros of f in $\mathbb{C}$. Since f is odd, the zeros are distributed symmetrically about 0, and so p is either odd or even (depending on whether $f(0) = 0$ or ∞). Similarly, q is either even or odd. □

Corollary 5.2.5. *If $f \in R_3$ is odd and $f(0) = 0$, then f can be written in the form*

$$f(x) = \frac{Ax^3 + Bx}{Cx^2 + 1}$$

for some $A, B, C \in \mathbb{C}$.

Proposition 5.2.6. *The image of the map η_0 of* (5.1.2) *intersects every orbit in R_3.*

Proof. First, note that $\eta_0(0, 0) = x^3$ is in the small orbit and $\eta_0(1, 0) = x^3 - x$ is in the non-closed orbit. Therefore we only need to consider orbits outside the null fibre.

Given $f \in R_3$ outside the null fibre, Lemma 5.2.2 ensures that there is a Möbius transformation α such that the critical points of $f \circ \alpha$ are balanced. (In fact α is the inverse of a transformation taking the critical points to a balanced quadruple. If one of the critical points of f is ∞, then before applying the lemma we precompose f with a suitable transformation so that the resulting critical points are all finite.)

Now the finite critical points of f are the zeros of f', that is, the roots of the numerator of f'. For $f = p/q$ we have $f' = (p'q - pq')/p^2$. After balancing the critical points, the numerator of $(f \circ \alpha)'$ will be of the form $C(x^2 - A^2)(x^2 - B^2)$ for some $A, B, C \in \mathbb{C}^*$. The denominator is a perfect square. Therefore $(f \circ \alpha)'$ is an even function.

Let β be a Möbius transformation taking $f(\alpha(0))$ to 0, so that $\beta \circ f \circ \alpha$ has the same critical points as $f \circ \alpha$. Let $U = \mathbb{C} \setminus \{x \in \mathbb{C} : x \text{ or } -x \text{ is a pole of } \beta \circ f \circ \alpha\}$. Then $\beta \circ f \circ \alpha|_U$ satisfies the conditions of Lemma 5.2.3, and is therefore an odd function. By continuity, it follows that any poles of $\beta \circ f \circ \alpha$ must be symmetrically distributed about 0, and $\beta \circ f \circ \alpha$ is odd. By Corollary 5.2.5 we have

$$(\beta \circ f \circ \alpha)(x) = \frac{Ax^3 + Bx}{Cx^2 + 1}$$

for some $A, B, C \in \mathbb{C}$. Also, $f \in R_3$ implies $A \neq 0$. Therefore we can postcompose with the Möbius transformation $x \mapsto x/A$ to see that f is in the same orbit as $\eta_0(-B/A, -C)$. □

Corollary 5.2.7. *The image of the map ϕ of* (5.1.1) *is exactly R_3.*

Now we describe the map $\psi\colon \mathbb{C}^8 \to \phi^{-1}(R_3)$. Since the exponential map $\exp\colon \mathbb{C}^6 \to G$ is surjective, we simply need to find a surjective holomorphic map $\chi\colon \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{ab = 1\}$ which is dominating at 0, and then we can take $\psi = \chi \times \exp$.

Following Buzzard and Lu [**5**, page 645], define a map $\omega\colon \mathbb{C}^2 \to \mathbb{C}$ by

$$\begin{aligned} \omega(x, y) &= \begin{cases} \dfrac{e^{xy} - 1}{x} & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases} \\ &= y + \frac{xy^2}{2} + \frac{x^2 y^3}{3!} + \cdots . \end{aligned}$$

From the first form of the definition, we can see that for fixed x, the image of $y \mapsto \omega(x, y)$ is $\mathbb{C} \setminus \{-1/x\}$. From the series expression we can see that ω is holomorphic, and we can calculate derivatives

$$\begin{aligned} \left.\frac{\partial \omega}{\partial x}\right|_{x=0} &= y^2/2, \\ \frac{\partial \omega}{\partial y} &= e^{xy}. \end{aligned}$$

Proposition 5.2.8. *The map $\chi\colon \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{ab = 1\}$ defined by*

$$\chi(x, y) = \left(x, \frac{1 - e^{xy}}{x} \right) = (x, -\omega(x, y))$$

is surjective and dominating at 0.

Proof. Recall that for fixed a, the image of $y \mapsto \omega(a, y)$ is $\mathbb{C} \setminus \{-1/a\}$. Thus if $ab \neq 1$, then there exists y such that $\omega(a, y) \neq -b$, and then $\chi(a, y) = (a, b)$. Hence χ is surjective.

To prove dominability, we need to verify that the vectors $\partial\chi/\partial x$ and $\partial\chi/\partial y$ evaluated at $(x, y) = (0, 0)$ span $\mathbb{C}^2$. But we have $\partial\chi/\partial x|_{x=0} = (1, y^2/2)$ and $\partial\chi/\partial y = (0, e^{xy})$. Thus the derivatives evaluated at $(0, 0)$ are $(1, 0)$ and $(0, 1)$. □

Corollary 5.2.9. *With χ as above, $\psi = \chi \times \exp$ and ϕ as defined in* (5.1.1)*, the composition $\phi \circ \psi$ is surjective.*

5.3. Proof of transversality. We wish to show that the map ϕ of (5.1.1) is dominating. Recall that $\phi\colon \mathbb{C}^8 \to \mathbb{P}^7$ is built from maps $\eta\colon \mathbb{C}^2 \to \mathbb{P}^7$ and $\exp\colon \mathbb{C}^6 \to G$. For convenience, we repeat the definitions here.

$$\eta_0(a, b) = \frac{x^3 - ax}{-bx^2 + 1} = (1\colon 0\colon -a\colon 0\colon 0\colon -b\colon 0\colon 1),$$
$$\eta(a, b) = \eta_0(a + a_0, b + b_0)^g \text{ for some constants } a_0, b_0 \in \mathbb{C} \text{ and } g \in G,$$
$$\phi(s, t) = \eta(s)^{\exp(t)}, \qquad s \in \mathbb{C}^2,\ t \in \mathbb{C}^6.$$

Since exp is dominating at the identity, it follows that the image of $d\phi_0$ contains the tangent space to the orbit through $\eta(0, 0)$. So to prove that ϕ is dominating, it is sufficient to show that the image of $d\eta$ is transverse to the tangent space of the orbit. More precisely, we will show that the images of $d\phi_0$ and $d\eta_{(0,0)}$ together span the tangent space $T\mathbb{P}^7_f$, where $f = \phi(0, 0)$.

Proposition 5.3.1. *Let $a, b \in \mathbb{C}$, $ab \neq 1$, and $f = \eta_0(a, b)$. Suppose $f \in R_3$. Then the tangent space $T\mathbb{P}^7_f$ is spanned by the tangent space to the orbit through f together with the image of $d\eta_0(a, b)$.*

Proof. For convenience, we will work in affine coordinates: rational functions of the form
$$\frac{a_0x^3 + a_1x^2 + a_2x + a_3}{b_0x^3 + b_1x^2 + b_2x + 1}$$
will be written as $(a_0, a_1, a_2, a_3; b_0, b_1, b_2)$. In this notation, we have
$$\eta_0(a, b) = (1, 0, -a, 0; 0, -b, 0).$$
We can identify $T\mathbb{P}^7_f$ with $\mathbb{C}^7$, and $d\eta_{(a,b)}$ is the subspace
$$\{(0, 0, u, 0; 0, v, 0) : u, v \in \mathbb{C}\}. \tag{5.3.1}$$
The main part of the proof consists in finding a set of vectors spanning the tangent space to the fibre. Such a set can be realised as derivatives of infinitesimal generators of the group.

The Lie algebra $\mathrm{sl}_2(\mathbb{C})$ is generated by the three matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Infinitesimal generators of $\mathrm{PSL}_2(\mathbb{C})$ are given by their exponentials:

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^-t \end{pmatrix} \qquad e^{Bt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad e^{Ct} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

or, interpreted as Möbius transformations:

$$e^{At} \colon x \mapsto e^{2t}x \qquad e^{Bt} \colon x \mapsto x + t \qquad e^{Ct} \colon x \mapsto \frac{x}{tx+1}.$$

As infinitesimal generators of the group $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$, we can use ordered pairs (g^{-1}, id) and (id, g), where g is one of e^{At}, e^{Bt} or e^{Ct}. Thus we simply need to calculate the six vectors

$$\frac{d}{dt}(\eta_0(a,b) \circ e^{At})|_{t=0}, \qquad \frac{d}{dt}(e^{At} \circ \eta_0(a,b))|_{t=0},$$

and the corresponding vectors for e^{Bt} and e^{Ct}.

The first of those six vectors is computed as follows:

$$\eta_0(a,b) \circ e^{At} \colon x \mapsto \frac{e^{6t}x^3 - ae^{2t}x}{-be^{4t}x^2 + 1} = (e^{6t}, 0, -ae^{2t}, 0; 0. - be^{4t}, 0),$$

and so the derivative with respect to t, evaluated at $t = 0$, is

$$(6, 0, -2a, 0; 0, -4b, 0).$$

The remaining cases are handled similarly. The end result of the calculation is that the tangent space to the fibre is spanned by the rows of the following matrix:

$$\begin{pmatrix} 6 & 0 & -2a & 0 & 0 & -4b & 0 \\ 0 & 3 & 0 & -a & 0 & 0 & -2b \\ 0 & -2a & 0 & 0 & -b & 0 & 3 \\ 2 & 0 & -2a & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & a \end{pmatrix}$$

and it is straightforward to verify that the rows together with the vectors of (5.3.1) span $\mathbb{C}^7$. □

Since the above calculation does not depend on the choice of a and b, we have a dominating map for every $f \in R_3$, proving Theorem 1.7.

Appendix A. Table of Möbius transformations

Let $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then for each sequence of three distinct elements z_1, z_2, z_3 of $\{0, 1, \infty, t\}$ there is a unique Möbius transformation g sending 0, 1 and ∞ to z_1, z_2 and z_3 respectively. Table 1 lists the form of g for all 24 possible choices of (z_1, z_2, z_3). The fourth column

TABLE 1. The 24 Möbius transformations of Appendix A

$g(0,1,\infty)$	$g(x)$	$g(t)$	t_0	permutation
$(0,1,\infty)$	x	t	any	id
$(0,1,t)$	$tx/(x+t-1)$	$t^2/(2t-1)$	$1/2$	$(t\infty)$
$(0,\infty,1)$	$x/(x-1)$	$t/(t-1)$	2	(1∞)
$(0,\infty,t)$	$tx/(x-1)$	$t^2/(t-1)$	$e^{\pm\pi i/3}$	$(1\infty t)$
$(0,t,1)$	$tx/(tx+1-t)$	$t^2/(t^2-t+1)$	$e^{\pm\pi i/3}$	$(1t\infty)$
$(0,t,\infty)$	tx	t^2	-1	$(1t)$
$(1,0,\infty)$	$1-x$	$1-t$	$1/2$	(01)
$(1,0,t)$	$(tx-t)/(x-t)$	∞	any	$(01)(\infty t)$
$(1,\infty,0)$	$1/(1-x)$	$1/(1-t)$	$e^{\pm\pi i/3}$	(01∞)
$(1,\infty,t)$	$(tx-1)/(x-1)$	$t+1$	-1	$(01\infty t)$
$(1,t,0)$	$t/((1-t)x+t)$	$1/(2-t)$	2	$(01t\infty)$
$(1,t,\infty)$	$(t-1)x+1$	t^2-t+1	$e^{\pm\pi i/3}$	$(01t)$
$(\infty,0,1)$	$(x-1)/x$	$(t-1)/t$	$e^{\pm\pi i/3}$	$(0\infty 1)$
$(\infty,0,t)$	$t(x-1)/x$	$t-1$	2	$(0\infty t1)$
$(\infty,1,0)$	$1/x$	$1/t$	-1	(0∞)
$(\infty,1,t)$	$(tx+1-t)/x$	$(t^2-t+1)/t$	$e^{\pm\pi i/3}$	$(0\infty t)$
$(\infty,t,0)$	t/x	1	any	$(0\infty)(1t)$
$(\infty,t,1)$	$(x+t-1)/x$	$(2t-1)/t$	$1/2$	$(0\infty 1t)$
$(\mathrm{t},0,1)$	$t(x-1)/(tx-1)$	$t/(t+1)$	-1	$(0t\infty 1)$
$(\mathrm{t},0,\infty)$	$t(1-x)$	$t(1-t)$	$e^{\pm\pi i/3}$	$(0t1)$
$(\mathrm{t},1,0)$	$t/((t-1)x+1)$	$t/(t^2-t+1)$	$e^{\pm\pi i/3}$	$(0t\infty)$
$(\mathrm{t},1,\infty)$	$(1-t)x+t$	$2t-t^2$	2	$(0t)$
$(\mathrm{t},\infty,0)$	$t/(1-x)$	$t/(1-t)$	$1/2$	$(0t1\infty)$
$(\mathrm{t},\infty,1)$	$(x-t)/(x-1)$	0	any	$(0t)(1\infty)$

contains the values t_0 such that $g(t_0) \in \{0,1,\infty,t_0\}$, and the last column gives the permutation induced by g on the set $\{0,1,\infty,t\}$ when $t = t_0$. This table is used in the proof of Theorem 4.6.3.

Appendix B. The Oka property for the complement of an affine plane conic

It is well known that the complement of a smooth quadric surface in $\mathbb{P}^n$ is a homogeneous manifold, and is therefore Oka. In affine space, the situation is more complicated. Here we discuss only the case of smooth quadratic curves in $\mathbb{C}^2$, as mentioned at the end of Section 5.1. There is scope for further investigation in higher dimensions.

In $\mathbb{C}^2$ there are two isomorphism classes of smooth conics, depending on the behaviour at infinity. One class is represented by the curve $y = x^2$. For the complement of this curve, the projection onto the x-axis has fibre $\mathbb{C}^*$. It is readily seen that this projection is a trivial fibre bundle. Thus the complement is a fibre bundle with Oka base and Oka fibre, and is therefore an Oka manifold.

The more interesting case is the complement of $xy = 1$. Here the projection map onto a coordinate axis is not a fibration, because both $\mathbb{C}^*$ and $\mathbb{C}$ occur as fibres. Therefore it is rather more difficult to prove the Oka property. We can give a reasonably short proof by constructing a suitable covering space which "untwists" the fibres, and using the fact that a manifold is Oka if and only if it has an Oka covering space. We construct sprays (in the sense of Gromov) on the covering space, proving that it is subelliptic, which implies the Oka property. Refer to [**21**, Section 5] for the relevant definitions.

Write $X = \mathbb{C}^2 \setminus \{xy = 1\}$ and $Z = \{(x, y, z) \in \mathbb{C}^3 : e^z = xy - 1\}$. Define $\pi\colon Z \to X$ by $\pi(x, y, z) = (x, y)$. Then π is a covering map, so Z is Oka if and only if X is Oka.

Define maps $s_1, s_2, s_3\colon Z \times \mathbb{C} \to Z$ by

$$\begin{aligned} s_1(x, y, z, t) &= (e^t x, e^{-t} y, z), \\ s_2(x, y, z, t) &= \left(\frac{1 + e^{z+ty}}{y}, y, z + ty\right), \\ s_3(x, y, z, t) &= \left(x, \frac{1 + e^{z+tx}}{x}, z + tx\right). \end{aligned}$$

Note that s_2 and s_3 are holomorphic everywhere; the behaviour near $x = 0$ or $y = 0$ is similar to that of the map ω discussed immediately before Proposition 5.2.8.

It is straightforward to check from the definitions that the s_j are sprays and that they dominate at every point of Z. Hence Z is subelliptic, and so Z, and therefore X, is Oka.

Appendix C. Sage code for computations

Although the calculations in this paper are easy enough to verify by hand once the answer is known, the computer has been very useful as an exploratory tool in trying out various possibilities. The availability of computer algebra systems means that the research proceeded more quickly than would otherwise have been possible.

This appendix reproduces some of the code that was used to find the results of Sections 4.1, 4.3, 4.4 and 5.3. Sage [**76**] was chosen because its open-source nature makes it particularly easy for others to verify and reuse the code below; there are of course many commercial packages that can perform the same calculations.

In the listing below, lines beginning with the '`#`' character are comments.

```
# Sage code for some calculations in R_3

# Part 1: look at the standard forms and the quotient map,
# as described in Sections 4.3 and 4.4.

# First make a number field with our ‘special value’:
# alpha=e^{\pi i/3} is a root of z^2-z+1,
# but it’s more convenient to express things in terms of a=-(1+alpha),
# which is a root of z^2+3z+3.
# Use ‘aa’ for this specific value of a.
# In Sage, ‘QQ’ represents the rational numbers;
# we define ‘RR’ to be the quadratic field extension containing ‘aa’.
var('z')
RR.<aa>=QQ.extension(z^2+3*z+3)
alpha=-(1+aa)

# Now we need a polynomial ring:
R.<x,t,a,x1,x2,x3,x4,x5,x6>=RR[]
# Use x1-x6 as arguments of symmetric functions: see below.
# Note that the symbol R isn’t used below;
# the point of the previous command is to define the variables x,t,a,...
# and the rules for factorising polynomials in those variables.

# The standard form of an R3 element, as in Lemma 4.3.3:
f=x^2*(x+a)/((2*a+3)*x-a-2)

# Relationships between a, mu, lambda given by Lemma 4.3.3.
# Note that ‘lambda’ is a reserved word in the Python programming language,
# so we shouldn’t use it as the name of a variable,
# hence the names ‘lambdaval’ etc.
def muval(aval):
    return -aval*(aval+2)/(2*aval+3)

def lambdaval(aval):
    return muval(aval)^3/(aval+2)^2

def aval(muval,lambdaval):
    return (muval^3+3*muval*lambdaval-4*lambdaval)/(2*lambdaval*(1-muval))

# Fourth critical point and critical value:
m=muval(a)
l=lambdaval(a)
# ‘m’ and ‘l’ are abbreviations for mu and lambda respectively.

#  What happens to a when we shuffle lambda and mu?
f0=a
f1=aval(1/m,1/l)
f2=aval(1-m,1-l)
f3=aval(m/(m-1),l/(l-1))
f4=aval(1/(1-m),1/(1-l))
f5=aval((m-1)/m,(l-1)/l)
# f0 through f5 are the expressions of Lemma 4.4.3.

# What about symmetric functions of those f0-f5?
# Define s1-s6 to be the elementary symmetric functions of x1-x6
# Oddly, there doesn’t seem to be a Sage builtin that does it,
# so I need to create them by hand.
```

```
# It's easy to make s1 and s6: just add/multiply all the variables.
varlist=[x1,x2,x3,x4,x5,x6]
s1=sum(varlist)
s6=prod(varlist)
# For s2 through s5, we use the 'combinations' function to generate
# the needed monomials.
# Unfortunately 'combinations' can't take variables as arguments,
# so we need make a list of coefficients
# and then substitute the variables 'by hand'.
otherfuncs=[0,0,0,0,0,0]
for i in [2,3,4,5]:
    coefflist=combinations(range(6),i)
    for index in coefflist:
        otherfuncs[i]=otherfuncs[i]+prod(varlist[n] for n in index)
s2=otherfuncs[2]
s3=otherfuncs[3]
s4=otherfuncs[4]
s5=otherfuncs[5]
# Now s1 through s6 contain the elementary symmetric functions
# of the variables x1 through x6.
# We'll use this to verify (4.4.1).
# A similar process applied to the lambda values of (4.1.1)
# gives (4.1.2).

# Calculate symmetric functions of the six a-values f1 through f6:
s1a=s1.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)
s2a=s2.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)
s3a=s3.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)
s4a=s4.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)
s5a=s5.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)
s6a=s5.subs(x1=f1,x2=f2,x3=f3,x4=f4,x5=f5,x6=f0)

# Write these in lowest terms;
# Sage needs a little help here.
s2a=(s2a.numerator()/2^20)/(s2a.denominator()/2^20)
s3a=(s3a.numerator()/2^40)/(s3a.denominator()/2^40)
s4a=(s4a.numerator()/2^40)/(s4a.denominator()/2^40)
s5a=(s5a.numerator()/2^20)/(s5a.denominator()/2^20)
s6a=(s6a.numerator()/2^20)/(s6a.denominator()/2^20)
# Results:
# s2a=\frac{-a^{6}-9a^{5}+135a^{3}+360a^{2}+351a+117}
#          {a^{4}+6a^{3}+13a^{2}+12a+4}
# s1a=-144/16
# 6s2a+s3a=135
# 13s2a-s4a=360
# 12s2a+s5a=351
# s5a=s6a

# Something strange: s2a(0)=s2a(-3)=s2a(-3/2)=117/4
# Can s2a send anything else to 117/4?
s2apoly=117*s2a.denominator()-4*s2a.numerator()
factor(s2apoly)
# The result is 4*a^2*(a+3/2)^2*(a+3)^2
# so the answer is no, those are the only three values sent to 117/4.

# Does s2a distinguish the orbits?
calcdiff=s2a.subs(a=x1)-s2a.subs(a=x2)
```

```
factor(calcdiff)
# This produces the equation of Proof 4.4.7.

# Part 2: Find vectors spanning tangent spaces to orbits of R_3,
# as described in Section 5.3.

# Need to redefine the polynomial ring R,
# since Sage isn't comfortable mixing the algebraic number aa
# with the transcendental number e.
R.<x,t,a,b,K>=QQ[]

# Infinitesimal generators for the tangent space
eat=x*e^(2*t)
# 'eat' means e^{at} where a is an element of the Lie algebra
# as in Proof 5.3.1; similarly ebt and ect.
ebt=x+t
ect=x/(t*x+1)

def find_tangents(gamma):
# Given a rational function gamma, calculate the derivatives,
# evaluated at t=0, of gamma composed with each of eat, ebt, ect
# for both pre- and postcomposition.
# nb the constant term of the denominator of gamma must be nonzero.
    answerlist=[]
    for funcpair in [(gamma,eat),(gamma,ebt),(gamma,ect),
                     (eat,gamma),(ebt,gamma),(ect,gamma)]:
        comp=symbolic_expression(funcpair[0].subs(x=funcpair[1]))
           # need to convert to symbolic_expression
           # in order to use rational_simplify
        comp=comp.rational_simplify()
        numerator_coeffs=[0,0,0,0]
        denominator_coeffs=[0,0,0,0]
        for term in comp.numerator().coeffs(x):
            numerator_coeffs[term[1]]=term[0]
        numerator_coeffs.reverse()
        for term in comp.denominator().coeffs(x):
            denominator_coeffs[term[1]]=term[0]
        denominator_coeffs.reverse()
        coefflist=[term/denominator_coeffs[-1]
                   for term in numerator_coeffs+denominator_coeffs[:-1]]
        compd=[term.derivative(t).subs(t=0) for term in coefflist]
        answerlist.append(compd)
    return answerlist

eta_zero=(x^3-a*x)/(-b*x^2+1) # This is the function of (5.1.2).
find_tangents(eta_zero) # This outputs the matrix of Proof 5.3.1.
```

Appendix D. Index of notation

R_d	space of rational functions of degree d
G	the group $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$, acting on R_3
π	the categorical quotient map $R_3 \to \mathbb{C}$
f, g	denote elements of R_3
g	sometimes denotes an element of G, and sometimes a Möbius transformation
α, β	denote Möbius transformations
$(z_1, z_2; z_3, z_4)$	cross-ratio
λ, μ	values of cross-ratio
σ_k	elementary symmetric functions
s_k, s	symmetrised versions of cross-ratio
a	the parameter of Lemma 4.3.3
f_a	the function of Definition 4.3.1
f^g	the image of $f \in R_3$ under the action of $g \in G$
$f^{(\sigma)}$	see Lemma 4.4.2
σ, ρ	orderings of critical values and points (Section 4.4)
η, η_0	the maps $\mathbb{C}^2 \to \mathbb{P}^7$ of (5.1.2) and following
exp	the exponential map for G
ϕ, ψ	the two maps that are composed in Proposition 5.1.1
ω	the "Buzzard–Lu map", defined after Corollary 5.2.7
χ	the first component of ψ, defined in Proposition 5.2.8
A, B, C	infinitesimal generators of the Lie algebra $\mathrm{sl}_2(\mathbb{C})$

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