Holomorphic flexibility properties of complements and mapping spaces

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Contents

Abstract

The classical Oka principle in complex analysis is a heuristic, supported by theorems of Oka, Grauert and others, to the effect that certain holomorphically defined problems involving Stein manifolds have only topological obstructions to their solution. Gromov's influential 1989 paper on the Oka principle introduced the class of so-called *elliptic manifolds*, and gave an Oka principle for maps from Stein manifolds into elliptic manifolds.

More recently, Forstnerič and Lárusson have introduced the category of *Oka manifolds* and *Oka maps*, which fit naturally into an abstract homotopy-theoretic framework; every elliptic manifold in Gromov's sense is also an Oka manifold. Examples of Oka manifolds are complex Lie groups and their homogeneous spaces (which are also elliptic); the complement in $\mathbb{P}^n$ of an algebraic subvariety of codimension at least 2; Hirzebruch surfaces; and more generally any fibre bundle whose base and fibre are Oka.

The Oka property can be thought of as a sort of anti-hyperbolicity; the notion of Kobayashi hyperbolicity expresses a type of holomorphic rigidity, and conversely, Oka manifolds are those that enjoy a high degree of holomorphic flexibility. Other flexibility properties enjoyed by Oka manifolds include strong dominability: for every Oka manifold X and every $p \in X$ there exists a holomorphic map $\mathbb{C}^n \to X$ which maps 0 to p and is a submersion at 0; and $\mathbb{C}$-connectedness: every pair of points can be joined by an entire curve.

The aim of this thesis is to provide new examples of Oka manifolds, and to shed light on the relationship between the Oka property and other types of holomorphic flexibility. Naturally occurring candidates for examples include complements of hypersurfaces in $\mathbb{P}^n$, especially low-degree or non-algebraic hypersurfaces (in contrast with Kobayashi's conjecture that the complement of a generic high-degree hypersurface should be hyperbolic), and spaces of holomorphic maps.

This thesis contains three chapters. The first chapter outlines the historical development of Oka theory, gives an overview of the remaining chapters, and suggests some directions for future research.

Chapter 2 is a paper entitled *Oka properties of some hypersurface complements*, to appear in Proceedings of the American Mathematical Society. There are two main results: a characterisation of when a

complement in $\mathbb{P}^n$ of hyperplanes is Oka, and the result that the complement of the affine graph of a meromorphic function is Oka, subject to some restrictions. The proof of the second result involves an extension to Gromov's technique of localisation of algebraic subellipticity.

Chapter 3 is a paper entitled *Holomorphic flexibility properties of the space of cubic rational maps*. Define R_d to be the space of rational functions of degree d on the Riemann sphere. Geometric invariant theory can be used to explore the structure of R_d: the Möbius group acts on R_d by precomposition and postcomposition. The two-sided action on R_2 is transitive, implying that R_2 is an Oka manifold. The action on R_3 has $\mathbb{C}$ as its categorical quotient; Section 3.4 gives an explicit formula for the quotient map and describes its structure in some detail. Furthermore, R_3 is strongly dominable and $\mathbb{C}$-connected.

Signed statement

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Published or submitted works included in this thesis:

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