Chiral-Scale Perturbation Theory About an Infrared Fixed Point

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“An idea that is not dangerous is unworthy of being called an idea at all."

— Oscar Wilde: *The Critic as Artist*

Abstract

This work explores the infrared behaviour of the strong running coupling α_s in Quantum Chromodynamics (QCD). We propose that α_s runs non-perturbatively to an infrared fixed point α_{IR} for three light quark flavours u, d, s. At the fixed point, we show that the quark condensate spontaneously breaks scale and chiral $SU(3)_L \times SU(3)_R$ symmetry. Consequently, the low-lying spectrum contains nine pseudo-Nambu-Goldstone bosons: π, K, η and a scalar-isoscalar QCD dilaton σ. We argue that σ may be identified with the $f_0(500)$ resonance, a pole at a complex mass with real part $\lesssim m_K$. For low-energy expansions in α_s about α_{IR}, we replace chiral $SU(3)_L \times SU(3)_R$ perturbation theory with a new model-independent theory $\chi\mathrm{PT}_\sigma$ based on approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry.

We examine the phenomenological consequences which arise from this framework by constructing effective Lagrangians which simulate strong, weak, and electromagnetic interactions. We also study the convergence properties of the effective theory, wherein we find that $\chi\mathrm{PT}_\sigma$ converges much better than $\chi\mathrm{PT}_3$ in the presence of both scalar-isoscalar channels and $O(m_K)$ extrapolations in momentum. We achieve this without spoiling the successful leading order predictions of $\chi\mathrm{PT}_3$ elsewhere.

In our phenomenological investigations, we show that the $\Delta I = 1/2$ rule for non-leptonic K-decays emerges as a consequence of $\chi\mathrm{PT}_\sigma$, with a $K_S\sigma$ coupling fixed by data for $\gamma\gamma \to \pi\pi$ and $K_S \to \gamma\gamma$. This constitutes our most important result.

We also apply the electromagnetic trace anomaly to QCD at the infrared fixed point and obtain the estimate $R_{\mathrm{IR}} \approx 5$ for the non-perturbative Drell-Yan ratio $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ at α_{IR}.

Statement of Originality

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968. I also give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library catalogue and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

List of publications, workshop proceedings, and presentations based on this thesis.

- R. J. Crewther and L. C. Tunstall, *"Origin of the $\Delta I = 1/2$ Rule for Kaon Decays: QCD Infrared Fixed Point"*, arXiv:1203.1321 [hep-ph] (submitted to Physical Review D).
- R. J. Crewther and L. C. Tunstall, *"Infrared Fixed Point in the Strong Running Coupling: Unraveling the $\Delta I = 1/2$ Puzzle in K-Decays"*, arXiv:1306.4445 [hep-ph] (Contribution to the proceedings of the workshop "Determination of the Fundamental Parameters of QCD",

Nanyang Technological University, Singapore, March 18-22, 2013, to be published in Mod. Phys. Lett. A).

- L. C. Tunstall, *"QCD Dilatons and the Origin of the* $\Delta I = 1/2$ *Rule"*, talk and poster presented at CoEPP Summer School and Workshop, Lorne, VIC, Australia, February 20-24, 2012.
- L. C. Tunstall, *"Origin of the* $\Delta I = 1/2$ *Rule for Kaon Decays: QCD Infrared Fixed Point"*, talks given at Universidad de los Andes and Universidad Nacional de Colombia, Bogotá, Colombia, August 1-2, 2012.

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Contents

Chapter 1

Prelude

There is nowadays a consensus of theoretical and experimental evidence that Quantum Chromodynamics (QCD), the $SU(3)$ gauge theory of coloured quarks and gluons, may be regarded as *the* theory of the strong interactions. At energies much higher than the scale of hadrons ~ 1 GeV, the theory becomes asymptotically free and perturbation theory in the strong running coupling $\alpha_s = g^2/4\pi$ is applicable. In this domain, QCD explains a wide range of phenomena including jets, scaling violations in deep inelastic scattering, and the production of vector bosons in Drell-Yan processes. At low-energy scales $\mu \ll m_{t,c,b}$ however, the situation is considerably more complex: the theory becomes strongly coupled, the physical states (hadrons) are comprised of quarks and gluons confined in the form of $SU(3)$ colour singlet states,[1] and chiral symmetry for the light quarks u, d, s is believed to be spontaneously broken by the formation of a quark condensate $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$. Under these conditions, the relationship between the elementary degrees of freedom and the physical spectrum cannot be discerned through perturbative expansions in α_s, and thus non-perturbative techniques are required.

One such technique (which will be analysed in Chapter 2 of this thesis) is chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT_3, which is the relevant framework to undertake a systematic analysis of the low-energy Green functions and scattering amplitudes of QCD. As an effective field theory, χPT_3 describes the interactions between the pseudoscalar octet of mesons π, K, η, and is easily extended to include

[1]Furthermore, there are no thresholds for the excitation of coloured bound states of quarks and gluons.

octet baryons. The light mass of these mesons — relative to some typical hadronic scale like the mass of the ρ meson or proton — identifies π, K, η as the pseudo-Nambu-Goldstone bosons of spontaneously broken chiral $SU(3)_L \times SU(3)_R$ symmetry. Consequently, low-energy amplitudes involving these states may be expanded as an asymptotic series in powers of $O(m_K)$ momentum and light quark masses $m_{u,d,s} = O(m_K^2)$, with $m_{u,d}/m_s$ held fixed. A convenient method to calculate the terms in this series is to construct the most general effective Lagrangian consistent with the underlying symmetries of QCD. This forms the starting point of χPT$_3$ analyses, and the method has made considerable progress in charting the low-energy structure of QCD.

It has been observed [1] however, that χPT$_3$ expansions are afflicted with a peculiar malady, *viz.*, they typically diverge whenever both the $J^{PC} = 0^{++}$ channel is present and $O(m_K)$ extrapolations in momentum are made. Well studied examples include the branching ratio $\mathcal{B}(K_L \to \pi^0\gamma\gamma)$, with a chiral one-loop prediction [2] which is approximately a factor of three smaller than the measured value [3]; the cross-section for $\gamma\gamma \to \pi^0\pi^0$ with $O(m_K)$ momentum, where the leading order prediction [4, 5] of a linear rise in energy from threshold is not compatible with the data from the Crystal Ball experiment [6]; and the dominance of $\pi\pi$ interactions [7, 8] in the final-state of K_{e4} decays [9] and non-leptonic K [10, 11] and η [12, 13] decays. The problem is that in order to obtain a good representation of the data, the next-to-leading order amplitudes must be *larger* than the leading order prediction — the hallmark of a diverging series. The discrepancy is extreme for $K_S \to \pi\pi$, where in order to explain the observed factor of 22 enhancement in the $I = 0$ channel ($\Delta I = 1/2$ rule), a scalar amplitude generated at next-to-leading order must be 70 times the expected $\lesssim 30\%$ correction.

In this thesis, we solve the convergence problem of χPT$_3$ expansions by modifying the *leading order* of the three-flavour theory [14]. Our solution is based on an old idea [15, 16] that the chiral condensate $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$ may also be a condensate for scale transformations in the chiral $SU(3) \times SU(3)$ limit. In Chapter 3, we show that this scenario can occur in QCD if at low-energies, heavy quarks t, b, c decouple and α_s of the resulting three-flavour theory runs *non-perturbatively* to an infrared fixed point α_{IR}. At the fixed point, the QCD β-function vanishes, so the gluonic term $\sim G^a_{\mu\nu} G^{a\mu\nu}$ in the strong trace anomaly is absent at α_{IR}. It follows that in the chiral

$SU(3)_L \times SU(3)_R$ limit, $\langle \bar{q}q \rangle_{\rm vac}$ induces *nine* Nambu-Goldstone bosons: π, K, η and a 0^{++} QCD dilaton σ. With a mass set by the strange quark, the obvious candidate for σ is the $f_0(500)$ resonance, which arises as a pole of complex mass with real part $\lesssim m_K$ [17, 18].

In Chapter 4, we discuss the construction of a model-independent chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$ for low-energy amplitudes expanded in α_s about $\alpha_{\rm IR}$. Its effective Lagrangian summarises soft-$\{\pi, K, \eta, \sigma\}$ meson theorems for approximate chiral $SU(3)_L \times SU(3)_R$ and scale symmetry, with results for strong interactions similar to those found originally [15, 16]. Effective weak operators are then added to simulate non-leptonic K-decays. The main result is a simple and (in our view) appealing explanation of the $\Delta I = 1/2$ rule for kaon decays: In *lowest* order of $\chi\mathrm{PT}_\sigma$, there is a dilaton pole diagram $K_S \to \sigma \to \pi\pi$ which accounts for the dominant $I = 0$ amplitude. This allows the direct **8** and **27** $K\pi\pi$ vertex amplitudes g_8 and g_{27} to be of *similar* magnitude.

We are also able to extract the $\sigma\gamma\gamma$ coupling from $\gamma\gamma \to \pi^0\pi^0$ and relate it to the electromagnetic trace anomaly [19, 20] at the QCD infrared fixed point $\alpha_s = \alpha_{\rm IR}$. Our conclusions for $\gamma\gamma \to \pi^0\pi^0$ and $K_S \to \pi\pi$ with $g_8 \sim g_{27}$ are consistent with the standard explanation [21, 22] of $K_S \to \gamma\gamma$.

We conclude in Chapter 5 with a summary of results and a discussion on the significance of the work and the limitations of $\chi\mathrm{PT}_\sigma$. Directions for future research are presented.

Chapter 2

Chiral Perturbation Theory

In this chapter we discuss the salient features of chiral $SU(3)_L \times SU(3)_R$ perturbation theory $\chi\mathrm{PT}_3$, with digressions to the $SU(2)_L \times SU(2)_R$ theory $\chi\mathrm{PT}_2$ where required. The intent is to establish our notation and conventions, and most importantly, examine the underlying principles upon which $\chi\mathrm{PT}_3$ is based. Section 2.1 is a review of well known facts about chiral symmetry. In Section 2.2 we discuss the notion of perturbing about a Nambu-Goldstone symmetry and note the connection to effective chiral Lagrangians. Section 2.3 reviews the construction of the lowest order Lagrangian for the strong interactions. The functional methods pioneered by Gasser and Leutwyler [13] are reviewed in Section 2.4 and the next-to-leading order Lagrangian is presented. This is followed by a discussion on the scale which governs the convergence of $\chi\mathrm{PT}_3$ expansions. We review the construction of the weak effective Lagrangian in Section 2.5 and emphasise the role of vacuum alignment in eliminating $|K\rangle \to |\mathrm{vac}\rangle$ amplitudes in leading order. (This minimisation procedure plays an important role in Chapter 4 where we present our original research.) The oldest problem in particle physics — the $\Delta I = 1/2$ puzzle in non-leptonic decays — is reviewed. We conclude in Section 2.6 with a discussion on the lightest scalar-isoscalar resonance $f_0(500)$ and draw attention to the fact that $\chi\mathrm{PT}_3$ expansions *diverge* when both this channel is present and $O(m_K)$ extrapolations in momentum are made.

2.1 Chiral Symmetry

Chiral symmetry plays an important role in making sense of QCD at low energy scales $\mu \ll m_{t,b,c}$, where the three-flavour theory is described by the Lagrangian[1]

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{q=u,d,s} \bar{q}(i\not{D} - m_q)q \\ &= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{q=u,d,s} \{\bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - m_q(\bar{q}_L q_R + \bar{q}_R q_L)\}\,. \end{aligned} \tag{2.1}$$

Here $q_{L/R} = \frac{1}{2}(1 \mp \gamma_5)q$ and $[D_\mu, D_\nu] = igG^a_{\mu\nu}T^a$, where D_μ is the covariant derivative and $\{T^a\}$ generate the colour $SU(3)$ gauge group. In the limit where the quark masses vanish $m_q \to 0$, the left- and right-handed quark fields decouple from one another

$$\mathcal{L} \to \mathring{\mathcal{L}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{q=u,d,s} \{\bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R\}\,, \tag{2.2}$$

and thus QCD becomes invariant under the *global* $SU(3)_L \times SU(3)_R$ transformations

$$q_L \to Lq_L\,, \quad L \in SU(3)_L \qquad \text{and} \qquad q_R \to Rq_R \quad R \in SU(3)_R\,. \tag{2.3}$$

The conserved Noether currents associated with $SU(3)_L \times SU(3)_R$ are

$$J^{i\mu}_L = \tfrac{1}{2}\bar{q}_L\gamma^\mu\lambda^i q_L\,, \qquad \text{and} \qquad J^{i\mu}_R = \tfrac{1}{2}\bar{q}_R\gamma^\mu\lambda^i q_R\,, \qquad i = 1,\ldots,8\,, \tag{2.4}$$

where λ^i are the Gell-Mann matrices with normalization $\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}$. If we define the vector- and axial-vector currents $V^{i\mu}$ and $A^{i\mu}$ through the linear combinations

$$V^{i\mu} = J^{i\mu}_L + J^{i\mu}_R = \tfrac{1}{2}\bar{q}\gamma^\mu\lambda^i q\,, \tag{2.5}$$

$$A^{i\mu} = J^{i\mu}_L - J^{i\mu}_R = \tfrac{1}{2}\bar{q}\gamma^\mu\gamma_5\lambda^i q\,, \tag{2.6}$$

[1] Ghost and gauge-fixing terms are understood. We have also neglected to include the θ parameter associated with the strong-CP problem as this term is known to be extremely small $|\theta| < 10^{-10}$ from measurements of the electric-dipole moment of the neutron [23].

then the chiral charges $Q_V^i = \int \mathrm{d}^3x\, V^{i0}$ and $Q_A^i = \int \mathrm{d}^3x A^{i0}$ form a closed Lie algebra,

$$[Q_V^i, Q_V^j] = i f^{ijk} Q_V^k\,, \qquad [Q_A^i, Q_A^j] = i f^{ijk} Q_V^k\,, \qquad [Q_V^i, Q_A^j] = i f^{ijk} Q_A^k\,. \tag{2.7}$$

In the 1960s, these commutation relations were the cornerstone of current algebra analyses [24] and were exploited along with general principles like partial conservation of $A^{i\mu}$ (PCAC) at a time when there was no field theory of strong interactions. Then as now, it was evident that the chiral symmetry generated by (2.7) is not manifestly realised in the hadronic spectrum. The observed hadrons could be arranged into multiplets of approximate flavour $SU(3)_V$ symmetry, but the odd-parity partners were (are) conspicuously absent. It was also noted that the octet of pseudoscalar mesons π, K, η were much lighter than other hadrons like the ρ meson or the proton.

To reconcile these empirical facts, one concludes that the axial transformations generated by Q_A^i are not a symmetry of the ground state, and thus the chiral $SU(3)_L \times SU(3)_R$ symmetry is spontaneously broken

$$SU(3)_L \times SU(3)_R \to SU(3)_V\,. \tag{2.8}$$

In QCD, this pattern of spontaneous symmetry-breaking is understood to be due to the formation of a quark condensate

$$\lim_{m_q \to 0} \langle [Q_A^i, \bar{q}\gamma_5 \lambda^i q] \rangle_{\mathrm{vac}} \propto \langle \bar{q}q \rangle_{\mathrm{vac}} \neq 0\,, \qquad q = u, d, s\,. \tag{2.9}$$

Since there are eight broken axial charges, Goldstone's theorem implies there should be a massless $SU(3)_V$ octet of Nambu-Goldstone (NG) bosons with odd-intrinsic parity and zero baryon number. The obvious candidates for these states are π, K, η, whose small but non-zero masses can be understood as due to an explicit breaking of $SU(3)_L \times SU(3)_R$ symmetry through the quark masses $m_{u,d,s}$. Naturally, all of this was well known to practitioners of current algebra, but the calculations became prohibitively complex as the number of external legs or intermediate states increased. It was clear that a method to analyse corrections to the low-energy theorems was needed.

Such a method exists and is known as chiral $SU(3)_L \times SU(3)_R$ perturbation theory[1] χPT$_3$. It is an *effective field theory* which describes the low-energy interactions of π, K, η in a systematic and precise manner. Despite the name, χPT$_3$ is a non-perturbative technique for it does not rely on expansions in the strong running coupling $\alpha_s = g^2/4\pi$. Instead, the method relies on the notion of perturbation theory about a NG-symmetry [30, 31, 32, 33], *viz.,* low-energy scattering amplitudes and matrix elements can be described by an asymptotic series[2]

$$\mathcal{A} = \{\mathcal{A}_{\mathrm{LO}} + \mathcal{A}_{\mathrm{NLO}} + \mathcal{A}_{\mathrm{NNLO}} + \ldots\} \tag{2.10}$$

in powers and logarithms of $O(m_K)$ momentum and quark masses $m_{u,d,s} = O(m_K^2)$, with $m_{u,d}/m_s$ held fixed. The amplitudes may be calculated from an effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$, whose construction is guided by a "folk" theorem due to Weinberg [34]:

> **Theorem.** *For a given set of asymptotic states, the most general Lagrangian containing all terms allowed by the assumed symmetries will yield in perturbation theory the most general S-matrix elements consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles.*

A proof for the general $SU(N_f)_L \times SU(N_f)_R$ case has been provided by Leutwyler [35]. The remainder of this chapter concerns the construction of $\mathcal{L}_{\mathrm{eff}}$ for the low-energy interactions of the Standard Model of particle physics.

2.2 Perturbations about a Goldstone Symmetry

A large part of this thesis is concerned with the structure of χPT expansions of the type given by Eq. (2.10). The purpose of the present section is to review the main ideas which give rise to the notion that NG-symmetries may be studied as perturbations in some symmetry breaking parameter. Following Dashen and Weinstein [32] (see also [25]) we consider the case of chiral $SU(2)_L \times SU(2)_R$ symmetry in the

[1] For reviews see e.g. [25, 26, 27, 28, 29].

[2] Recall that a series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ constructed from a sequence of functions $\{\varphi_n\}$ gives an asymptotic expansion of the function $f(x)$ if for every N, we have $f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = O(\varphi_{N+1})$ as $x \to 0$.

isospin limit $m_u = m_d$. (The generalisation to the $SU(3)_L \times SU(3)_R$ case is straightforward.) In general, the QCD Hamiltonian is of the form

$$\mathcal{H} = \mathcal{H}_{\text{inv}} + \hat{m}\mathcal{H}' , \tag{2.11}$$

where $\mathcal{H}_{\text{inv}}$ is $SU(2)_L \times SU(2)_R$ invariant while the explicit symmetry-breaking term $\mathcal{H}' = \int \mathrm{d}^3x(\bar{u}\,u + \bar{d}\,d)$ belongs to the $(2,\bar{2}) \oplus (\bar{2},2)$ representation. Since the explicit breaking parameter $\hat{m} = \frac{1}{2}(m_u + m_d)$ is small relative to the mass scale of (say) the ρ meson, we seek a perturbative expansion in $\hat{m}$ about the chiral limit $\hat{m} \to 0$. There is however, a complication since the physical spectrum contains a triplet of massless NG bosons π^+, π^0, π^- in this limit. This is a problem because the Gell-Mann-Oakes-Renner relation $m_\pi^2 \propto \hat{m}$ [36] implies that the expansion parameter appears in matrix elements involving the divergence of the axial-vector current $\partial^\mu A^i_\mu = m_\pi^2 F_\pi \pi$. As a result, there is an ambiguity in the ordering of terms such as

$$\frac{m_\pi^2 F_\pi}{q^2 - m_\pi^2} \sim \begin{cases} O(\hat{m}) & \text{for} \quad q^2 \gg m_\pi^2 , \\ O(1) & \text{for} \quad 0 < q^2 \ll m_\pi^2 . \end{cases} \tag{2.12}$$

Dashen and Weinstein [32] showed that by carefully removing the π-poles, one can resolve this ambiguity, and thus construct a well defined perturbation theory. A simple example [25, 32] suffices to show how such a procedure works. First, we denote the Fourier transform of the axial-vector current and its divergence as

$$A^i_\mu(q) \equiv \int \mathrm{d}^4x\, e^{iq\cdot x} A^i_\mu(x), \qquad \text{and} \qquad \mathrm{d}A^i(q) \equiv \int \mathrm{d}^4x\, e^{iq\cdot x} \partial^\mu A^i_\mu(x). \tag{2.13}$$

Evidently, matrix elements between hadronic states α, β obey the relation

$$\langle\alpha|\mathrm{d}A^i(q)|\beta\rangle = -iq^\mu \langle\alpha|A^i_\mu(q)|\beta\rangle , \tag{2.14}$$

both sides of which contain poles due to the pions,

$$\langle\alpha|\mathrm{d}A^i(q)|\beta\rangle\Big|_{\text{pole}} = \frac{i m_\pi^2 F_\pi}{q^2 - m_\pi^2} \langle\alpha + \pi^i(q)|S|\beta\rangle , \tag{2.15}$$

$$\langle\alpha|A^i_\mu(q)|\beta\rangle\Big|_{\text{pole}} = -\frac{q_\mu F_\pi}{q^2 - m_\pi^2} \langle\alpha + \pi^i(q)|S|\beta\rangle , \tag{2.16}$$

for scattering matrix S. Subtracting the poles from both sides of Eq. (2.14), one finds for the remainder

$$F_\pi \langle \alpha + \pi^i(q)|S|\beta\rangle = q^\mu \, \langle \alpha|A^i_\mu(q)|\beta\rangle\Big|_{\text{rem.}} + \langle \alpha|\mathrm{d}A^i|\beta\rangle\big|_{\text{rem.}}\,, \tag{2.17}$$

and thus the second term on the right-hand side is $O(\hat{m})$ unambiguously. In general, the matrix element $\langle \alpha|A^i_\mu(q)|\beta\rangle\Big|_{\text{rem.}}$ may have an $O(q^{-1})$ singularity as $q_\mu \to 0$, but the prefactor of q^μ ensures that this term is either calculable or vanishes in the $\hat{m} \to 0$ limit. The result is a soft-π theorem,

$$\lim_{\substack{q\to 0\\ \hat{m}\to 0}} F_\pi \langle \alpha + \pi^a(q)|S|\beta\rangle = \{\text{calculable}\} + O(q^2) + O(\hat{m})\,. \tag{2.18}$$

As shown by Dashen and Weinstein [32], and later made systematic by Weinberg [34] and others [13, 37], the above low-energy theorem (2.18) can also be obtained by constructing the most general effective Lagrangian allowed by chiral $SU(2)_L \times SU(2)_R$ symmetry up to the desired order in q^2 and $\hat{m}$, and computing the relevant amplitude in the tree approximation. Because of its convenience, it is the latter approach which has become the standard for performing calculations and is commonly referred to as chiral $SU(2)_L \times SU(2)_R$ perturbation theory $\chi\mathrm{PT}_2$.

2.3 Effective Lagrangians for Strong Interactions

To examine the consequences of the assumed spontaneous symmetry-breaking pattern

$$SU(3)_L \times SU(3)_R \to SU(3)_V\,, \tag{2.19}$$

it is convenient to introduce an $SU(3)$ field $U = U(\phi)$, where the octet of pseudoscalar NG bosons ϕ_i parametrise the coset space $(SU(3)_L \times SU(3)_R)/SU(3)_V$ with group action

$$U \to RUL^\dagger\,, \quad \text{where} \qquad R \in SU(3)_R \quad \text{and} \quad L \in SU(3)_L\,. \tag{2.20}$$

For stable vacua $\langle U \rangle_{\text{vac}} = I$, the ϕ_i can be treated as field fluctuations, to wit $U = \exp(i\lambda_i\phi_i/F_\pi)$, where $F_\pi \simeq 93$ MeV is the pion decay constant[1] and

$$\frac{\lambda_i\phi_i}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}. \tag{2.21}$$

In accord with the dictates of Weinberg's theorem [34], one constructs the most general effective Lagrangian $\mathcal{L}_{\text{eff}}$ consistent with the symmetries of QCD. In χPT$_3$, this is achieved by arranging the terms in $\mathcal{L}_{\text{eff}}$ in increasing powers of derivatives $\partial = O(m_K)$ and $O(m_q) = O(m_K^2)$ quark masses,[2]

$$\mathcal{L}_{\text{eff}}[U, U^\dagger] = \sum_{n=1}^{\infty} \mathcal{L}_{2n} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots. \tag{2.22}$$

In the leading order (LO) of χPT$_3$, the strong interactions of π, K, η mesons are described by

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \text{Tr}(MU^\dagger + UM^\dagger), \tag{2.23}$$

where the second term on the right-hand side belongs to the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation, M is proportional to the quark mass matrix

$$M = \tfrac{1}{2}F_\pi^2 B_0 \,\text{diag}(m_u, m_d, m_s), \tag{2.24}$$

and the low-energy constant B_0 is related to the quark condensate

$$-F_\pi^2 B_0 = \langle \bar{u}u \rangle_{\text{vac}} = \langle \bar{d}d \rangle_{\text{vac}} = \langle \bar{s}s \rangle_{\text{vac}}. \tag{2.25}$$

From the chiral symmetry-breaking term in (2.23) follow a number of important results, including the Gell-Mann-Oakes-Renner relations [36] which relate the quark

[1] As a low-energy constant, F_π is not fixed by chiral symmetry. The quoted value is obtained from the leptonic weak decay $\pi^+ \to \mu^+\nu_\mu$.

[2] The absence of odd-numbers of derivatives is a simple consequence of the intrinsic odd-parity of the NG bosons ϕ_i.

and pseudo-NG boson masses,

$$\begin{aligned}
&m_{\pi^\pm}^2 = 2\hat{m}\,B_0\,, && m_{\pi^0}^2 = 2\hat{m}\,B_0 - \epsilon + O(\epsilon^2)\,,\\
&m_{K^\pm}^2 = (m_u + m_s)B_0\,, && m_{K^0}^2 = (m_d + m_s)B_0\,,\\
&m_\eta^2 = \tfrac{2}{3}(\hat{m} + 2m_s)B_0 + \epsilon + O(\epsilon^2)\,,
\end{aligned} \tag{2.26}$$

where

$$\hat{m} = \tfrac{1}{2}(m_u + m_d)\,, \qquad \epsilon = \frac{B_0}{4}\frac{(m_u - m_d)^2}{m_s - \hat{m}}\,. \tag{2.27}$$

Evidently, in the chiral $SU(3)_L \times SU(3)_R$ limit the meson masses vanish and π, K, η become genuine NG bosons. By combining Eq. (2.25) with the expression for $m_{\pi^\pm}^2$ in (2.26), it is possible to relate the meson and quark masses directly to quark condensate [36]:

$$F_\pi^2 m_{\pi^\pm}^2 = -\hat{m}\langle \bar{u}\,u + \bar{d}\,d\rangle_{\text{vac}}\,. \tag{2.28}$$

Similarly, if we neglect the tiny $O(\epsilon)$ isospin-breaking effects in (2.26), the Gell-Mann-Okubo sum rule [38, 39]

$$3m_\eta^2 = 4m_K^2 - m_\pi^2 \tag{2.29}$$

is obtained, which predicts the physical η mass to within 4% and provides a stringent test on the mode of quark condensation [40].

2.4 Functional Methods and Gauge Interactions

The most convenient way to simulate electromagnetic (or weak semi-leptonic) interactions in χPT$_3$ is to make use of the background field method [41, 42]. Pioneered in χPT$_3$ by Gasser and Leutwyler [13, 37], this formalism has the added benefit of providing a) the means to calculate Green functions and their associated Ward identities in a manifestly chiral invariant way, and b) a method to study higher-order unitarity corrections in a systematic manner.

The starting point is to extend the chiral symmetric Lagrangian $\mathring{\mathcal{L}}$ of Eq. (2.2) to include Hermitian matrix-valued external fields $v_\mu(x), a_\mu(x), s(x), p(x)$,

$$\mathring{\mathcal{L}} \to \mathcal{L}_{\text{ext}} = \mathring{\mathcal{L}} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - ip\gamma_5)q\,. \tag{2.30}$$

In the presence of these external fields, the generating functional $Z[v,a,s,p]$ is defined in terms of the transition amplitude from the vacuum state in the asymptotic past $|\Omega_{\text{in}}\rangle$ to the vacuum state in the asymptotic future $|\Omega_{\text{out}}\rangle$,

$$\begin{aligned}\exp\{iZ[v,a,s,p]\} &= \langle\Omega_{\text{out}}|\Omega_{\text{in}}\rangle_{v,a,p,s,} \\ &= \int [Dq][D\bar{q}][DA_\mu]\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{ext}}\right\}. \end{aligned} \tag{2.31}$$

If one perturbs Z about the point $v_\mu = a_\mu = s = p = 0$, then $\mathcal{L}_{\text{ext}}$ reduces to $\mathring{\mathcal{L}}$, and thus the chiral-limit Green functions of QCD are obtained. On the other hand, it is a simple matter to simulate non-zero quark masses *and* electroweak gauge interactions by setting $p = 0$ and expanding about

$$r_\mu = v_\mu + a_\mu = -eQA_\mu + \dots, \tag{2.32}$$

$$l_\mu = v_\mu - a_\mu = -eQA_\mu - \frac{e}{\sqrt{2}\sin\theta_W}(W_\mu^+ T_+ + \text{h.c.}) + \dots, \tag{2.33}$$

$$s = \text{diag}(m_u, m_d, m_s) + \dots, \tag{2.34}$$

where Q is the quark-charge matrix and the relevant Cabibbo-Kobayashi-Maskawa (CKM) matrix elements V_{ij} are included within T_+:

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.35}$$

Note that the external field formalism formally promotes $SU(3)_L \times SU(3)_R$ to a *local* symmetry, so that R, L become spacetime dependent matrices. Consequently, the left- and right-handed gauge fields l_μ and r_μ transform non-covariantly under $SU(3)_L \times SU(3)_R$,

$$r_\mu \to R r_\mu R^\dagger + iR\partial_\mu R^\dagger, \tag{2.36}$$

$$l_\mu \to L l_\mu L^\dagger + iL\partial_\mu L^\dagger, \tag{2.37}$$

$$s + ip \to R(s+ip)L^\dagger. \tag{2.38}$$

This leads to the introduction of a covariant derivative

$$\nabla_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu , \qquad \nabla_\mu U \to R \nabla_\mu U L^\dagger , \tag{2.39}$$

and field strength tensors

$$f_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] , \tag{2.40}$$
$$f_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] . \tag{2.41}$$

At low energies, the Green functions of QCD are expanded in powers of the external momenta and quark masses. In the path integral representation (2.31), this is equivalent to expanding in derivatives of the external fields. It follows that the low-energy representation of Z is given by [13]

$$e^{iZ} = \int [DU] \exp\left\{ i \int \mathrm{d}^4 x\, \mathcal{L}_{\text{eff}} \right\} , \tag{2.42}$$

and thus the generating functional factorises

$$Z = Z_2 + Z_4 + \dots , \tag{2.43}$$

where $Z_2 = \int \mathrm{d}^4 x\, \mathcal{L}_2$ denotes the classical action. The locally invariant Lagrangian $\mathcal{L}_2$ is obtained from Eq. (2.23) by replacing $\partial \to \nabla$ and $M \to \chi \equiv \frac{1}{2} F_\pi^2 B_0 (s + ip)$:

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \mathrm{Tr}(\nabla_\mu U \nabla^\mu U^\dagger) + \mathrm{Tr}(\chi U^\dagger + U \chi^\dagger) . \tag{2.44}$$

Electromagnetic transitions can then be calculated in LO by simply expanding the external fields in $\mathcal{L}_2$ about the physical point (2.34).

2.4.1 Next-to-Leading Order Effects

In the tree approximation, one can simply read off the scattering amplitudes $\mathcal{A}$ by inspection of vertices contained in $\mathcal{L}_2$. However, these amplitudes are purely real and thus to preserve the unitarity of the theory

$$\mathfrak{Im}\,\mathcal{A} \sim |\mathcal{A}|^2, \tag{2.45}$$

chiral loop diagrams are required at next-to-leading order (NLO) in the derivative and quark mass expansion (2.22). Indeed, the non-renormalizable nature of $\mathcal{L}_{\text{eff}}$ manifests itself through the fact that at NLO, new vertices different from those of $\mathcal{L}_2$ are introduced into the effective theory. Therefore, the full theory requires an infinite set of counterterms. Nevertheless, at a *fixed-order* in χPT_3 the divergences can be absorbed through appropriate counterterms, and the truncated theory renormalized. It is here that the functional methods described above become indispensable.

At NLO in χPT_3, the generating functional Z receives contributions from the following.

(a) One-loop graphs constructed from vertices in $\mathcal{L}_2$.

(b) The most general effective Lagrangian $\mathcal{L}_4$ in the tree approximation.

(c) The Wess-Zumino-Witten (WZW) construction of the chiral anomaly [43, 44].

The explicit form of contributions (a) and (b) have been calculated by Gasser and Leutwyler [13], with the result for the latter given by

$$\begin{aligned}
\mathcal{L}_4 = {} & L_1 \text{Tr}(\nabla_\mu U \nabla^\mu U^\dagger)^2 + L_2 \text{Tr}(\nabla_\mu U \nabla_\nu U^\dagger)\text{Tr}(\nabla^\mu U \nabla^\nu U^\dagger) \\
& + L_3 \text{Tr}(\nabla_\mu U \nabla^\mu U^\dagger \nabla_\nu U \nabla^\nu U^\dagger) + L_4 \text{Tr}(\nabla_\mu \nabla^\mu U^\dagger)\text{Tr}(\chi U^\dagger + U\chi^\dagger) \\
& + L_5 \text{Tr}(\nabla_\mu U \nabla^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)) + L_6 \text{Tr}(\chi U^\dagger + U\chi^\dagger)^2 \\
& + L_7 \text{Tr}(\chi U^\dagger - U\chi^\dagger)^2 + L_8 \text{Tr}(U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger) \\
& - i L_9 \text{Tr}(f^R_{\mu\nu} \nabla^\mu U \nabla^\nu U^\dagger + f^L_{\mu\nu} \nabla^\mu U^\dagger \nabla^\nu U) + L_{10} \text{Tr}(U f^L_{\mu\nu} U^\dagger f^{\mu\nu}_R) \\
& + H_1 \text{Tr}(f^R_{\mu\nu} f^{\mu\nu}_R + f^L_{\mu\nu} f^{\mu\nu}_L) + H_2 \text{Tr}(\chi\chi^\dagger).
\end{aligned} \tag{2.46}$$

The numerical values of the low-energy constants L_i are not determined by chiral symmetry alone and parametrize our limitations in solving the dynamics of QCD directly. They have been determined either empirically [13], through models which resemble QCD [45], via meson-resonance saturation [46], or on the lattice [47]. The external field terms involving H_i are of no physical relevance [13].

For a given amplitude with L loops and N_d vertices with d powers of $O(m_K)$ momentum, it is convenient to keep track of the chiral dimension D [34],

$$D = 2L + 2 + \sum_{d\in 2\mathbb{N}} 2(d-2)N_d\,. \tag{2.47}$$

This allows us to arrange the successive terms in $\mathcal{L}_{\text{eff}}$ as follows,

$$\begin{aligned}
D=2: &\quad \{\; L=0\,,\quad d=2\,,\quad Z_2 \qquad = \int \mathrm{d}^4x\, \mathcal{L}_2\,; \\
D=4: &\quad \begin{cases} L=0\,, & d=4\,,\quad Z_4^{\text{tree}} \quad = \int \mathrm{d}^4x\, \mathcal{L}_4 + Z_{WZW}\,, \\ L=1\,, & d=2\,,\quad Z_2^{1-\text{loop}} = \int \mathrm{d}^4x\, \mathcal{L}_2^{1-\text{loop}}\,. \end{cases}
\end{aligned}$$

The effect of terms with $D \geq 6$ is less important provided the momentum of a given amplitude satisfies

$$\{\text{momentum}\}/\chi_{\text{ch}} \ll 1\,, \tag{2.48}$$

where the infrared mass scale $\chi_{\text{ch}} \approx 1$ GeV is set by the chiral condensate $\langle\bar{q}q\rangle_{\text{vac}}$. This forms the basis behind the expectation that NLO corrections should be 30% at most,

$$|\mathcal{A}_{\text{NLO}}/\mathcal{A}_{\text{LO}}| \lesssim 0.3\,. \tag{2.49}$$

From Eq. (2.47) we see that D increases with L, so to maintain a fixed mass dimension for a given amplitude, each loop comes with a factor of F_π^{-2}. Combined with the standard geometric loop factor $(4\pi)^{-2}$, this leads to an estimate for the scale of chiral symmetry-breaking [48, 49],

$$\chi_{\text{ch}} = 4\pi F_\pi \simeq 1.2\,\text{GeV}\,. \tag{2.50}$$

Note that χ_{ch} also sets the scale of hadrons which don't belong to the NG sector $\{\pi, K, \eta\}$. For example, the Goldberger-Treiman relation for the nucleon mass M_N,

$$g_{\pi NN} F_\pi \simeq g_A M_N\,, \tag{2.51}$$

clearly shows that M_N remains massive in the chiral $SU(3)_L \times SU(3)_R$ limit. Given this fact, it is odd that the scale $\Lambda_{\rm QCD} \approx 200$ MeV for ultraviolet expansions in the asymptotic free domain

$$\{\text{momentum}\}/\Lambda_{\rm QCD} \gg 1\,, \tag{2.52}$$

is commonly taken as the relevant scale for all applications of QCD. Although strong gluon fields are presumably responsible for $\Lambda_{\rm QCD}$ and $\chi_{\rm ch}$, this does not imply that the ratio

$$\chi_{\rm ch}/\Lambda_{\rm QCD} \approx 5 \tag{2.53}$$

must be unity.

2.5 Effective Lagrangians for Weak Interactions

In the electroweak sector of the standard model, the strangeness-changing $|\Delta S|=1$ non-leptonic transitions are induced by the current-current interaction[1]

$$\mathcal{L}_W(x) = \frac{G_F}{\sqrt{2}} \sum_{q=u,c,t} V^*_{qs} V_{qd} \int \mathrm{d}^4 y\, \Delta(y) T\{(\bar{s}_L \gamma^\mu q_L)_{x+y/2} (\bar{q}_L \gamma_\mu d_L)_{x-y/2}\} + \text{h.c.}\,, \tag{2.54}$$

where $G_F \simeq 10^{-5} M_p^{-2}$ is the Fermi coupling, V is the CKM matrix, and

$$\Delta(y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 q\, e^{iq\cdot y} \frac{1}{1-q^2/m_W^2} \tag{2.55}$$

is the Lorentz-scalar part of the W-boson propagator in the 't Hooft-Feynman gauge. The heavy states W, t, c, b can be decoupled from the theory at low energies $\mu \ll m_{t,c,b}$ using operator product expansion techniques. The result is that the slightly non-local interaction (2.54) is expanded in a set of local four-quark operators $\mathcal{O}_i$ and scale dependent Wilson coefficients $c_i(\mu)$,

$$\mathcal{L}_W(x) \sim \mathcal{L}^{\Delta S=1}_{\rm eff}(x,\mu) = \frac{G_F}{2\sqrt{2}} V^*_{us} V_{ud} \sum_i c_i(\mu) \mathcal{O}_i(x) + \text{h.c.}\,. \tag{2.56}$$

The precise form of the $c_i(\mu)$ and $\mathcal{O}_i(x)$ is given in [51, 52, 53].

In terms of effective degrees of freedom π, K, η, the chiral structure of the weak

[1] See e.g. [50, Sec. VIII-2] for a review.

currents constrains the number of allowed operators. Consider for example [50] the product $(\bar{s}_L\gamma^\mu u_L)(\bar{u}_L\gamma_\mu d_L)$, which contains two $SU(3)_L$ octets of differing isospin and hence the decomposition

$$SU(3)_L: \qquad (\mathbf{8}_L \otimes \mathbf{8}_L)|_{\text{sym}} = \mathbf{8}_L \oplus \mathbf{27}_L\,, \tag{2.57}$$

$$\text{Isospin}: \qquad \mathbf{1} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}} \oplus \frac{\mathbf{3}}{\mathbf{2}}\,. \tag{2.58}$$

Other products in (2.56) have either the same chiral/isospin structure as above or are pure octet and isospin-$\frac{1}{2}$. Thus $\mathcal{L}_{\text{eff}}^{\Delta S=1}$ transforms in the $(\mathbf{8}_L, \mathbf{1}_R) \oplus (\mathbf{27}_L, \mathbf{1}_R)$ representation of $SU(3)_L \times SU(3)_R$. These symmetry properties must also be reflected in the effective weak Lagrangian of χPT$_3$, to wit the LO expression

$$\mathcal{L}_{\text{weak}}[U, U^\dagger] = g_8 Q_8 + g_{27} Q_{27} + Q_{mw} + \text{h.c.} \tag{2.59}$$

contains an octet operator [54]

$$Q_8 = \mathcal{J}_{13}\mathcal{J}_{21} - \mathcal{J}_{23}\mathcal{J}_{11}\,, \quad \mathcal{J}_{ij} = (U\partial_\mu U^\dagger)_{ij} \tag{2.60}$$

the U-spin triplet component [51, 55] of a **27** operator

$$Q_{27} = \mathcal{J}_{13}\mathcal{J}_{21} + \tfrac{3}{2}\mathcal{J}_{23}\mathcal{J}_{11} \tag{2.61}$$

and a weak mass operator [56]

$$Q_{mw} = \text{Tr}(\lambda_6 - i\lambda_7)\big(g_M M U^\dagger + \bar{g}_M U M^\dagger\big)\,. \tag{2.62}$$

In general, the low-energy coefficients g_i, g_M and $\bar{g}_M$ are complex.

It is important to note that the weak mass operator Q_{mw} introduces tadpole graphs which destabilise the ground state $|\Omega\rangle$ of the theory. Crewther [55] has emphasised that the correct procedure is to consider field fluctuations about the true vacuum $|\text{vac}\rangle$, obtained by minimising the combined strong and weak interaction potential

$$\mathcal{V}[U, U^\dagger] = -\text{Tr}MU^\dagger - Q_{mw} + \text{h.c.}\,. \tag{2.63}$$

At LO in G_F, this is achieved by noting that the mass matrix

$$m = M + (g_M \lambda_{6-i7} + \bar{g}_M^* \lambda_{6+i7})M = \tfrac{1}{2} F_\pi^2 B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 2g_M \\ 0 & 2\bar{g}_M^* & m_s \end{pmatrix}, \tag{2.64}$$

can be diagonalised through a suitable chiral $SU(3)_L \times SU(3)_R$ rotation [55],

$$m = L_0 M R_0^\dagger + O(G_F^2 M). \tag{2.65}$$

As a result, the potential is given by

$$\begin{aligned} \mathcal{V} &= -\mathrm{Tr}(mU^\dagger + Um^\dagger) \\ &= -\mathrm{Tr}(M\tilde{U}^\dagger + \tilde{U}M^\dagger) + O(G_F^2 M), \end{aligned} \tag{2.66}$$

where $\tilde{U}$ is chirally rotated with respect to U,

$$\tilde{U} = L_0^\dagger U R_0. \tag{2.67}$$

Evidently, it is the rotated field $\tilde{U}$ which satisfies $\langle \tilde{U} \rangle_{\text{vac}} = I$ and thus allows us to adopt the parametrisation

$$\tilde{U}(\phi) = \exp(i\lambda_i \phi_i / F_\pi), \qquad \text{where} \qquad \langle \phi_i \rangle_{\text{vac}} = 0. \tag{2.68}$$

Note that in the true ground state $|\text{vac}\rangle$, all trace of Q_{mw} has been removed from $\mathcal{V}$ at LO, i.e. one has the no-tadpoles theorem [55]

$$\langle K | \mathcal{H}_{\text{weak}} | \text{vac} \rangle = O(m_s^2 - m_d^2). \tag{2.69}$$

Thus, in the LO of $\chi\mathrm{PT}_3$ there are only *two* effective operators which contribute to physical processes,

$$\mathcal{L}_{\text{weak}} \to \tilde{\mathcal{L}}_{\text{weak}}[\tilde{U}, \tilde{U}^\dagger] = g_8 \tilde{Q}_8 + g_{27} \tilde{Q}_{27} + \text{h.c.}, \tag{2.70}$$

where the tilde indicates that these operators are now functions of $\tilde{U}$.

2.5.1 The $\Delta I = 1/2$ Puzzle

Starting from Eq. (2.70), one may study the rich phenomenology of K-mesons. There is however a severe problem — so old that new solutions are rarely attempted — associated with the decays of the short- and long-lived states

$$K_S \to \pi\pi\,, \qquad K_L \to \pi\pi\pi\,. \tag{2.71}$$

Experimentally, there is a large enhancement of the isospin-$\frac{1}{2}$ decays. This phenomenon is particularly striking in the S-wave $\pi\pi$ mode, where the measured rates [3] exhibit the ratios

$$\gamma_{+-} = \frac{\Gamma(K_S \to \pi^+\pi^-)}{\Gamma(K^+ \to \pi^+\pi^0)} \simeq 463\,, \qquad \gamma_{00} = \frac{\Gamma(K_S \to \pi^0\pi^0)}{\Gamma(K^+ \to \pi^+\pi^0)} \simeq 205\,, \tag{2.72}$$

which are in gross disagreement with the naive expectations $\gamma_{+-} \sim \mathrm{O}(1) \sim \gamma_{00}$ from perturbative electroweak calculations. It is useful to translate the above in terms of isospin amplitudes A_I for the $\pi\pi$ final-state. The $I = 1$ state is forbidden by Bose symmetry and thus the transition amplitudes can be parametrized as [57]

$$\mathcal{A}(K_S \to \pi^+\pi^-) = \frac{2}{\sqrt{3}} A_0 e^{i\delta_0} + \sqrt{\frac{2}{3}} A_2 e^{i\delta_2}\,, \tag{2.73}$$

$$\mathcal{A}(K_S \to \pi^0\pi^0) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} - \frac{2}{\sqrt{3}} A_2 e^{i\delta_2}\,, \tag{2.74}$$

$$\mathcal{A}(K^+ \to \pi^+\pi^0) = \sqrt{\frac{3}{2}} A_2 e^{i\delta_2}\,, \tag{2.75}$$

where the $\pi\pi$-scattering phase shifts δ_I arise as a consequence of Watson's theorem. Comparison with the data in (2.72) leads to the $\Delta I = 1/2$ rule for kaons[1]

$$\mathfrak{Re}|A_0/A_2| \simeq 22\,, \tag{2.76}$$

whose origin remains a mystery despite five decades of theoretical investigation.

In the leading order of $\chi\mathrm{PT}_3$, A_0 and A_2 are given by the coefficients g_8 and g_{27} of Eq. (2.70) which are clearly *not* fixed by chiral symmetry. The problem then is to

[1] A similar rule is observed in the non-leptonic hyperon decays [58, 59].

explain why

$$|g_8/g_{27}| \simeq 22 \tag{2.77}$$

is so large compared with simple quark-model estimates which work for $\Delta I = 3/2$ amplitudes. It has been suggested [53] that penguin diagrams may be responsible for this 'octet dominance'. Other approaches to the problem make use of a variety of techniques including the many colour N_c limit [60, 61, 62, 63, 64], QCD sum rules [65, 66, 67, 68], and direct evaluation on the lattice [69, 70, 71], with varying degrees of success.

2.6 The Lowest QCD Resonance: Problems with Chiral $SU(3)_L \times SU(3)_R$ Expansions?

The indication that the hadronic spectrum may contain a *light* $J^{PC} = 0^{++}$ resonance pre-dates QCD, and was originally introduced in the 1960s to improve the description of NN-scattering by models based on the 'one boson exchange' potential [72]. At the time, there were additional hints of a broad state $\epsilon(700)$ hiding in the S-wave of $\pi\pi$-scattering. The large width however, posed a serious technical difficulty in interpreting the results from phase-shift analyses of the data. Determinations of the 0^{++} pole, deep in the complex s-plane, were either model-dependent or sensitive to a chosen parametrisation of the data on the real axis [73, 74]. The $\epsilon(700)$ remained in the PDG tables until 1974 [75], and as a result it became generally accepted that the hadronic spectrum did not contain 0^{++} states below ≈ 1 GeV. In 1996, Törnqvist and Roos [76] introduced the $f_0(500)$ resonance (a reborn $\epsilon(700)$), but it was still clear that the interpretation of the data was model-dependent at best.

The situation changed dramatically in 2006, when Caprini *et al.* [17] observed that the general principles of quantum field theory, *viz.*, unitarity, analyticity, and crossing symmetry, allow the Roy equations [77] to be complexified. As a result, the mass and width of the $f_0(500)$ resonance could be determined precisely. For $\pi\pi$-scattering, these requirements allow a representation of the S-matrix element[1]

$$S_\ell^I = \eta_\ell^I \exp(2i\delta_\ell^I), \tag{2.78}$$

[1] We have for isospin I and angular momentum ℓ, the elasticity η_ℓ^I and phase-shift δ_ℓ^I.

in terms of a twice-subtracted dispersion relation. The scattering amplitude is analytic in the s-plane except for cuts along the real axis:

$$\begin{cases} \text{Right-hand cut:} & 4m_\pi^2 < s < \infty\,, \\ \text{Left-hand cut:} & -\infty < s < 0\,. \end{cases} \tag{2.79}$$

By projecting S^I_ℓ onto partial waves t^I_ℓ, one obtains a set of integral equations derived by Roy [77], which for $I = 0 = \ell$ are given by

$$t^0_0(s) = a + (s - 4m_\pi^2)b + \sum_{I=0}^{2}\sum_{\ell=0}^{\infty}\int_{4m_\pi^2}^{\infty} \mathrm{d}s'\, K^{0I}_{0\ell}(s,s')\,\mathfrak{Im}\, t^I_\ell(s')\,. \tag{2.80}$$

The kernels $K^{I'I}_{\ell'\ell}$ are known algebraic expressions of s, s' and m_π [17], and thus the input into (2.80) is just the imaginary part of t^I_ℓ (determined from data) and the two subtraction constants a and b. The latter can be related to the S-wave scattering lengths

$$a = a^0_0\,, \qquad b = (2a^0_0 - 5a^2_0)/12m_\pi^2\,, \tag{2.81}$$

and are known from $\chi\mathrm{PT}_2$ calculations to great precision [78]. With these inputs, the solution to (2.80) gives the full $\pi\pi$-scattering amplitude. As shown by Roy [77], (2.80) is valid for real s in the interval $-4m_\pi^2 < s < 60m_\pi^2$. The key insight by Caprini *et al.* [17] was to note that the Roy equations may be analytically continued to *complex* values of s, thereby evading the aforementioned difficulties in performing extrapolations from the real axis. They made use of the fact that unitarity relates S-matrix elements on the first I and second II Riemann sheets,

$$S^0_0(s)^{II} = 1/S^0_0(s)^I\,, \tag{2.82}$$

and thus determined the $f_0(500)$ pole by solving the Roy equations (2.80) subject to the constraint $S^0_0(s)^I = 0$. Their *model-independent* results for the complex pole mass and residue,

$$m_{f_0} = 441^{+16}_{-8} - i\,272^{+9}_{-12.5}\,\mathrm{MeV}\,, \qquad |g_{f_0\pi\pi}| = 3.31^{+0.35}_{-0.15}\,\mathrm{GeV}\,, \tag{2.83}$$

have been confirmed by subsequent analyses [18] which relax the chiral constraints on a_0^0 and a_0^2. We note that in all determinations of this kind, the real part of m_{f_0} is less than m_K.

Now that the existence of a light 0^{++} resonance has been established, can one do away with the octet dominance hypothesis (2.77) and argue that the large $I = 0$ enhancement in $K_S \to \pi\pi$ is due to a dominant f_0-pole? This line of investigation was considered long ago [79, 80, 81, 82] and the formalism of χPT$_3$ was extended to include a scalar field in a chiral invariant fashion. The problem remains [1] that *lowest order* predictions from χPT$_3$ are in general found wanting for amplitudes which involve both a 0^{++} channel and $O(m_K)$ extrapolations in momenta, i.e. χPT$_3$ expansions *diverge*. Notable processes where χPT$_3$ expansions fail include the following.

1. The transition $K_L \to \pi^0\gamma\gamma$, with a chiral one-loop prediction for the rate [2] which is approximately 1/3 the experimental value [3].

2. The cross-section for $\gamma\gamma \to \pi^0\pi^0$, where the lowest order prediction [4, 5] of a linear rise in energy from threshold (Fig. 2.1) is incompatible with the data from the Crystal Ball experiment [6].

Figure 2.1: Plot of the chiral one-loop prediction [4, 5] (dashed line) for the cross-section $\sigma(\gamma\gamma \to \pi^0\pi^0)$ as a function of the momentum transfer $\sqrt{s}$. Also shown is the data from Crystal Ball [6], although we note that the data point at $\sqrt{s} = 0.275$ GeV is not consistent with the available phase space.

3. $K_{\ell 4}$ decays [9] and non-leptonic K [10, 11] and η [12, 13] decays are known to receive final-state $\pi\pi$ interactions [7, 8] which are important or even dominant relative to purely chiral contributions [1, 7, 9, 10, 11, 12, 13]

To examine the problem in more detail, consider the process $K_L \to \pi^0\gamma\gamma$ at NLO in $\chi\mathrm{PT}_3$, with amplitude

$$\mathcal{A}(K_L \to \pi^0\gamma\gamma) = \{\mathcal{A}_{\mathrm{LO}} + \mathcal{A}_{\mathrm{NLO}}\}_{\chi\mathrm{PT}_3}\,. \tag{2.84}$$

Since $\mathcal{A}_{\mathrm{LO}}$ predicts too small a rate, depending on the relative phase, a reasonable fit to the data can be obtained if

$$|\mathcal{A}_{\mathrm{NLO}}|_{\chi\mathrm{PT}_3} \simeq \sqrt{2}|\mathcal{A}_{\mathrm{LO}}|_{\chi\mathrm{PT}_3}\,. \tag{2.85}$$

Clearly, (2.85) is at odds with the usual expectation (Sec. 2.4.1) that NLO corrections in $\chi\mathrm{PT}_3$ are at most 30%,

$$|\mathcal{A}_{\mathrm{NLO}}/\mathcal{A}_{\mathrm{LO}}|_{\chi\mathrm{PT}_3} \lesssim 0.3\,. \tag{2.86}$$

The discrepancy is extreme for $K_S \to \pi\pi$, where in order to explain the observed factor of 22 enhancement in the $I=0$ channel, a dominant f_0 amplitude generated at NLO must be 70 times the expected $\lesssim 30\%$ correction.[1]

How are we to make sense of this? In any effective field theory, the validity of a chosen perturbative expansion depends crucially on a clear separation of scales between the heavy and effective degrees of freedom. The success of $\chi\mathrm{PT}_3$ then depends on the distinction between the light pseudo-NG bosons π, K, η and the non-Goldstone sector $\{f_0, \rho, \omega, K^*, N, \eta', \ldots\}$. Evidently, $\chi\mathrm{PT}_3$ expansions fail when the 0^{++} channel is present because f_0 sits right in the middle of $\{\pi, K, \eta\}$ (Fig. 2.2).

It is this problem which we seek to solve. In this thesis, we take the view that the systematic failure of $\chi\mathrm{PT}_3$ expansions in the 0^{++} channel can and should be corrected by modifying the *lowest order* of the three-flavour theory (with no change to $\chi\mathrm{PT}_2$). As we discuss in the next chapter, our solution is closely connected to the infrared behaviour of the strong running coupling α_s.

[1] Provided $|g_8| \simeq |g_{27}| \simeq O(1)$, as many calculations [60, 61, 62, 63, 64, 65, 66, 67, 68] indicate.

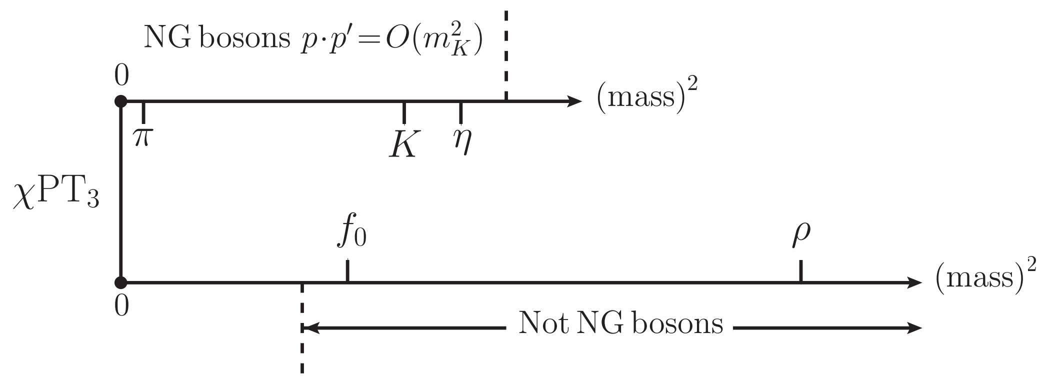


Figure 2.2: Spectrum of hadrons in χPT$_3$. Note that there is no scale separation because the non-NG boson $f_0(500)$ lies within the NG sector $\{\pi, K, \eta\}$.

Chapter 3

Asymptotia

This chapter reviews elements of the renormalization group, in particular, the notion of an effective charge $\bar{\alpha}$ in Quantum Electrodynamics (QED) (Section 3.1) and its generalisation $\bar{\alpha}_s$ in Quantum Chromodynamics (QCD) (Section 3.1.1). In the latter case, we present a selection of definitions for $\bar{\alpha}_s$ from the literature before extending our discussion in Section 3.2 to include non-perturbative definitions based on lattice QCD or solutions to the Dyson-Schwinger equations. In Section 3.3, we examine the theoretical evidence for “freezing" in the infrared, *viz.* the idea that QCD possesses an infrared fixed point. This enables us to present the underlying premise of this thesis that the strong coupling $\alpha_s(\mu)$ of three-flavour QCD may run non-perturbatively to a nontrivial, stable fixed point in the infrared limit $\mu \to 0$. The implications arising from this scenario are briefly discussed, with a detailed discussion reserved for Chapter 4.

3.1 Effective Charges

A well known feature of perturbative quantum field theory is the inherent ambiguity associated with the definitions of the renormalized couplings and masses. For instance, two different theorists may adopt different renormalization prescriptions $\mathcal{R}$ and $\mathcal{R}'$ for the same physics (described by the same Lagrangian $\mathcal{L}$). At a fixed-order in perturbation theory, the difference between $\mathcal{R}$ and $\mathcal{R}'$ may arise because different subtraction points are used for a given type of coupling constant, or the finite parts of the counterterms may be different. Stückelberg and Peterman [83] were the first to consider transformations of the kind $\mathcal{R} \to \mathcal{R}'$ and called these mappings elements of the *renormalization group*. Although it really has nothing to with group theory, the name arose because $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}''$ implies that $\mathcal{R} \to \mathcal{R}''$ belongs to this class of transformations. In this language, the requirement that physical observables be independent of a chosen prescription or scheme implies that $\mathcal{L}$ is renormalization group invariant.

As experiments grow ever more precise, the arbitrariness of the renormalization procedure can cause serious confusion in the interpretation of results based on truncated perturbative expansions. Take for instance QCD [84], where the choice of renormalization scale μ in the strong running coupling $\alpha_s(\mu^2)$ is the main source of uncertainty in perturbative predictions for observables such as the Drell-Yan ratio

$$R(Q^2) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \,. \tag{3.1}$$

In any scale-setting procedure, one must guess the effect of the omitted higher-order terms. Typically, this is done by setting $\mu = Q$ of order the momentum transfer in the physical process and then varying the scale over some range of momenta. The problem with this approach is that the resulting prediction is strongly sensitive to the choice of renormalization-scheme (RS).

There is however, a way to reformulate perturbation theory in a RS-invariant way, even though the theory is not solved to all orders. The price to be paid for such a formulation is that results now become *process dependent*. The originators of this approach were Gell-Mann and Low [85], who in a remarkably prescient paper on the short-distance behaviour of Quantum Electrodynamics (QED), established the relevance of the renormalization group to the *asymptotic* behaviour of physical processes. A key part of their analysis was the introduction of a quantity known as an *effective charge*, which is an all-order resummation of perturbation theory. In general, effective charges provide a *physical* definition for the fundamental coupling of a quantum field theory, and as such they possess the desirable properties of being *independent* of the RS and analytic when defined relative to spacelike momentum scales.

For example, in QED the renormalization of the dressed photon propagator[1] $\Pi_{\mu\nu}(q)$ involves a function $d(q^2;\alpha,m)$,

$$\Pi_{\mu\nu}(q) = -i\frac{g_{\mu\nu}}{q^2+i\epsilon}d(q^2;\alpha,m) + q_\mu q_\nu \,\text{terms}\,, \tag{3.2}$$

where $\alpha = \alpha(\mu^2) = e^2(\mu^2)/4\pi$ and $m = m(\mu^2)$ refer to the renormalized coupling and electron mass.[2] By summing the Dyson series of one-particle-irreducible (1PI) insertions of the photon self-energy $\Pi(q^2;\alpha,m)$, one obtains a definition of the effective charge $\bar{\alpha}(q^2;\alpha,m)$ for the theory,

$$\bar{\alpha}(q^2;\alpha,m) \equiv \alpha d(q^2;\alpha,m) = \frac{\alpha}{1-\Pi(q^2;\alpha,m)}\,. \tag{3.3}$$

This represents a physical coupling since $\bar{\alpha}$ can be determined by measuring the potential $V(r)$ (or rather, its Fourier transform) between two heavy test charges.

From the definition (3.3), we can show that $\bar{\alpha}$ possesses the following important properties [85]. Firstly, electric charge normalization $\Pi(0;\alpha,m) = 0$ implies that $\bar{\alpha}$ reproduces the fine-structure constant at zero-momentum,

$$\lim_{q^2\to 0}\bar{\alpha}(q^2;\alpha,m) = \alpha \simeq 1/137\,. \tag{3.4}$$

Secondly, since Π is gauge-invariant to all orders of perturbation theory, so too is $\bar{\alpha}$. Thirdly, the QED Ward identity $Z_1 = Z_2$ for wave function renormalization implies that the combination

$$\alpha D_{\mu\nu}(q) = \alpha_0 D_{0,\mu\nu}(q) \tag{3.5}$$

is not renormalized. Thus $\bar{\alpha}$ is also renormalization group invariant,

$$\begin{aligned}\bar{\alpha}(q^2;\alpha,m) &= \alpha d(q^2;\alpha,m) = \alpha_0 d_0(q^2;\alpha_0,m_0)\\ &= \text{independent of } \mu \text{ and scheme } \mathcal{R}.\end{aligned} \tag{3.6}$$

The existence of the massless theory — valid perturbatively, and assumed to be true in the non-perturbative case — means that we may adopt with impunity a mass-

[1] Longitudinal terms $\propto q_\mu q_\nu$ can be eliminated by a gauge transformation, so they are unphysical.
[2] Bare quantities will be denoted e_0, m_0 etc.

independent scheme such as minimal subtraction (MS or $\overline{\text{MS}}$) in dimensional regularization and hence obtain the 'improved' Callan-Symanzik equation due to 't Hooft [86] and Weinberg [87]

$$\left\{\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha)\frac{\partial}{\partial\alpha}+\gamma_m(\alpha)m\frac{\partial}{\partial m}\right\}\bar{\alpha}(q^2;\alpha,m)=0\,, \tag{3.7}$$

where the β-function and anomalous mass dimension γ_m,

$$\beta(\alpha)=\mu^2\frac{\partial\alpha}{\partial\mu^2}\qquad\text{and}\qquad\gamma_m(\alpha)=\mu^2\frac{\partial\ln m}{\partial\mu^2}\,, \tag{3.8}$$

are determined by the RS, but are independent of the physical quantity $\bar{\alpha}$. In the Euclidean region, the asymptotic limit $-q^2/m^2\to\infty$ corresponds to massless QED, and thus the asymptotic function $\bar{\alpha}_{\text{as}}$ — obtained by subtracting terms of the form $(m^2/q^2)\ln^n(-q^2/m^2)$ at each order of perturbation theory — depends on only one kinematical scale q,

$$\bar{\alpha}_{\text{as}}=\bar{\alpha}_{\text{as}}(t,\alpha)\,,\qquad\text{where}\qquad t=\ln(-q^2/\mu^2)\,. \tag{3.9}$$

In this limit, the Callan-Symanzik equation (3.7) becomes

$$\left\{\frac{\partial}{\partial t}-\beta(\alpha)\frac{\partial}{\partial\alpha}\right\}\bar{\alpha}_{\text{as}}(t,\alpha)=0\,, \tag{3.10}$$

whose solution may be found by introducing a *running coupling* $\alpha_t=\alpha_t(t)$, defined by

$$t=\ln(-q^2/\mu^2)=\int_{\alpha}^{\alpha_t}\frac{dz}{\beta(z)}\,. \tag{3.11}$$

It follows from the boundary condition $\alpha_t(t=0)=\alpha$, that the general solution to (3.10)

$$\bar{\alpha}_{\text{as}}(t,\alpha)=\bar{\alpha}_{\text{as}}(0,\alpha_t(t))\,, \tag{3.12}$$

relates the effective charge to the running coupling in the asymptotic limit:

$$\lim_{-q^2/m^2\to\infty}\bar{\alpha}(q^2;\alpha,m)=\alpha_t(t)\,. \tag{3.13}$$

As is well known, the β-function provides important information on the asymptotic behaviour of a theory in the ultraviolet and infrared regimes. It is however, a RS-dependent quantity and thus it is often convenient to work with the renormalization group invariant, but process dependent *Gell-Mann-Low function* Ψ,

$$\Psi(\bar{\alpha}_{\rm as}) = q^2 \frac{\partial \bar{\alpha}_{\rm as}}{\partial q^2} \,. \tag{3.14}$$

3.1.1 Effective Charges for QCD

The Gell-Mann-Low method of effective charges described above has been generalised to QCD by Grunberg [88, 89, 90]. To minimise the number of technical complications, we shall restrict our discussion to massless QCD and we refer the reader to [89, Sec. V] for the more general case. The details of the approach can be summarised as follows.

By analogy with $\bar{\alpha}_{\rm as}$, we consider a dimensionless physical (or renormalization group invariant) quantity $\sigma(Q^2)$, which depends on only one kinematical scale Q^2. In general, QCD predictions for $\sigma(Q^2)$ at large Q^2 have the form

$$\sigma(Q^2) = A + B\alpha_s(\mu^2)[1 + \sigma_1(Q^2/\mu^2)\alpha_s(\mu^2) + \sigma_2(Q^2/\mu^2)\alpha_s^2(\mu^2) + O(\alpha_s^3)] \,, \tag{3.15}$$

where A and B are constants. The coefficients $\sigma_i(Q^2/\mu^2)$ depend on both the definition of $\alpha_s(\mu^2)$ and the scale-setting procedure for μ. Clearly, there is no ambiguity if the right-hand side of Eq. (3.15) is calculated to all-orders. For all practical purposes however, a truncation must be made and thus the prediction for $\sigma(Q^2)$ becomes RS-dependent.

Grunberg observed that this dependence can be greatly simplified if one makes use of the dimensional transmutation property of QCD [91], where $\alpha_s(\mu^2)$ is related to some low-energy scale Λ.[1] The key idea is to consider a different asymptotic expansion of $\sigma(Q^2) = F(Q^2/\Lambda^2)$ in powers of $1/\ln(Q^2/\Lambda^2)$. Summing the leading loga-

[1] Note that Λ is not *necessarily* the scale $\Lambda_{\rm QCD} \approx 200$ MeV extracted from perturbative expansions of β in the asymptotically free regime. In general, Λ can be any mass scale which is generated dynamically, e.g. the proton mass or pion decay constant F_π.

rithms via the renormalization group, one gets [89]

$$\sigma(Q^2) = A + B\left[\frac{1}{\beta_1 \ln(Q^2/\Lambda^2)}\right]\left[1 + \frac{\sigma_1(1) + C\beta_1^2 - \beta_2/\beta_1 \ln\ln(Q^2/\Lambda^2)}{\beta_1 \ln(Q^2/\Lambda^2)}\right] \tag{3.16}$$

where the QCD analogue of the β-function (3.8) reads

$$\beta(\rho) = \mu^2 \frac{\partial \rho}{\partial \mu^2} = -\beta_1 \rho^2 - \beta_2 \rho^3 + O(\rho^4), \tag{3.17}$$

where $\rho = \alpha_s(\mu^2)/4\pi$ and for N_f quark flavours, the first two coefficients

$$\beta_1 = 11 - \tfrac{2}{3}N_f, \qquad \text{and} \qquad \beta_2 = 108 - \tfrac{38}{3}N_f, \tag{3.18}$$

are scheme independent. The constant C is determined by the definition of Λ. The important thing to note here is that all trace of $\alpha_s(\mu^2)$ has vanished: the RS dependence of $\sigma(Q^2)$ is now characterised in terms of a single number λ, which parametrises the freedom to redefine $\Lambda \to \bar{\Lambda} = \lambda\Lambda$ (or $Q \to \bar{Q} = Q/\lambda$). By considering the expansion of the inverse function $F^{-1}(\sigma) = Q^2/\Lambda^2$, the remaining scheme dependence of Λ or Q can be eliminated by a simple rescaling. By definition, $F^{-1}(\sigma)$ depends solely on the physical quantity considered, i.e.

$$F^{-1}(\sigma) = \text{RS-invariant, even when the series (3.16) is truncated.} \tag{3.19}$$

We now seek an appropriate expansion parameter for Q^2/Λ^2. Since $\sigma(Q^2)$ is RS-invariant, Grunberg advocates for the particular choice where all higher-order corrections can be absorbed into the definition of an effective charge $\bar{\alpha}_s(Q^2)$ associated with $\sigma(Q^2)$:

$$\sigma(Q^2) = A + B\bar{\alpha}_s(Q^2). \tag{3.20}$$

As in the QED example, the prediction for $\bar{\alpha}_s(Q^2)$ can be obtained in terms of a generalised Gell-Mann-Low function $\bar{\beta}$, defined by

$$Q^2 \frac{\partial \bar{\rho}}{\partial Q^2} = \bar{\beta}(\bar{\rho}) = -\beta_1 \bar{\rho}^2 - \beta_2 \bar{\rho}^3 - \bar{\beta}_3 \bar{\rho}^4 + O(\bar{\rho}^5). \tag{3.21}$$

Note that the first two coefficients $\beta_{1,2}$ coincide with those from the QCD β-function

(Eq. (3.18)), but the higher-order ones $\beta_{i\geq 3}$ depend on the quantity $\sigma(Q^2)$ considered. The exact solution to (3.21) is given by [89]

$$\beta_1 \ln \frac{Q^2}{\Lambda^2} = \frac{1}{\bar{\rho}} + \frac{\beta_2}{\beta_1} \ln(\beta_1 \bar{\rho}) + K_1 + \int_0^{\bar{\rho}} dz \left[\frac{1}{z^2} - \frac{\beta_2}{\beta_1}\frac{1}{z} + \frac{\beta_1}{\bar{\beta}(z)} \right], \tag{3.22}$$

where K_1 is a constant determined from the NLO correction to $\bar{\rho}(Q^2)$ at $\mu^2 = Q^2$. Eq. (3.22) is the final necessary ingredient, which when compared with (3.20) yields the required expression for $F^{-1}(\sigma)$. So we see that the problem of RS dependence in perturbative QCD calculations can be overcome in a systematic manner. Instead of focusing on the 'best' RS, the pertinent question now becomes: what is the 'best' choice of an effective charge in QCD? The answer is yet to be established and some popular examples from the literature are given below.

(a) By close analogy with QED, one can define an effective charge $\alpha_V(Q^2)$ as the coefficient in the static limit of the scattering potential between two infinitely heavy quarks [92, 93]

$$V(Q^2) = -4\pi C_F \frac{\alpha_V(Q^2)}{Q^2}, \tag{3.23}$$

where the momentum transfer is Euclidean $Q^2 = -q^2 > 0$ and $C_F = 4/3$ is the Casimir operator in the fundamental representation of colour $SU(3)$.

(b) A direct attempt to generalise the resummation procedure of 1PI graphs in the photon propagator to the gluonic case is the pinch technique [94, 95, 96, 97, 98]. In QCD, the Ward identity result $Z_1 = Z_2$ does not hold and thus obtaining a RS- and gauge-invariant result after Dyson-resummation is rather complicated. The result from rearranging the contributions to scattering amplitudes is a struture of the form

$$\alpha_{\text{pinch}}(Q^2) = \frac{\alpha_s}{1 - \hat{\Pi}(Q^2)}. \tag{3.24}$$

(c) A physical coupling $\alpha_R(Q^2)$ can obtained directly from high precision measurements of the Drell-Yan ratio at momentum transfer Q^2,

$$R_{e^+e^-}(Q^2) = \left[\sum_i Q_i^2\right] \left[1 + \frac{\alpha_R(Q^2)}{\pi}\right], \tag{3.25}$$

where the sum is over the charges Q_i of the active quark flavours.

Similarly, for fixed values of the kinematical scale, hadronic τ-decays of the type $\tau^- \to \nu_\tau + \text{hadrons}$ allow one to define an effective charge α_τ [99] through the ratio of the hadronic to leptonic decay channels

$$R_\tau(m_{\tau'}^2) = R_\tau^0 \left[1 + \frac{\alpha_\tau(m_{\tau'}^2)}{\pi}\right], \tag{3.26}$$

where R_τ^0 is leading order QCD prediction and $0 < m_{\tau'} < m_\tau$ refers to some hypothetical τ mass.

While each of the above represents a perfectly well-defined, RS-invariant definition of the strong coupling, it is clear that in order to probe the infrared limit of QCD, genuine non-perturbative techniques are required. The keywords in this domain are lattice QCD and Dyson-Schwinger equations. Examples of both are discussed below.

3.2 Non-perturbative Determinations of the Strong Running Coupling

3.2.1 Schrödinger Functional Scheme

The Schrödinger functional was proposed by Lüscher *et al.* [100, 101] as a means to study the scaling properties of QCD in a Euclidean box of volume L^4, and hence obtain a strong running coupling $\alpha_{\mathrm{SF}}(L) = \bar{g}^2(L)/4\pi$ from numerical simulations on the lattice.

In the context of QCD, much of the theoretical effort has been directed towards the interpolation between the non-perturbative (large L) and perturbative (small L) regimes of the theory, with the aim to express (say) $\alpha_s(M_Z)$ entirely in terms of low-energy quantities such as the string tension or Sommer scale [100, 101]. On the other hand, the Schrödinger functional scheme has been used to explore the behaviour of strong-coupling extensions to the Standard Model such as technicolour, with the aim to determine the critical number of fermion flavours in some representation of $SU(N)$ which induce a scale invariant phase of the theory [102]. Due to computational limitations [103], it cannot be said that the deep infrared limit $L \to \infty$ has yet been probed in a realistic simulation of QCD with $N_f = 3$.

To see how $\bar{g}^2(L)$ is defined in this approach, we follow Lüscher *et al.* [101] and consider pure $SU(3)$ Yang-Mills theory[1] with gauge fields $A^a_\mu(x)$ and Euclidean action in the temporal gauge,

$$S[A] = -\frac{1}{4g_0^2}\int d^4x\, G^a_{\mu\nu} G^{a\mu\nu}\,, \tag{3.27}$$

where g_0 is the bare coupling and $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ is the gluon field strength. The Schrödinger functional refers to the path integral

$$\mathcal{Z}[C, C'] = \int D[A_\mu] e^{-S[A]}\,, \tag{3.28}$$

where periodic boundary conditions are imposed in all spatial directions and C, C' refer to fixed boundary values of $A^a_\mu(x)$ at times $t = 0$ and $t = T$ respectively. The boundary values are chosen such that the gauge field configuration (or induced background field) $B^a_\mu(x)$ which minimises (3.27) is stable, and unique up to gauge transformations. In the presence of this background field, $\mathcal{Z}$ can be calculated at weak coupling by using the saddle-point evaluation of the effective action $\Gamma[B] = -\ln \mathcal{Z}$ about $B^a_\mu(x)$. The result is an asymptotic series

$$\Gamma[B] = g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + g_0^2 \Gamma_2[B] + \ldots\,, \tag{3.29}$$

where the leading term is proportional to the action $\Gamma_0[B] = g_0^2 S[B]$. In general, there are many ways to define $\bar{g}^2(L)$ and most lattice studies choose B^a_μ to depend on a dimensionless parameter η. For practical reasons [101], this is usually implemented by requiring that the boundary values of the background field are spatially constant and diagonal,

$$C_k = \frac{i}{L}\begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} \quad \text{and} \quad C'_k = \begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix}, \quad k = 1,2,3. \tag{3.30}$$

[1] The inclusion of quarks in a lattice regularized Schrödinger functional is formally straightforward [104], but dependent on the way in which one simulates chiral fermions (see e.g. [102] for an analysis with staggered fermions).

The angles ϕ_α and ϕ'_α are constrained to be real and must sum to zero to ensure that the matrices are elements of $SU(3)$. A particular choice [105] which ensures that B^a_μ corresponds to a stable solution of the field equations is

$$\begin{aligned} \phi_1 &= \eta - \pi/3 & \phi'_1 &= -\eta - \pi \\ \phi_2 &= -\eta/2 & \phi'_2 &= \eta/2 + \pi/3 \\ \phi_3 &= -\eta/2 + \pi/3 & \phi'_3 &= \eta/2 + 2\pi/3 \end{aligned} \tag{3.31}$$

Subject to the boundary conditions specified above, the quantity

$$\Gamma'[B] = \frac{\partial}{\partial \eta}\Gamma[B] = -\frac{\partial}{\partial \eta}\ln \mathcal{Z} \tag{3.32}$$

is renormalization group invariant and thus the renormalized coupling

$$\frac{k}{\bar{g}^2(L)} = -\left.\frac{\partial}{\partial \eta}\ln \mathcal{Z}\right|_{\eta=0} \tag{3.33}$$

depends only on the box size L, i.e. $\bar{g}^2(L)$ (or equivalently, $\alpha_{\text{SF}}(L)$) defines a running coupling. The constant of proportionality $k = \Gamma_0[B]$ is chosen so that one recovers $\bar{g}^2 = g_0$ in the leading order of perturbation theory. From the definition (3.33), a β-function can be defined in the obvious manner

$$\beta(\alpha_{\text{SF}}) = -L\frac{\partial \alpha_{\text{SF}}}{\partial L}\,. \tag{3.34}$$

An important consistency check of the SF coupling is that it can be perturbatively related to other schemes, e.g. the $\overline{\text{MS}}$ scheme. If we define

$$\alpha_{\text{SF}}(Q) = \frac{\bar{g}^2(L)}{4\pi}, \qquad \text{where} \qquad Q = 1/L, \tag{3.35}$$

then the relation between the two approaches is given by [105]

$$\alpha_{\overline{\text{MS}}} = \alpha_{\text{SF}} + k_1\alpha^2_{\text{SF}} + O(\alpha^3_{\text{SF}}), \tag{3.36}$$

where both couplings are evaluated at the same momentum Q.

3.2.2 Dyson-Schwinger Equations

In this approach, non-perturbative definitions of α_s are given by constructing renormalization group invariant quantities from two- and three-point functions of the theory. The analytic expressions for these quantities (and hence α_s) are in turn obtained by solving truncated systems of Dyson-Schwinger equations. A well studied example is provided by the ghost-gluon (gh) vertex in Landau gauge $SU(N_c)$ Yang-Mills theory [106, 107, 108]. Here the renormalized ghost and gluon propagators are described in terms of the dressing functions $G(p^2)$ and $Z(p^2)$,

$$D_G(p^2) = -\frac{G(p^2)}{p^2}, \qquad D_{\mu\nu}(p^2) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Z(p^2)}{p^2}, \tag{3.37}$$

where p refers to Euclidean momentum. The propagators D_G and $D_{\mu\nu}$, and coupling g are related to their bare counterparts through the multiplicative renormalization constants

$$\tilde{Z}_3 D_G = D_{G,0}, \qquad Z_3 D_{\mu\nu} = D_{0,\mu\nu}, \qquad Z_g g = g_0. \tag{3.38}$$

In non-abelian gauge theories like QCD, the Ward identities must be replaced by Slavnov-Taylor identities such as

$$\tilde{Z}_1 = Z_g \tilde{Z}_3 Z_3^{1/2}, \tag{3.39}$$

for the gh vertex. This identity defines a renormalized coupling $\alpha_{gh}(\mu) = g^2(\mu)/4\pi$,

$$\alpha = \alpha_0 \frac{\tilde{Z}_3^2 Z_3}{\tilde{Z}_1^2}. \tag{3.40}$$

A running coupling can be defined by noting that the renormalized dressing functions (3.37) are related to their bare counterparts through

$$\begin{aligned} G_0(p^2) &= G(p^2)\tilde{Z}_3, \\ Z_0(p^2) &= Z(p^2) Z_2, \end{aligned} \tag{3.41}$$

and thus in Landau gauge $\tilde{Z}_1 = 1$, the quantity

$$\alpha G^2(p^2)Z(p^2) = \alpha_0 G_0^2(p^2)Z_0(p^2) \tag{3.42}$$

is renormalization group invariant. By evaluating the left-hand side of (3.42) at μ^2 and $\mu^2 = p^2$, one obtains from the gh vertex the definition [106],

$$\alpha_{gh}(p^2) = \alpha_{gh}(\mu^2)G^2(p^2,\mu^2)Z(p^2,\mu^2). \tag{3.43}$$

Similar definitions can be obtained for the three-gluon ($3g$) and four-gluon ($4g$) vertices, whose dressing functions H_1^{3g} and H_1^{4g} each define a running coupling [108]:

$$\alpha_{3g}(p^2) = \alpha_{3g}(\mu^2)\left[H_1^{3g}(p^2,\mu^2)\right]^2 Z^3(p^2,\mu^2), \tag{3.44}$$

$$\alpha_{4g}(p^2) = \alpha_{4g}(\mu^2)\left[H_1^{4g}(p^2,\mu^2)\right]^2 Z^4(p^2,\mu^2). \tag{3.45}$$

3.3 Varieties of Asymptotic Behaviour

As we have seen above, there is a large number of suitable definitions for α_s. However, once a particular definition is chosen, it is possible to formulate precise hypotheses about β (or $\bar{\beta}$) for the whole theory. It is the purpose of this section to examine the possibilities which may occur in QCD.

To be specific, we restrict our discussion to low energies $\mu \ll m_{t,b,c}$, where heavy quarks t, b, c are decoupled from the theory. There are then two logical possibilities[1] for the resulting three-flavour theory (Fig. 3.1):

1. **Growth without bound.** If the integral[2]

$$\int_{\alpha_0}^{\infty} \frac{dz}{\beta(z)} \tag{3.46}$$

[1]The analogous case for QED is discussed in [109].

[2]In a slight abuse of notation, we denote $\alpha_0 \equiv \alpha_s(\mu_0^2)$ for some reference scale μ_0.

Figure 3.1: Varieties of asymptotic behavior for the QCD β-function with three light quarks u, d, s. The dashed line shows the case when the running coupling $\alpha_s(\mu^2)$ undergoes continued growth with decreasing scale μ (scenario (1) in text), while the solid line shows $\alpha_s(\mu^2)$ flowing to an infrared fixed point α_{IR} (scenario (2)). For completeness, we also include the case where $\alpha_s(\mu^2)$ diverges at a finite value of μ (dotted line), but we emphasize that this scenario is of *no physical relevance* since it produces poles in Green's functions in the spacelike momentum region (i.e. a tachyon or "Landau ghost") [110, Chapter 6].

is divergent, then the solution to the renormalization group equation

$$\ln \mu^2/\mu_0^2 = \int_{\alpha_0}^{\alpha_s} \frac{dz}{\beta(z)} \tag{3.47}$$

implies that as μ decreases, α_s experiences continued growth, becoming infinite in the infrared limit $\ln \mu^2/\mu_0^2 \to -\infty$.

2. **Infrared fixed point at finite coupling.** In this scenario, the integral

$$\int_{\alpha_0}^{\alpha_{\mathrm{IR}}} \frac{dz}{\beta(z)} \tag{3.48}$$

diverges because of a zero in $\beta(z)$ at $z = \alpha_{\mathrm{IR}}$. As shown in Fig. 3.1, β is negative in the physical region

$$0 < \alpha_s < \alpha_{\mathrm{IR}}, \tag{3.49}$$

and positive thereafter. The finite value α_{IR} is known as an infrared fixed point[1] since Eq. (3.8) dictates that as μ decreases, α_s will increase in the physical region (3.49) and decrease for $\alpha_s > \alpha_{\rm IR}$. In either case, α_s runs to $\alpha_{\rm IR}$ in the infrared limit $\mu \to 0$. About this point, the β-function and the anomalous dimension of the quark mass operator $\gamma_m = \mu^2 \partial \ln m_q / \partial \mu^2$ may be expanded in the series

$$\beta(\alpha_s) = \beta'(\alpha_{\rm IR})(\alpha_s - \alpha_{\rm IR}) + O\big((\alpha_s - \alpha_{\rm IR})^2\big)\,, \tag{3.50}$$

$$\gamma_m(\alpha_s) = \gamma_m(\alpha_{\rm IR}) + \gamma_m'(\alpha_{\rm IR})(\alpha_s - \alpha_{\rm IR}) + O\big((\alpha_s - \alpha_{\rm IR})^2\big)\,, \tag{3.51}$$

where $\beta' = \partial\beta/\partial\alpha_s$ and $\gamma_m' = \partial\gamma_m/\partial\alpha_s$ are evaluated at $\alpha_{\rm IR}$.

It is worth emphasizing that it is unclear from the literature which scenario is actually realized in QCD, and in particular, how sensitive the results are to the number N_f of active quark flavours. Part of the problem resides in the fact that precise knowledge of β (or $\bar{\beta}$) is largely restricted to the perturbative domain where expansions about the ultraviolet fixed point $\alpha_s = 0$ converge sufficiently rapidly. For instance, Banks and Zaks [112] have made the observation that if N_f lies within the 'conformal window'

$$8\tfrac{1}{19} < N_f < 16\tfrac{1}{2}\,, \tag{3.52}$$

then at two-loop order, β possesses a so-called *perturbative* infrared fixed point

$$\alpha_{\rm IR}^{2-\rm loop} = -\beta_1/\beta_2\,, \tag{3.53}$$

where the coefficients β_i are those of Eq. (3.18). The difficulty with this picture is that for $N_f < 8\frac{1}{19}$ (i.e. QCD as currently understood), the infrared zero in β disappears and a Landau pole is generated (Fig. 3.1). Nevertheless, there is a large body of work [89, 95, 99, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122] based on perturbative extrapolations of this type, which makes use of effective charges or alternative schemes. Results from these approaches indicate (almost without exception) that α_s "freezes" in the infrared, i.e. QCD possesses an infrared fixed point as described by scenario (2).

[1]Note that to be considered a fixed point, $\alpha_{\rm IR}$ must produce a sufficiently strong zero so that (3.48) diverges. For example, a simple zero (as shown in Fig. 3.1) $\beta(\alpha_s) \sim \text{const.}(\alpha_s - \alpha_{\rm IR})$ is sufficient, but the weaker zero $\beta(\alpha_s) \sim \text{const.}|\alpha_s - \alpha_{\rm IR}|^{1/2}$ is not [111].

Naturally, a fully non-perturbative framework is required to settle the matter, and thus running couplings of the type described in Sec. 3.2.2 become the relevant objects of study. Unfortunately, the question about whether α_s behaves like scenario (2) or (1) is far from settled. Indeed, some results from the literature are contradictory.

For instance, from the SF definition of α_s, the result for $N_f = 0$ shows that β becomes linear at large α_s, i.e. the running coupling of pure Yang-Mills is unbounded (scenario (1)). This is to be compared with the results from Dyson-Schwinger analyses [108, 123] of the gh, $3g$, and $4g$ vertices of Eqs. (3.43-3.45), all of which find that α_s is finite in the infrared limit $p^2 \to 0$:

$$\lim_{p^2\to 0} \alpha_{gh}(p^2) = \frac{c_1}{N_c}, \qquad \lim_{p\to 0} \alpha_{3g}(p^2) = \frac{c_2}{N_c}, \qquad \lim_{p\to 0} \alpha_{4g}(p^2) = \frac{c_3}{N_c}, \tag{3.54}$$

where the c_i are non-zero constants whose value depends on the definition of the running coupling. Similarly, for $N_f \neq 0$, the conclusion from SF analyses [102] of $N_f \leq 8$ is that there is no infrared fixed point. How can this be reconciled with the $N_f = 3$ Dyson-Schwinger analysis [124] which arrives at the opposite conclusion?

The origin of this conflict can be traced to the fact that a non-perturbative *definition* for α_s must be chosen, and thus comparing results from different approaches is rarely straightforward. Despite the lack of consensus, we take the view that an infrared fixed point in QCD (scenario (2)) should be taken seriously. The underlying assumption which forms the basis of this thesis, is contained in the following statement.

> **Proposal.** There exists a *physical* definition of the strong running coupling α_s which is analytic, non-negative, and monotonic. At low-energies $\mu \ll m_{t,b,c}$ after heavy quarks have decoupled, the resulting three-flavour α_s runs *non-perturbatively* to a finite, non-trivial infrared fixed point α_{IR}.

The requirement of analyticity follows from the properties of physical amplitudes at spacelike momentum. Monotonicity is required so that 'false zeroes' are avoided. This refers to the disastrous case when α_s runs to a value α_* such that $\beta(\alpha_*) = 0$, *before* the infrared limit is reached. After α_*, a decrease in μ implies a *decrease* in the strength of α_s, reaching the 'true' fixed point at $\alpha_{\mathrm{IR}} \equiv \alpha_s(\mu^2 = 0)$. A simple example is shown in Fig. 3.2.

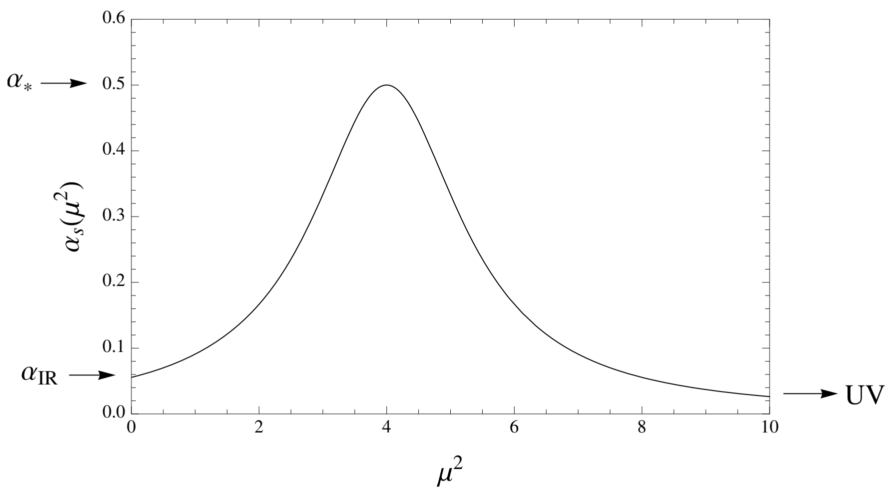


Figure 3.2: Plot of a strong running coupling $\alpha_s(\mu^2) = 1/[(\mu^2-4)^2+2]$ with the turning point $\alpha_* \equiv \alpha_s(\mu^2 = 4)$ giving rise to a 'false zero' in the QCD β-function. As the scale μ^2 is decreased from the ultraviolet (UV) region, α_s increases in strength as it approaches α_*, but switches and becomes weaker as it approaches the 'true' infrared fixed point at $\alpha_{\mathrm{IR}} \equiv \alpha_s(\mu^2 = 0)$.

In Chapter 4, we scrutinise the implications for low-energy phenomenology in the proposed picture. The key observation is that the gluonic term $\sim G^a_{\mu\nu} G^{a\mu\nu}$ in the trace anomaly of the energy-momentum tensor $\theta_{\mu\nu}$ [125, 126, 127, 128, 129],[1]

$$\theta^\mu_\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^{a\mu\nu} + \big(1 + \gamma_m(\alpha_s)\big) \sum_{q=u,d,s} m_q \bar{q} q \tag{3.55}$$

is absent at the fixed point α_{IR} and thus $\theta_{\mu\nu}$ becomes traceless in the chiral limit

$$\begin{aligned} \theta^\mu_\mu \Big|_{\alpha_s = \alpha_{\mathrm{IR}}} &= \big(1 + \gamma_m(\alpha_{\mathrm{IR}})\big)(m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s) \\ &\to 0 \ , \ SU(3)_L \times SU(3)_R \text{ limit} . \end{aligned} \tag{3.56}$$

Naively, one might consider this scenario a phenomenological disaster: does a trace-

[1]We have $[D_\mu, D_\nu] = i g G^a_{\mu\nu} T^a$ where D_μ is the covariant derivative, $\{T^a\}$ generate the gauge group, $\alpha_s = g^2/4\pi$ is the strong coupling, and $\beta = \mu^2 \partial \alpha_s / \partial \mu^2$ and $\gamma_m = \mu^2 \partial \ln m_q / \partial \mu^2$ refer to a mass-independent renormalization scheme with scale μ.

less θ^μ_μ not imply invariance under scaling transformations $\xi : x \to e^\xi x$ and hence a continuous mass spectrum? In QCD, the answer is in the negative since the attendant strong gluon fields form a quark condensate $\langle \bar{q}q \rangle_{\mathrm{vac}} \neq 0$. The notion that $\langle \bar{q}q \rangle_{\mathrm{vac}}$ may also be a scale condensate in the scaling limit (3.56) can be deduced from the fact that $\bar{q}q$ is not a singlet under scale transformations

$$i[D, \bar{q}q] = (3 - \gamma_m(\alpha_{\mathrm{IR}}))\bar{q}q\,, \tag{3.57}$$

where we have denoted D as the generator of the symmetry group. It follows that for scale current $D_\mu = x^\nu \theta_{\mu\nu}$, the time-ordered amplitude

$$\Gamma_\mu(q) = \int \mathrm{d}^4x\, e^{iq\cdot x}\, T\langle D_\mu(x)\bar{q}q(0)\rangle_{\mathrm{vac}}\,, \tag{3.58}$$

at α_{IR} is non-vanishing in the zero momentum limit:

$$\begin{aligned} \lim_{q\to 0} q^\mu \Gamma_\mu(q) &= i\langle [D, \bar{q}q]\rangle_{\mathrm{vac}} \\ &= (3 - \gamma_m(\alpha_{\mathrm{IR}})\langle \bar{q}q\rangle_{\mathrm{vac}} \\ &\neq 0\,. \end{aligned} \tag{3.59}$$

In this limit, we see that $\Gamma_\mu(q)$ has an $O(q^{-1})$ singularity, *viz.,* there is a massless 0^{++} QCD dilaton σ coupled to D_μ. This is nothing more than a statement of Goldstone's theorem and thus we conclude that at α_{IR}, there are *nine* NG bosons: π, K, η, σ.

The remainder of this thesis is devoted towards the development of an effective field theory which describes the interactions of this alternate NG sector.

Chapter 4

Chiral-Scale Perturbation Theory about an Infrared Fixed Point

With this chapter, we present the bulk of original research in this dissertation. Our intent is to examine in quantitative detail, the phenomenological consequences which arise from our proposal in Chapter 3 for an infrared fixed point α_{IR} in three-flavour QCD. In Section 4.1, we review a part of the history regarding a dilaton σ in the strong interactions and argue that, in our proposal, the $f_0(500)$ resonance is the most likely candidate for such a state. Section 4.2 introduces the combined expansion about the chiral $SU(3)_L \times SU(3)_R$ and $\alpha_s \to \alpha_{\mathrm{IR}}$ limits. We then review the rules for constructing chiral-scale operators, which feature in the construction of our subsequent effective Lagrangians.

In Section 4.3, we replace ordinary chiral $SU(3)_L \times SU(3)_R$ perturbation theory $\chi\mathrm{PT}_3$ with a new *model-independent* theory $\chi\mathrm{PT}_\sigma$ based on approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry. In leading order and next-to-leading order, we construct the most general effective Lagrangians for the strong interactions. The relationship to the $\chi\mathrm{PT}_3$ expressions is discussed. We present the equations of motion for $\chi\mathrm{PT}_\sigma$ and apply them to construct the 'improved' energy-momentum tensor in the effective theory. By extending the functional methods of Gasser and Leutwyler [13] to include local scale invariance, we calculate a closed form expression for the one-loop effective action.

Important low-energy constants of the new theory are fixed in Section 4.4, where we derive expressions for the σ mass and effective $\sigma\pi\pi$ coupling. We show that the quark mass anomalous dimension at α_{IR} can be related to the πN sigma commutator Σ_{ud} and comment on the apparent contradiction between the $\chi\mathrm{PT}_\sigma$ estimate for the strange sigma term Σ_s and lattice determinations. We then test the convergence of our chiral-scale expansion by adding dilaton loop diagrams to the standard analysis of $\pi\pi$-scattering by Manohar and Georgi [48, 49]. We find that there are *two* scales which govern the convergence of the chiral-scale expansion.

In Section 4.5, we derive an explicit relation between the non-perturbative Drell-Yan ratio and the effective $\sigma\gamma\gamma$ coupling. We then make use of dispersive analyses of data on $\gamma\gamma \to \pi^0\pi^0$ to obtain the prediction $R_{\mathrm{IR}} \approx 5$. To our knowledge, this is the very first determination of R_{IR} in a fully non-perturbative framework of QCD.

Section 4.6 concerns the weak interactions in $\chi\mathrm{PT}_\sigma$. We construct the leading order Lagrangian and show that vacuum alignment induces an effective $K_S\sigma$ coupling. By making use of data on $\gamma\gamma \to \pi^0\pi^0$ and $K_S \to \gamma\gamma$ we determine the value of this coupling and discover that the $\Delta I = 1/2$ rule emerges as a consequence of $\chi\mathrm{PT}_\sigma$. We believe that our proposed explanation for this long standing problem stands as our most important result.

4.1 Historical Overview and Modern Developments

The idea that scale invariance may be an approximate symmetry of the low-energy strong interactions is certainly not new. Before the formulation of QCD, there was some interest in the scale symmetric counterpart of PCAC, partially conserved dilatation current (PCDC), and its phenomenological consequences. It could be combined with existing techniques such as current algebra and low-energy effective Lagrangians, and hence the validity of a particular framework or model tested against experiment.

Naturally, chiral symmetry featured prominently in these investigations and considerable effort was directed towards the development of models which combined both symmetries in a consistent formalism. In this regard, the manner in which chiral symmetry was realized — that is, in the Wigner-Weyl or NG mode — led to important observable differences, for it is not possible for both modes to be realized simultaneously [130].

The starting point of this thesis lies in an approach [15, 16] which considered the possibility that $\langle\bar{q}q\rangle_{\text{vac}} \neq 0$ may also be a condensate for scale transformations in the chiral $SU(3)_L \times SU(3)_R$ limit. The resulting dilaton[1] σ was associated with the $\epsilon(700)$ resonance whose mass and width could — by analogy with π, K, η mesons — conceivably originate from the u, d, s current quark masses. Approximate chiral and scale invariance [15, 16] implied a dominant derivative $\sigma\pi\pi$ coupling

$$\mathcal{L}^{\text{preQCD}}_{\sigma\pi\pi} = F_\sigma^{-1}\sigma\big(|\partial\pi|^2 + O(m_\pi^2)|\pi|^2\big) \tag{4.1}$$

consistent with a broad 0^{++} resonance yet with a *small* effect on $\pi\pi$ scattering in the $SU(2)_L \times SU(2)_R$ limit $\partial = O(m_\pi)$, as observed. Here F_σ is the coupling of σ to the vacuum via the energy momentum tensor $\theta_{\mu\nu}$, 'improved' [137] when spin-0 fields are present:

$$\langle\sigma(q)|\theta_{\mu\nu}|\text{vac}\rangle = (F_\sigma/3)\big(q_\mu q_\nu - g_{\mu\nu}q^2\big)\,. \tag{4.2}$$

Subsequently it was shown [19, 20] that, given the Drell-Yann ratio

$$R(Q^2) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,, \tag{4.3}$$

there is a trace anomaly[2] $(R\alpha/6\pi)F_{\mu\nu}F^{\mu\nu}$, and hence a $\sigma\gamma\gamma$ coupling

$$\mathcal{L}^{\text{preQCD}}_{\sigma\gamma\gamma} = \frac{R\alpha}{6\pi F_\sigma}\sigma F_{\mu\nu}F^{\mu\nu}\,. \tag{4.4}$$

Interest in dilatons waned when QCD arrived in 1972-73: there had to be a gluonic version $\sim G^a_{\mu\nu}G^{a\mu\nu}$ of the trace anomaly and hence (it seemed) no scale invariant limit and no dilaton. The all-orders formula for QCD followed a few years later [125, 126, 127, 128, 129],

$$\theta^\mu_\mu = \frac{\beta(\alpha_s)}{4\alpha_s}G^a_{\mu\nu}G^{a\mu\nu} + \big(1+\gamma_m(\alpha_s)\big)\sum_{q=u,d,s} m_q\bar{q}q \tag{4.5}$$

[1]To avoid confusion with the linear σ-model [131], we reserve the term *dilaton* and notation σ for a NG boson due to *exact scale invariance* in some limit. Furthermore, we are *not* talking about scalar gluonium [132, 133, 134, 135], or "walking gauge theories" [136] where β is small but never zero.

[2]For electromagnetic field strength $F_{\mu\nu}$ and fine-structure constant $\alpha \simeq 1/137$.

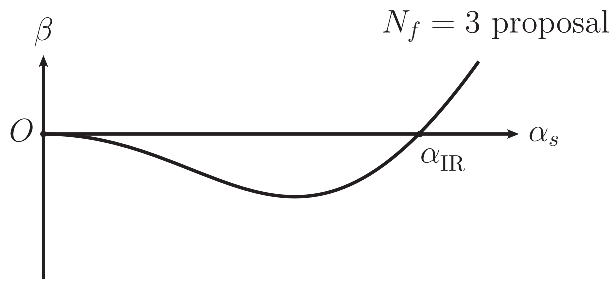


Figure 4.1: Proposed three-flavour β function with an infra-red fixed point α_{IR}.

by which time, the $\epsilon(700)$ resonance had faded away.

Nowadays however, the situation is rather different. The lowest QCD resonance has been reinstated in the PDG tables as $f_0(500)$ and its existence is now generally accepted.[1] As noted in Chapter 2, this change of fortune is largely due to the first-principles determination of its pole on the second sheet at [17, 18]

$$m_{f_0} = 441 - i\,272\,\mathrm{MeV}. \tag{4.6}$$

Consequently, precise knowledge of m_{f_0} has allowed the radiative coupling of f_0 to two photons to be determined with similar accuracy. Dispersive analyses [139] applied to data [6] on $\gamma\gamma \to \pi^0\pi^0$ allow one to extract the two photon width, with the updated determination [140] given by

$$\Gamma(f_0 \to \gamma\gamma) = 1.98^{+0.30}_{-0.24}\,\mathrm{keV}. \tag{4.7}$$

These developments surrounding f_0 and its effect in non-leptonic decays [81], provide the motivation on our part to propose the existence of an infrared fixed point α_{IR} in the three-flavour β function. As noted in Chapter 3, this conclusion refers to low energies, where for $N_f = 3$ light flavours (Fig. 4.1):

$$\beta\,(\alpha_{\mathrm{IR}}) = 0\,,\; N_f = 3\,. \tag{4.8}$$

[1]Although for some controversy on this interpretation, see [138].

At the fixed point, Eq. (4.5) implies

$$\begin{aligned}\theta^\mu_\mu\Big|_{\alpha_s=\alpha_{\rm IR}} &= \big(1+\gamma_m(\alpha_{\rm IR})\big)(m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s)\\ &\to 0\ ,\ SU(3)_L\times SU(3)_R \text{ limit}\end{aligned} \tag{4.9}$$

and hence a dilaton due to quark condensation, as in the pre-QCD theory [15, 16]. With the explicit breaking term in (4.9) dominated by the strange quark, the obvious candidate for this state is f_0 and its interactions with π, K, η are described by a chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$, based on expansions in α_s about $\alpha_{\rm IR}$. This proposed replacement for $\chi\mathrm{PT}_3$ possesses some desirable features, the foremost being:

1. Since f_0 is a broad flavour singlet coupled strongly to π, K, η mesons, f_0 pole terms dominate low-energy scattering of these mesons, including $\pi\pi$ scattering with[1] $O(m_K)$ momenta. We argued in Chapter 2 that in this respect, $\chi\mathrm{PT}_3$ *fails* because it classifies these pole terms as non-leading. That problem is solved in $\chi\mathrm{PT}_\sigma$ because $f_0 = \sigma$ is part of the NG sector and thus contributes in the leading order (LO) of $\chi\mathrm{PT}_\sigma$ expansions (Fig. 4.2). Note that this is achieved without upsetting successful LO $\chi\mathrm{PT}_3$ predictions for amplitudes which do not involve the 0^{++} channel; that is because the $\chi\mathrm{PT}_3$ Lagrangian equals the $\sigma\to 0$ limit of the $\chi\mathrm{PT}_\sigma$ Lagrangian. In NLO, new chiral loop diagrams involving σ need to be checked.

2. The $\Delta I = 1/2$ rule for K-decays emerges as a *consequence* of the effective theory, with a dilaton pole diagram (Fig. 4.3) accounting for the large $I=0$ amplitude in $K_S\to\pi\pi$. Here vacuum alignment of the effective potential induces K_S-σ mixing, with an effective coupling $g_{K_S\sigma}$ fixed by data on $\pi^0\pi^0\to\gamma\gamma$ and $K_S\to\gamma\gamma$.

3. The $\gamma\gamma$ channel constrains the radiative σ coupling, and hence by employing results such as (4.7), one obtains a prediction for the non-perturbative Drell-

[1]But not for $O(m_\pi)$ momenta in our scheme, because the $\sigma\pi\pi$ coupling turns out to be mostly derivative, as in Eq. (4.1). Ordinary chiral $SU(2)_L\times SU(2)_R$ perturbation theory $\chi\mathrm{PT}_2$, with pions as NG bosons and *no* dilaton, remains valid. Because of the relatively large term $m_s\bar{s}s$ in Eq. (3.55) for θ^μ_μ, $\chi\mathrm{PT}_2$ is not sensitive to the behaviour of β.

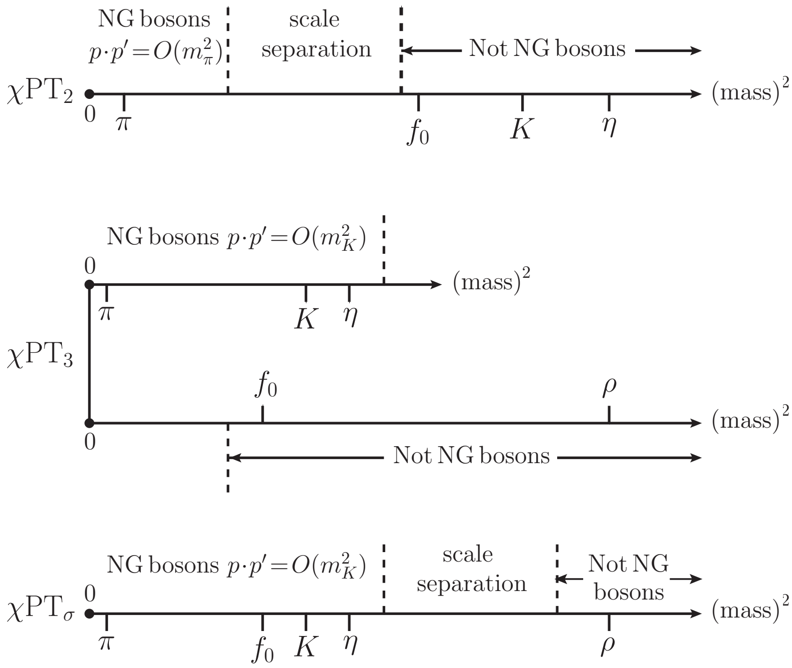


Figure 4.2: Scale separations between Nambu-Goldstone (NG) sectors and other hadrons for each type of chiral perturbation theory χPT discussed in this thesis. Note that scale separation in the two-flavour theory $\chi\mathrm{PT}_2$ (chiral $SU(2) \times SU(2)$, top diagram) is ensured by limiting extrapolations in momenta p, p' to $O(m_\pi)$ (not $O(m_K)$). In conventional three-flavour theory $\chi\mathrm{PT}_3$ (middle diagram), there is *no scale separation*: the non-NG boson $f_0(500)$ sits in the middle of the NG sector $\{\pi, K, \eta\}$. Our three-flavour proposal $\chi\mathrm{PT}_\sigma$ (bottom diagram) for $O(m_K)$ extrapolations in momenta implies a clear scale separation between the NG sector $\{\pi, K, \eta, \sigma = f_0\}$ and the non-NG sector $\{\rho, \omega, K^*, N, \eta', \ldots\}$.

Yan ratio at α_{IR}:

$$R_{\mathrm{IR}} \approx 5\,. \tag{4.10}$$

There are however, a number of subtleties involved in both the construction of $\chi\mathrm{PT}_\sigma$ and the derivation of results such as those listed above. Given the broad width of f_0, can one define a sensible power-counting scheme in the effective theory? If so, what sets the chiral symmetry-breaking scale of the theory? How is the gluonic anomaly incorporated? The purpose of this chapter is to answer precisely these questions and present a detailed exposition on the construction of $\chi\mathrm{PT}_\sigma$.

We shall also use $\chi\mathrm{PT}_\sigma$ to explore phenomenological implications beyond $K_S \to$

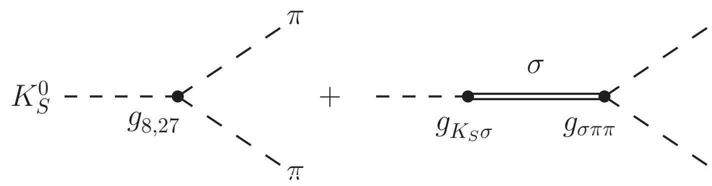


Figure 4.3: Role of a QCD dilaton $\sigma = f_0$ in $K_S \to \pi\pi$ with couplings $g_{K_S\sigma}$ and $g_{\sigma\pi\pi}$ derived from the effective theory $\chi\mathrm{PT}_\sigma$.

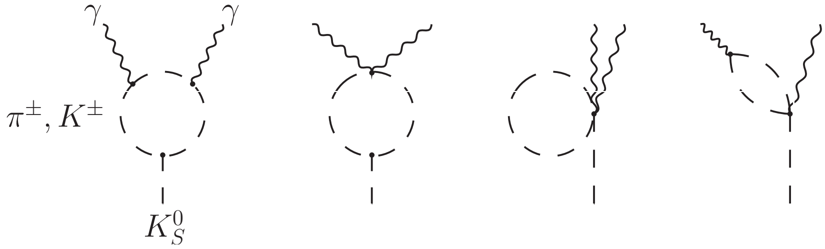


Figure 4.4: Example of leading order $\pi^\pm, K^\pm$ loop graphs for $K_S \to \gamma\gamma$, which are finite in the chiral limit [21, 22].

$\pi\pi$. For example, in the two-photon channel, we encounter the surprising property that charged π, K loops — in $K_S \to \gamma\gamma$ (Fig. 4.4) for example — are finite at one-loop, and thus enter at the *same order* as σ-pole diagrams. These graphs do not affect the electromagnetic trace anomaly at α_{IR}, but the pre-QCD result (4.4) for the effective $\sigma\gamma\gamma$ coupling must be amended by the replacement $R_{\mathrm{IR}} \to R_{\mathrm{IR}} - \frac{1}{2}$:

$$g_{\sigma\gamma\gamma} = \frac{(R_{\mathrm{IR}} - \frac{1}{2})\alpha}{3\pi F_\sigma}\,. \tag{4.11}$$

We present a derivation of this result in Sec. 4.5.1.

4.2 Broken Scale Invariance

Our first step is to construct a chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$ for low-energy amplitudes expanded about the combined limit

$$m_{u,d,s} \sim 0 \qquad \text{and} \qquad \alpha_s \lesssim \alpha_{\mathrm{IR}}\,. \tag{4.12}$$

This is the relevant infrared regime where amplitudes are expanded in powers and logarithms of the external momentum,

$$O(m_K)\,\{\text{momentum}\} \ll \chi_{\text{ch}}\,. \tag{4.13}$$

Recall that in χPT_3, χ_{ch} is $4\pi F_\pi$; a similar result is derived for χPT_σ in Sec. 4.4.3.

To calculate the terms in the perturbation series, we need to construct the most general effective Lagrangian consistent with approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry. This is achieved by employing a set of rules [16, 141, 142] which generate chiral-scale effective Lagrangians from chiral invariant operators. (Explicit symmetry-breaking terms are easily accommodated within the formalism.) These rules were developed long ago and are based on the formal theory of conformal invariance [130, 141, 143, 144]; in particular, the use of the non-linear realization to study spontaneous symmetry breaking.[1]

Scale transformations (or dilatations) correspond to changes in the spacetime coordinates,

$$\xi : x_\mu \to e^{\xi} x_\mu\,, \qquad \xi \in \mathbb{R}\,, \tag{4.14}$$

under which particle fields $\varphi(x)$ transform as follows

$$\xi : \varphi(x) \to e^{\xi d}\varphi(e^{\xi}x)\,, \tag{4.15}$$

where d is the scale dimension of φ. If we denote D as the dilatation generator, then the infinitesimal version of (4.15) reads

$$\delta_\xi \varphi = i[D,\varphi] = (d + x\cdot\partial)\varphi\,. \tag{4.16}$$

For Lagrangian operators of dimension d, the above transformation law generalizes in the obvious way:

$$\delta_\xi \mathcal{L}_d = \partial^\mu(x_\mu \mathcal{L}_d) + (d-4)\mathcal{L}_d\,, \tag{4.17}$$

and thus we recover the well known result that — up to a total divergence — only $d = 4$ operators are allowed in a scale-invariant theory. Note that in these theories,

[1] For an elegant exposition, see [145, Chapter 3].

scale invariance of the free-field massless Lagrangians

$$\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \qquad \text{and} \qquad \tfrac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi \tag{4.18}$$

becomes manifest if one assigns $d_\phi = 1$ and $d_\psi = \frac{3}{2}$ for the boson and fermion fields respectively.

To obtain dynamical information on (exact or broken) scale symmetry we need the corresponding current. It is well known that the energy-momentum tensor is not uniquely defined in a quantum field theory. Callan, Coleman, and Jackiw [137] advocated for a redefinition, or so-called 'improvement', of the symmetric Belinfante tensor which allows one to define a dilatation current

$$D_\mu(x) = x^\nu\theta_{\mu\nu}(x)\,. \tag{4.19}$$

The advantage of this procedure is that the divergence of D_μ is given purely by a sum of scale-breaking $d \neq 4$ Lagrangian operators $\mathcal{L}_d$

$$\partial^\mu D_\mu(x) = \theta^\mu_\mu(x) = \sum_d (d-4)\mathcal{L}_d\,. \tag{4.20}$$

Notice that the connection between exact scale invariance and a vanishing trace θ^μ_μ is now manifest. Furthermore, note that the added terms required to obtain the 'improved' $\theta_{\mu\nu}$ do not contribute to space integrals of moments like $\theta_{0\mu}$ and hence leave the Poincaré generators and their commutation relations intact [146].

The situation when the right-hand side of (4.20) is non-zero forms the basis for what is known as the PCDC hypothesis [146, 147]. By analogy with PCAC, one assumes that θ^μ_μ acts as an interpolating field for f_0, with θ^μ_μ matrix elements dominated by the 0^{++} pole. Many of the pre-QCD results [15, 16] involving dilatons were derived under this hypothesis.

In non-linear realizations of scale symmetry, a chiral *invariant* field σ appears as terms $\sim \partial\sigma$ in covariant derivatives. It transforms as

$$\xi : \sigma \to \sigma + F_\sigma \log|\det(\partial x'/\partial x)| \tag{4.21}$$

under conformal transformations $x \to x'$. The infinitesimal transformation law is then given by

$$\delta_\xi \sigma = F_\sigma + x \cdot \partial \sigma \,, \tag{4.22}$$

and thus the field $\exp(\sigma/F_\sigma)$ transforms covariantly with $d = 1$:

$$\delta_\xi [\exp(\sigma/F_\sigma)] = (1 + x \cdot \partial) \exp(\sigma/F_\sigma) \,. \tag{4.23}$$

By combining powers of e^{σ/F_σ} with chiral Lagrangian operators such as

$$\mathcal{K}[U, U^\dagger] = \tfrac{1}{4} F_\pi^2 \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \,, \tag{4.24}$$

or the dilaton kinetic energy $\mathcal{K}_\sigma = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma$, chiral-scale operators of the desired dimension are formed. For example, these operators become scale invariant with $d = 4$ after the rescaling

$$\mathcal{K} \to \mathcal{K} e^{2\sigma/F_\sigma} \qquad \text{and} \qquad \mathcal{K}_\sigma \to \mathcal{K}_\sigma e^{2\sigma/F_\sigma} \,. \tag{4.25}$$

Note that σ and ϕ_i do not have conventional scaling properties. Indeed, by (4.21), σ scales inhomogeneously, while the dimension of ϕ_i (and hence U) must be zero to ensure that the chiral charge algebra of Eq. (2.7) remains consistent under scale transformations[1] [16, 142].

4.3 Chiral-Scale Lagrangian

We would now like to construct an effective field theory of approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry in the physical region

$$0 < \alpha_s < \alpha_{\mathrm{IR}} \,. \tag{4.26}$$

In the infrared regime $\alpha_s \lesssim \alpha_{\mathrm{IR}}$, the most general effective Lagrangian is composed of three pieces

$$\mathcal{L}[\sigma, U, U^\dagger] = :\mathcal{L}^{d=4}_{\mathrm{inv}} + \mathcal{L}^{d>4}_{\mathrm{anom}} + \mathcal{L}^{d<4}_{\mathrm{mass}}: \,, \tag{4.27}$$

[1] Alternatively, if one considers the general transformation law $\xi : \phi_i(x) \to e^{\xi d}\phi_i(e^\xi x)$, then for $d \neq 0$, the dilatation current is not chiral invariant [148].

where each term is characterised by the manner in which they preserve or break chiral and scale symmetry, and the colons refer to normal ordered operators. Here $\mathcal{L}_{\text{inv}}$ is an $SU(3)_L \times SU(3)_R$ singlet and carries dimension $d = 4$, while $\mathcal{L}_{\text{mass}}$ contains the quark mass matrix (2.24)

$$M = \tfrac{1}{2}F_\pi^2 B_0 \text{diag}(m_u, m_d, m_s), \tag{4.28}$$

and hence belongs to the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation. The operator dimension of $\mathcal{L}_{\text{mass}}$ satisfies $1 \le d_{\text{mass}} < 4$ as in the pre-QCD theory[1] [15, 16] but with

$$d_{\text{mass}} = 3 - \gamma_m(\alpha_{\text{IR}}), \tag{4.29}$$

as a result of expanding about α_{IR} (cf. Eq. (3.51)). The term $\mathcal{L}_{\text{anom}}$ simulates the gluonic anomaly, whose dimension is found by noting that the operator insertion of $G^a_{\mu\nu}G^{a\mu\nu}$ corresponds to the $\beta\partial/\partial\alpha_s$ term in the Callan-Symanzik equation

$$\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \gamma_m(\alpha_s)\sum_q m_q\frac{\partial}{\partial m_q}\Big\}\mathcal{A} = 0 \tag{4.30}$$

for renormalization-group invariant QCD amplitudes $\mathcal{A}$. Taking $\partial/\partial\alpha_s$, we find

$$\Big\{\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \beta'(\alpha_s)\Big\}\frac{\partial\mathcal{A}}{\partial\alpha_s} = -\sum_q m_q\frac{\partial^2\{\gamma_m(\alpha_s)\mathcal{A}\}}{\partial m_q \partial\alpha_s}, \tag{4.31}$$

so for $\alpha_s \lesssim \alpha_{\text{IR}}$, $\mathcal{L}_{\text{anom}}$ has a positive anomalous dimension equal to the slope of β at the fixed point:

$$d_{\text{anom}} = 4 + \beta'(\alpha_{\text{IR}}) > 4. \tag{4.32}$$

As $\alpha_s \to \alpha_{\text{IR}}$, the gluonic anomaly vanishes, however this does not uniquely determine the chiral-scale power counting of terms in $\mathcal{L}_{\text{anom}}$. In principle, we could have constructed a chiral-scale perturbation theory with m_σ and m_K as independent expansion parameters, but that would make sense only if there were a fourth light quark or different low-energy scales for chiral and scale expansions. Fig. 4.2 pro-

[1]The upper bound on d_{mass} is required so that PCAC holds when $SU(3)_L \times SU(3)_R$ symmetry is spontaneously broken; the lower bound is a consequence of the Källén-Lehmann spectral representation of the two-point function [149].

vides clear confirmation that the choice $m_\sigma = O(m_K)$ is sensible.

The above considerations, combined with the machinery developed in Sec. 4.2, lead to an explicit formula for the $\chi\mathrm{PT}_\sigma$ Lagrangian (4.27) in LO:

$$
\begin{aligned}
\mathcal{L}^{d=4}_{\mathrm{inv,LO}} &= \{c_1\mathcal{K} + c_2\mathcal{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\}e^{2\sigma/F_\sigma}\,, \\
\mathcal{L}^{d>4}_{\mathrm{anom,LO}} &= \{(1-c_1)\mathcal{K} + (1-c_2)\mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\}e^{(2+\beta')\sigma/F_\sigma}\,, \\
\mathcal{L}^{d<4}_{\mathrm{mass,LO}} &= \mathrm{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,,
\end{aligned}
\tag{4.33}
$$

where β' and γ_m are the anomalous dimensions $\beta'(\alpha_{\mathrm{IR}})$ and $\gamma_m(\alpha_{\mathrm{IR}})$ of Eqs. (4.32) and (4.29). Note that in the limit $\sigma \to 0$, (4.33) reduces to $\mathcal{L}_2$ of Eq. (2.23), and thus LO predictions for $\chi\mathrm{PT}_3$ without a 0^{++} channel are preserved. (As claimed.)

Characteristically of effective field theories, the low-energy constants c_i are not fixed a priori by symmetry arguments alone. However, the requirement of a stable vacuum in the σ direction (no tadpoles) implies that c_3 and c_4 depend on how the field σ is chosen. For expansions about $\sigma = 0$, all terms linear in σ must cancel:

$$
\begin{aligned}
4c_3 + (4+\beta')c_4 &= (\gamma_m - 3)\langle \mathrm{Tr}(MU^\dagger + UM^\dagger)\rangle_{\mathrm{vac}} \\
&= (\gamma_m - 3)F_\pi^2\big(m_K^2 + \tfrac{1}{2}m_\pi^2\big)\,.
\end{aligned}
\tag{4.34}
$$

Because of our requirement $\mathcal{L}_{\mathrm{anom}} = O(\partial^2, M)$, both c_3 and c_4 are $O(M)$.

4.3.1 Local Scale Invariance

We are now in the position to extend the functional methods of $\chi\mathrm{PT}_3$ to include the additional invariance under scaling transformations (4.14). This can be achieved by noting that under local resizings, the particle fields $\varphi(x)$ and (flat) metric[1] $g_{\mu\nu}$ obey the following transformation laws (see e.g. [150])

$$
\varphi(x) \to e^{d\,\xi(x)}\varphi(x)\,, \qquad g^{\mu\nu} \to e^{2\xi(x)}g^{\mu\nu}\,,
\tag{4.35}
$$

[1] Our metric signature is $(+---)$.

where d is the scale dimension (or conformal weight) and $\xi(x)$ the conformal factor. The local nature of (4.35) implies the introduction of a covariant derivative

$$D_\mu \varphi = \partial_\mu \varphi + d S_\mu \varphi \,, \tag{4.36}$$

which transforms as $D_\mu \varphi \to e^{d\xi(x)} D_\mu \varphi$ provided the vector field S_μ transforms as

$$S_\mu \to S_\mu - \partial_\mu \xi(x) \,. \tag{4.37}$$

In the case of QCD, we then need to extend the chiral symmetric Lagrangian $\mathring{\mathcal{L}}$ of Eq. (2.2) to include the additional external field S_μ,

$$\mathcal{L}_{\text{ext}} = \mathring{\mathcal{L}} + \bar{q}\gamma^\mu (v_\mu + a_\mu \gamma_5) q - \bar{q}(s - i p \gamma_5) q + S_\mu D^\mu \,, \tag{4.38}$$

so that the generating functional $Z[v, a, s, p, S]$ is now defined through the path integral

$$\begin{aligned} \exp\{iZ[v,a,s,p,S]\} &= \langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle_{v,a,p,s,S} \\ &= \int [Dq][D\bar{q}][DA_\mu] \exp\left\{ i \int \mathrm{d}^4 x \, \mathcal{L}_{\text{ext}} \right\}. \end{aligned} \tag{4.39}$$

Expanding about the point $v_\mu = a_\mu = s = p = S = 0$ then gives us the Green functions of QCD in the chiral-scale limit (4.9). For the low-energy representation, it is very convenient to introduce the field

$$X = F_\sigma e^{\sigma/F_\sigma} \,, \tag{4.40}$$

so that $D_\mu X = (\partial_\mu + S_\mu) X$ tranforms covariantly with weight $d = 1$. Note that the chiral $SU(3)_L \times SU(3)_R$ covariant derivative ∇_μ associated with the U fields is unchanged by (4.35) since U is a singlet under scale transformations.

Just like the external fields r_μ and l_μ contained in ∇_μ, the external field S_μ is counted as $O(\partial = m_K)$ in chiral-scale power counting, with field strength

$$S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu \,. \tag{4.41}$$

The low-energy representation of Z is then given by

$$e^{iZ} = \int [DU][DX] \exp\left\{ i \int \mathrm{d}^4 x \, \mathcal{L}_{\text{eff}} \right\}, \tag{4.42}$$

and thus the generating functional factorises in the same fashion as $\chi\mathrm{PT}_3$,

$$Z = Z_2 + Z_4 + \ldots . \tag{4.43}$$

In this formalism, the scale dimension of a given operator can be deduced by explicitly writing out the metric contractions; for example, the $O(\partial^2)$ terms $(D_\mu X)^2$ and $\mathrm{Tr}(\nabla_\mu U \nabla^\mu U^\dagger)$ have conformal weight 4 and 2 respectively,

$$(D_\mu X)^2 = \underbrace{g^{\mu\nu}}_{d=2} \underbrace{D_\mu X}_{d=1} \underbrace{D_\nu X}_{d=1}, \qquad \mathrm{Tr}(\nabla_\mu U \nabla^\mu U^\dagger) = \underbrace{g^{\mu\nu}}_{d=2} \mathrm{Tr}(\underbrace{\nabla_\mu U}_{d=0} \underbrace{\nabla_\nu U^\dagger}_{d=0}). \tag{4.44}$$

A simple rescaling of the low-energy coefficients $c_1, \ldots, c_4$ appearing in $\mathcal{L}_{\text{str}}$

$$\begin{gathered} z_1 = \frac{c_1}{F_\sigma^2}, \qquad z_2 = c_2, \qquad z_3 = \frac{c_3}{F_\sigma^4}, \\ z_4 = \frac{1-c_1}{F_\sigma^{2+\beta'}}, \quad z_5 = \frac{1-c_2}{F_\sigma^{\beta'}}, \quad z_6 = \frac{c_4}{F_\sigma^{4+\beta'}}, \quad z_7 = \frac{1}{F_\sigma^{3-\gamma_m}}, \end{gathered} \tag{4.45}$$

allows us to rewrite the σ dependence in terms of X, to wit the LO Lagrangian of *local* scale and chiral $SU(3)_L \times SU(3)_R$ symmetry is

$$\mathcal{L}_{\text{LO}}^{d=4} = z_1 \mathcal{K} X^2 + z_2 \tfrac{1}{2}(D_\mu X)^2 + z_3 X^4, \tag{4.46}$$

$$\mathcal{L}_{\text{LO}}^{d>4} = z_4 \mathcal{K} X^{2+\beta'} + z_5 \tfrac{1}{2}(D_\mu X)^2 X^{\beta'} + z_6 X^{4+\beta'}, \tag{4.47}$$

$$\mathcal{L}_{\text{LO}}^{d<4} = z_7 \mathrm{Tr}(\chi U^\dagger + U \chi^\dagger) X^{3-\gamma_m}. \tag{4.48}$$

4.3.2 Equations of Motion

The equations of motion for the U and X (or σ) fields play a useful role in constructing the trace of $\theta_{\mu\nu}$ in the effective theory and also in applications of the background field method. Here we simply list the results; a derivation is provided in Appendix

B. For the U field we get at LO (noting the additional term involving $\partial_\mu X$),

$$\begin{aligned}0 = {} & [(\nabla^\mu \nabla_\mu U)U^\dagger - U(\nabla^\mu \nabla_\mu U^\dagger)][z_1 X^2 + z_4 X^{2+\beta'}] \\ & + \frac{4}{F_\pi^2} z_7 [U\chi^\dagger - \chi U^\dagger + \tfrac{1}{3}\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)] X^{3-\gamma_m} \\ & + [(\nabla^\mu U)U^\dagger - U(\nabla^\mu U^\dagger)][2z_1 X + (2+\beta')z_4 X^{1+\beta'}]\partial_\mu X\,, \end{aligned} \tag{4.49}$$

while the X field satisfies

$$\begin{aligned}0 = {} & (D_\mu D^\mu X)[z_2 + z_5 X^{\beta'}] + \beta' z_5 \tfrac{1}{2}(D_\mu X)^2 X^{\beta'-1} - \mathcal{K}[2z_1 X + (2+\beta')z_4 X^{1+\beta'}] \\ & - 4z_3 X^3 - (4+\beta')z_6 X^{3+\beta'} - (3-\gamma_m) z_7 \mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) X^{2-\gamma_m}\,. \end{aligned} \tag{4.50}$$

4.3.3 Trace Anomaly in the Effective Theory

Because of Eqs. (4.20) and (4.32), the critical exponent β' normalises the gluonic term in the trace of the effective energy-momentum tensor,

$$\theta^\mu_\mu\Big|_{\text{eff}} = :\beta' \mathcal{L}^{d>4}_{\text{anom}} - (1+\gamma_m)\mathcal{L}^{d<4}_{\text{mass}}: \tag{4.51}$$

We derive the explicit expression for (4.51) by adapting the prescription of Coleman *et al.* [137] to nonlinear realizations of scale symmetry. Consequently, derivative terms in the *canonical* energy-momentum tensor are scaled by the factors

$$\psi_1 = z_1 X^2 + z_4 X^{2+\beta'} \qquad \text{and} \qquad \psi_2 = z_2 + z_5 X^{\beta'}\,, \tag{4.52}$$

while the 'improvement' term for spin-0 fields takes the form

$$\mathcal{I}_{\mu\nu}[X] = \frac{1}{6}(g_{\mu\nu} D^2 - D_\mu D_\nu)\left\{ z_2 X^2 + \frac{2}{2+\beta'} z_5 X^{2+\beta'} \right\}. \tag{4.53}$$

The coefficients in $\mathcal{I}_{\mu\nu}$ are fixed by the requirement that the trace be expressed in terms of explicit scale-breaking operators (Eq. 4.20). By combining these factors in the appropriate manner, we arrive at the LO expression for the 'improved' energy-momentum tensor

$$\begin{aligned}\theta_{\mu\nu}\Big|_{\text{eff}} = :\{\tfrac{1}{2}F_\pi^2\text{Tr}(\nabla_\mu U \nabla_\nu U^\dagger) - g_{\mu\nu}\mathcal{K}\}\psi_1 - g_{\mu\nu}\text{Tr}z_7(\chi U^\dagger + U\chi^\dagger) \\ + \mathcal{I}_{\mu\nu} + \{D_\mu X D_\nu X - g_{\mu\nu}\tfrac{1}{2}(D_\alpha X)^2\}\psi_2 - g_{\mu\nu}\{z_3 + z_6 X^{\beta'}\}X^4 :,\end{aligned} \tag{4.54}$$

with the trace obtained via the X equations of motion:

$$\begin{aligned}\theta^\mu_\mu\Big|_{\text{eff}} &= :\beta'\mathcal{L}^{d>4}_{\text{anom}} - (1+\gamma_m)\mathcal{L}^{d<4}_{\text{mass}} : \\ &= :\beta'\{z_4\mathcal{K}X^{2+\beta'} + z_5\tfrac{1}{2}(D_\mu X)^2 X^{\beta'} + z_6 X^{4+\beta'}\} \\ &\quad - (1+\gamma_m)z_7\text{Tr}(MU^\dagger + UM^\dagger)X^{3-\gamma_m} : .\end{aligned} \tag{4.55}$$

4.3.4 The Next-to-Leading Order Lagrangian

The technique used to obtain Eq. (4.33) from χPT_σ also works for $O(\partial^4, M\partial^2, M^2)$ terms in $\mathcal{L}$ (and in $\mathcal{L}_{\text{weak}}$ below). At next to leading order, the most general effective Lagrangian

$$\mathcal{L}_{\text{NLO}}[U, U^\dagger, X] =: \mathcal{L}^{d=4}_{\text{NLO}} + \mathcal{L}^{d>4}_{\text{NLO}} + \mathcal{L}^{d<4}_{\text{NLO}} : \tag{4.56}$$

contains 38 distinct terms, where $\mathcal{L}^{d=4}_{\text{NLO}}$ is assigned unprimed low-energy constants $\{l_i, h_i, s_i\}$, $\mathcal{L}^{d>4}_{\text{NLO}}$ primed $\{l'_i, h'_i, s'_i\}$, and $\mathcal{L}^{d<4}_{\text{NLO}}$ double-primed $\{l''_i, h''_i, s''_i\}$. To keep track of the scale dimensions we write out the metric factors for each term:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{NLO}}^{d=4} &= l_1[g^{\mu\nu}\mathrm{Tr}(\nabla_\mu U \nabla_\nu U^\dagger)]^2 + g^{\mu\alpha}g^{\nu\lambda}\big[l_2\mathrm{Tr}(\nabla_\mu U\nabla_\nu U^\dagger)\mathrm{Tr}(\nabla_\alpha U\nabla_\lambda U^\dagger) \\
&\quad + l_3\mathrm{Tr}(\nabla_\mu U\nabla_\alpha U^\dagger\nabla_\nu U\nabla_\lambda U^\dagger)\big] + g^{\mu\nu}\big[l_4\mathrm{Tr}(\nabla_\mu U\nabla_\nu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\
&\quad + l_5\mathrm{Tr}(\nabla_\mu U\nabla_\nu U^\dagger(\chi U^\dagger + U\chi^\dagger))\big]X^2 + \big[l_6\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)^2 \\
&\quad + l_7\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)^2 + l_8\mathrm{Tr}(U\chi^\dagger U\chi^\dagger + \chi U^\dagger\chi U^\dagger)\big]X^4 \\
&\quad - g^{\mu\alpha}g^{\nu\lambda}\big[i\,l_9\mathrm{Tr}(f^R_{\mu\nu}\nabla_\alpha U\nabla_\lambda U^\dagger + f^L_{\mu\nu}\nabla_\alpha U^\dagger\nabla_\lambda U) - l_{10}\mathrm{Tr}(Uf^L_{\mu\nu}U^\dagger f^R_{\alpha\lambda}) \\
&\quad - h_1\mathrm{Tr}(f^R_{\mu\nu}f^R_{\alpha\lambda} + f^L_{\mu\nu}f^L_{\alpha\lambda})\big] + h_2\mathrm{Tr}(\chi\chi^\dagger)X^4 + h_3 g^{\mu\alpha}g^{\nu\lambda}S_{\mu\nu}S_{\alpha\lambda} \\
&\quad + h_4 g^{\mu\alpha}g^{\nu\lambda}S_{\mu\nu}\mathrm{Tr}(f^R_{\alpha\lambda} + f^L_{\alpha\lambda}) + s_1 g^{\mu\nu}D_\mu X D_\nu X\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\
&\quad + s_2 g^{\mu\alpha}g^{\nu\lambda}S_{\mu\nu}\mathrm{Tr}(\nabla_\alpha U\nabla_\lambda U^\dagger), && (4.57) \\
\mathcal{L}_{\mathrm{NLO}}^{d>4} &= X^{\beta'}\mathcal{L}_{\mathrm{NLO}}^{d=4}\Big|_{\{l_i,h_i,s_i\}\longleftrightarrow\{l'_i,h'_i,s'_i\}}, && (4.58) \\
\mathcal{L}_{\mathrm{NLO}}^{d<4} &= X^{1-\gamma_m}\big[l''_4\mathrm{Tr}(\nabla_\mu U\nabla^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\
&\quad + l''_5\mathrm{Tr}(\nabla_\mu U\nabla^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger))\big] + X^{3-\gamma_m}\big[l''_6\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)^2 \\
&\quad + l''_7\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)^2 + l''_8\mathrm{Tr}(U\chi^\dagger U\chi^\dagger + \chi U^\dagger\chi U^\dagger) + h''_2\mathrm{Tr}(\chi\chi^\dagger)\big]. && (4.59)
\end{aligned}
$$

The requirement that the NLO expression above reduces to the $\chi\mathrm{PT}_3$ result [13] in the $\sigma \to 0$ limit (or equivalently $X \to F_\sigma$) implies that the coefficients are not totally independent. By comparison with $\mathcal{L}_4$ from Eq. (2.46), we find

$$
\begin{aligned}
l_i + l'_i F_\sigma^{\beta'} &= L_i\,, \qquad i = 1,2,3,9,10\,, && (4.60) \\
h_1 + h'_1 F_\sigma^{\beta'} &= H_1\,, && (4.61) \\
l_i + l'_i F_\sigma^{\beta'} + l''_i F_\sigma^{1-\gamma_m} &= L_i\,, \qquad i = 4,\dots,8\,, && (4.62) \\
h_2 + h'_2 F_\sigma^{\beta'} + h''_2 F_\sigma^{1-\gamma_m} &= H_2\,. && (4.63)
\end{aligned}
$$

Note that to the extent allowed by data for soft-σ amplitudes, $O(\partial^4)$ coefficients can be predicted by saturation by non-Goldstone resonances — like $\chi\mathrm{PT}_3$ [46], but with f_0 *excluded.*

4.3.5 The One-Loop Effective Action

We now extend the method of Gasser and Leutwyler [13] so that the effective action is computed by expanding about *spacetime-dependent* solutions $\bar{U}(x) = u^2(x)$ and $\bar{X}(x)$ to the classical equations of motion (4.49-4.50). In $\chi\mathrm{PT}_\sigma$, we consider the expansion in terms of fluctutation fields $\xi = \xi^i \lambda^i$ and ρ,

$$U = u e^{i\xi} u = u(1 + i\xi - \tfrac{1}{2}\xi^2 + \ldots)u\,, \tag{4.64}$$

$$X = \bar{X} e^{\rho} = \bar{X}(1 + \rho + \tfrac{1}{2}\rho^2 + \ldots)\,, \tag{4.65}$$

where cubic or higher order terms in the fluctuation fields are not shown explicitly. To quadratic order in ξ and ρ, the following Lagrangian operators read (where d is some scale dimension)

$$\begin{aligned}
\mathrm{Tr}(\nabla_\mu U \nabla^\mu U^\dagger) X^d &= \mathrm{Tr}(\nabla_\mu \bar{U} \nabla^\mu \bar{U}^\dagger)\bar{X}^d + i L_\xi \bar{X}^d + d\rho\,\mathrm{Tr}(\nabla_\mu \bar{U}\nabla^\mu \bar{U}^\dagger)\bar{X}^d \\
&\quad + i d\rho L_\xi \bar{X}^d + Q_\xi \bar{X}^d + \tfrac{1}{2} d^2\rho^2 \mathrm{Tr}(\nabla_\mu \bar{U}\nabla^\mu \bar{U}^\dagger)\bar{X}^d\,, \qquad (4.66)\\
\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) X^d &= \mathrm{Tr}(\chi \bar{U}^\dagger + \bar{U}\chi^\dagger)\bar{X}^d + i\,\mathrm{Tr}(\xi\Sigma^-)\bar{X}^d \\
&\quad + d\rho\,\mathrm{Tr}(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger)\bar{X}^d + i d\rho\,\mathrm{Tr}(\xi\Sigma^-)\bar{X}^d - \tfrac{1}{2}\mathrm{Tr}(\xi^2\Sigma^+)\bar{X}^d \\
&\quad + \tfrac{1}{2} d^2\rho^2 \mathrm{Tr}(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger)\bar{X}^d\,, \qquad (4.67)\\
\tfrac{1}{2}(D_\mu X)^2 X^d &= \tfrac{1}{2}(D_\mu \bar{X})^2 \bar{X}^d + (2+d)\rho\tfrac{1}{2}(D_\mu\bar{X})^2\bar{X}^d + (D_\mu\rho D^\mu \bar{X})\bar{X}^{d+1} \\
&\quad + \tfrac{1}{2}(d+2)^2\rho^2\tfrac{1}{2}(D_\mu\bar{X})^2\bar{X}^d + (2+d)\rho(D_\mu\rho D^\mu\bar{X})\bar{X}^{1+d} \\
&\quad + \tfrac{1}{2}(D_\mu\rho)^2\bar{X}^{2+d}\,, \qquad (4.68)
\end{aligned}$$

where we have defined the following,

$$\begin{aligned}
\Sigma^\pm &= u\chi^\dagger u \pm u^\dagger \chi u^\dagger\,, \\
L_\xi &= \mathrm{Tr}(\nabla_\mu \bar{U}^\dagger \nabla^\mu (u\xi u)) - \mathrm{Tr}(\nabla_\mu(u^\dagger \xi u^\dagger)\nabla^\mu \bar{U})\,, \qquad (4.69)\\
Q_\xi &= \mathrm{Tr}(\nabla_\mu(u^\dagger\xi u^\dagger)\nabla^\mu(u\xi u)) - \tfrac{1}{2}\mathrm{Tr}(\nabla_\mu\bar{U}^\dagger\nabla^\mu(u\xi^2 u)) - \tfrac{1}{2}\mathrm{Tr}(\nabla_\mu\bar{U}\nabla^\mu(u^\dagger\xi^2 u^\dagger))\,. \qquad (4.70)
\end{aligned}$$

The terms linear in ξ and ρ vanish from the lowest order action by the equations of motion; see Appendix C. However, one must keep track of the mixed terms of $O(\xi\rho)$.

Collecting each of the quadratic pieces gives

$$S\Big|_{\xi^2} = \int \mathrm{d}^4x \left\{ \tfrac{1}{4}F_\pi^2 Q_\xi (z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'}) - z_7\tfrac{1}{2}\mathrm{Tr}(\xi^2\Sigma^+)\bar{X}^{3-\gamma_m} \right\}, \tag{4.71}$$

$$S\Big|_{\xi\rho} = -i\int \mathrm{d}^4x\,\rho \left\{ \tfrac{1}{4}F_\pi^2 L_\xi (2z_1\bar{X}^2 + (2+\beta')z_4\bar{X}^{2+\beta'}) + (3-\gamma_m)z_7\mathrm{Tr}(\xi\Sigma^-)\bar{X}^{3-\gamma_m} \right\}, \tag{4.72}$$

$$\begin{aligned} S\Big|_{\rho^2} = \int \mathrm{d}^4x \Big\{ &-\tfrac{1}{2}\rho D^\mu D_\mu\rho (z_2\bar{X}^2 + z_5\bar{X}^{2+\beta'}) + \tfrac{1}{2}\rho^2 \big[\bar{\mathcal{K}}(4z_1\bar{X}^2 + z_4(2+\beta')^2\bar{X}^{2+\beta'}) \\ &+ 16z_3\bar{X}^4 + (4+\beta')^2 z_6\bar{X}^{4+\beta'} - \tfrac{1}{2}(D^\mu D_\mu\bar{X})(2z_2\bar{X} + (2+\beta')z_5\bar{X}^{1+\beta'}) \\ &+ \tfrac{1}{2}(D_\mu\bar{X})^2(2z_2 + (2+\beta')z_5\bar{X}^{\beta'}) + (3-\gamma_m)^2\mathrm{Tr}(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger)\bar{X}^{3-\gamma_m}\big] \Big\} \\ = \int \mathrm{d}^4x \Big\{ &-\tfrac{1}{2}(z_2\bar{X}^2 + z_5\bar{X}^{2+\beta'})\rho D^\mu D_\mu\rho + \tfrac{1}{2}\Lambda_\rho\rho^2 \Big\}, \end{aligned} \tag{4.73}$$

where Λ_ρ denotes the coefficient of the quadratic term in ρ which does not involve any derivatives. Following [13], we can write the expressions (4.71-4.72) in a more compact formalism by introducing the anti-hermitian matrices

$$\Gamma_\mu = \tfrac{1}{2}[u^\dagger, \partial_\mu u] - \tfrac{1}{2}iu^\dagger r_\mu u - \tfrac{1}{2}iul_\mu u^\dagger, \tag{4.74}$$

$$\Delta_\mu = \tfrac{1}{2}u^\dagger\nabla_\mu\bar{U}u^\dagger = -\tfrac{1}{2}u\nabla_\mu\bar{U}^\dagger u, \tag{4.75}$$

and defining a covariant derivative for ξ:

$$d_\mu\xi = \partial_\mu\xi + [\Gamma_\mu, \xi]. \tag{4.76}$$

In this notation, one finds

$$L_\xi = -4\mathrm{Tr}(\Delta_\mu d^\mu\xi), \tag{4.77}$$

$$Q_\xi = \mathrm{Tr}(d_\mu\xi d^\mu\xi - [\Delta_\mu, \xi][\Delta^\mu, \xi]), \tag{4.78}$$

and by shifting the derivative d_μ through integration by parts, we find

$$S\Big|_{\xi^2} = -\int \mathrm{d}^4x \left\{ \tfrac{1}{4}F_\pi^2 \mathrm{Tr}(\xi d^\mu d_\mu \xi + [\Delta_\mu, \xi][\Delta^\mu, \xi])(z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'}) \right.$$
$$+ z_7\tfrac{1}{2}\mathrm{Tr}(\xi^2\Sigma^+)\bar{X}^{3-\gamma_m} - \tfrac{1}{8}F_\pi^2\mathrm{Tr}(\xi^2)\big[(2z_1 + (2+\beta')(1+\beta')z_4\bar{X}^{\beta'})(\partial_\mu\bar{X})^2$$
$$\left. - (2z_1\bar{X} + (2+\beta')z_4\bar{X}^{1+\beta'})\partial^\mu\partial_\mu\bar{X}\big] \right\}, \tag{4.79}$$

$$S\Big|_{\xi\rho} = -i\int \mathrm{d}^4x\, \rho \left\{ F_\pi^2 \mathrm{Tr}(\Delta_\mu d^\mu \xi)(2z_1\bar{X}^2 + (2+\beta')z_4\bar{X}^{2+\beta'}) \right.$$
$$\left. - (3-\gamma_m)z_7\mathrm{Tr}(\xi\Sigma^-)\bar{X}^{3-\gamma_m} \right\}. \tag{4.80}$$

By writing out the components ξ^i

$$\xi = \xi^i\lambda^i, \qquad \mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}, \tag{4.81}$$

and defining

$$A^{ij}\xi^j = \frac{F_\pi^2}{4}(z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'})d^\mu d_\mu \xi^i + \Lambda^{ij}\xi^j, \tag{4.82}$$

$$\Lambda^{ij} = \frac{F_\pi^2}{8}\left\{ \mathrm{Tr}([\lambda^i, \Delta_\mu][\lambda^j, \Delta^\mu])(z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'}) + \frac{z_7}{F_\pi^2}\mathrm{Tr}(\{\lambda^i, \lambda^j\}\Sigma^+)\bar{X}^{3-\gamma_m} \right.$$
$$- 2\delta^{ij}\big[(2z_1 + (2+\beta')(1+\beta')z_4\bar{X}^{\beta'})(\partial_\mu\bar{X})^2$$
$$\left. - (2z_1\bar{X} + (2+\beta')z_4\bar{X}^{1+\beta'})\partial^\mu\partial_\mu\bar{X}\big] \right\}, \tag{4.83}$$

$$B = (z_2\bar{X}^2 + z_5\bar{X}^{2+\beta'})D^\mu D_\mu \rho - \Lambda_\rho\,, \tag{4.84}$$

$$C^i = -2(3-\gamma_m)z_7\mathrm{Tr}(\lambda^i\Sigma^-)\bar{X}^{3-\gamma_m}\,, \tag{4.85}$$

$$D^{i\mu} = 2F_\pi^2\mathrm{Tr}(\Delta^\mu\lambda^i)(2z_1\bar{X}^2 + (2+\beta')z_4\bar{X}^{2+\beta'}), \tag{4.86}$$

the quadratic terms in ξ and ρ can be collected in the formal expression

$$S^{1-\mathrm{loop}} = -\tfrac{1}{2}\int \mathrm{d}^4x \left\{ \xi^i A^{ij}\xi^j + \rho B\rho + i\rho[\xi^i C^i + D^{i\mu}d_\mu\xi^i] \right\}, \tag{4.87}$$

where C^i and $D^{i\mu}$ are vectors which are independent of ξ and ρ. Collecting the fluctuation fields into a vector $\boldsymbol{v}^\intercal = (\xi^a, \rho)$ further simplifies the action,

$$S^{1-\text{loop}} = -\tfrac{1}{2}\int \mathrm{d}^4x\, \boldsymbol{v}^\intercal \boldsymbol{O} \boldsymbol{v}\,, \qquad \boldsymbol{O} = \begin{pmatrix} A^{ij} & iC^i \\ iD^{j\mu}d_\mu & B \end{pmatrix}. \tag{4.88}$$

Now, the 1-loop effective action $Z^{1-\text{loop}}$ is in terms of a generating functional,

$$e^{iZ_{1-\text{loop}}} = \int [\mathcal{D}\xi][\mathcal{D}\rho]\exp\Big\{-\tfrac{1}{2}i\int \mathrm{d}^4x\, \boldsymbol{v}^\intercal \boldsymbol{O} \boldsymbol{v}\Big\}, \tag{4.89}$$

so Wick rotating and performing the Gaussian integral gives the formal result

$$Z^{1-\text{loop}} = \tfrac{1}{2}i\ln\det\boldsymbol{O}\,, \tag{4.90}$$

thereby reducing the analysis to calculating the eigenvalues of $\boldsymbol{O}$.

We have not been able to regularize $\det\boldsymbol{O}$. The standard heat kernel method is typically applied to polynomial field theories, without the added complication arising from the off-diagonal terms in $\boldsymbol{O}$. One possible way forward is to block expand the determinant, i.e.

$$\det\boldsymbol{O} = B\det A^{ij} + D^{j\mu}d_\mu\det C^i\,, \tag{4.91}$$

and then isolate the divergences from $\det A^{ij}$ and $\det C^i$ through standard methods. Clearly this warrants further investigation.

4.4 Strong Interactions

In phenomenological applications of $\chi\mathrm{PT}_\sigma$ presented in this thesis, we either work in the tree approximation or study one-loop amplitudes which are finite in the chiral-scale limit. Consequently, either (4.33) or (4.46-4.48) may be used, although we feel that the former description makes the NG nature of σ rather clear. For this reason, we shall adopt (4.33) as the starting point in all the phenomenological analyses we discuss.

To that end, we take $\mathcal{L}_{\text{str}}$ and expand e^{σ/F_σ} to collect terms quadratic in σ and thus obtain the dilaton mass

$$m_\sigma^2 F_\sigma^2 = F_\pi^2(m_K^2 + \tfrac{1}{2}m_\pi^2)(3-\gamma_m)(1+\gamma_m) - \beta'(4+\beta')c_4\,. \tag{4.92}$$

Note that at $\alpha_{\rm IR}$, σ becomes a true Goldstone boson in the $SU(3)_L \times SU(3)_R$ limit because of our requirement that $c_4 = O(M)$. Similarly, we find for the effective $g_{\sigma\pi\pi}$ coupling

$$\mathcal{L}_{\sigma\pi\pi} = \big\{\big(2+(1-c_1)\beta'\big)|\partial\boldsymbol{\pi}|^2 - (3-\gamma_m)m_\pi^2|\boldsymbol{\pi}|^2\big\}\sigma/(2F_\sigma)\,. \tag{4.93}$$

Both (4.92) and (4.93) resemble pre-QCD results [15, 16, 142, 151] but have extra gluonic terms proportional to β'. Precluding any accidental fine-tuning in the theory, we assume that the unknown coefficient $2+(1-c_1)\beta'$ in Eq. (4.93) does not vanish. This preserves the key feature of the original work, that $\mathcal{L}_{\sigma\pi\pi}$ is mostly *derivative*: for soft $\pi\pi$ scattering (energies $\sim m_\pi$), the dilaton pole amplitude is negligible because the $\sigma\pi\pi$ vertex is $O(m_\pi^2)$, while the $\sigma\pi\pi$ vertex for an on-shell dilaton

$$g_{\sigma\pi\pi} = -\big(2+(1-c_1)\beta'\big)\,m_\sigma^2/(2F_\sigma) + O(m_\pi^2) \tag{4.94}$$

is $O(m_\sigma^2)$, consistent with σ being identified with the broad resonance f_0.

4.4.1 Sigma Terms

It is instructive to examine how the decomposition of the nucleon's mass differs between χPT$_3$ and χPT$_\sigma$. In χPT$_3$, there is no infrared fixed point and thus no limit in which the gluonic piece of the trace anomaly may be considered small. This fact underlies the widely held belief that the gluonic anomaly is responsible for most of the nucleon's mass,

$$M_N = \langle N|\theta_\mu^\mu|N\rangle \underset{\chi\mathrm{PT}_3}{=} \frac{\beta(\alpha_s)}{4\alpha_s}\langle N|G_{\mu\nu}^a G^{a\mu\nu}|N\rangle + O\big(m_K^2\big)\,. \tag{4.95}$$

Note that this interpretation *relies* on the implicit assumption that $f_0(500)$ pole amplitudes $\sim 1/m_{f_0}^2$ provide a negligible contribution to M_N. Since the f_0 mass is much lighter than other members of the non-Goldstone sector, a negligible effect can be obtained only if f_0 couples weakly to the operators $G_{\mu\nu}^a G^{a\mu\nu}$ and $\bar{q}q$. On the other hand, we have seen in Chapter 2 that the small f_0 mass means that χPT$_3$ has no

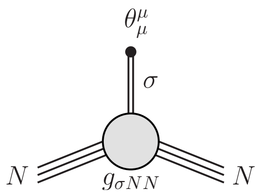


Figure 4.5: Dominant σ-pole diagram in $\chi\mathrm{PT}_\sigma$ for $\langle N|\theta^\mu_\mu|N\rangle$.

scale separation, which is a problem because f_0 couples strongly to other particles.

Now consider the relation between hadronic masses and θ^μ_μ in $\chi\mathrm{PT}_\sigma$. Since $\beta(\alpha_s)$ is small near α_{IR}, the gluonic trace anomaly is small *as an operator*, but it can produce large amplitudes when coupled to dilatons. For example, consider how M_N arises in $\chi\mathrm{PT}_\sigma$ (Fig. 4.5). From Eq. (4.2) we see that like other pseudo-NG bosons, σ couples to the vacuum via the divergence of its symmetry current $\partial^\mu D_\mu = \theta^\mu_\mu$,

$$\langle\sigma|\theta^\mu_\mu|\mathrm{vac}\rangle = -m_\sigma^2 F_\sigma = O(m_\sigma^2)\,,\ m_\sigma \to 0\,. \tag{4.96}$$

The nucleon remains massive in the scaling limit because the σNN coupling

$$\mathcal{L}_{\sigma NN} = g_{\sigma NN}\sigma\bar{N}N \tag{4.97}$$

and crossing symmetry imply that the m_σ dependence in (4.96) is cancelled by a σ-pole at zero momentum transfer (Fig. 4.5). Explicitly, we have

$$\begin{aligned} M_N = \langle N|\theta^\mu_\mu|N\rangle &= \langle NN|\theta^\mu_\mu|\mathrm{vac}\rangle \\ &\simeq \langle NN|\sigma\rangle\, \frac{i}{q^2 - m_\sigma^2}\bigg|_{q^2=0} \langle\sigma|\theta^\mu_\mu|\mathrm{vac}\rangle \\ &\neq 0\,,\ m_\sigma \to 0\,. \end{aligned} \tag{4.98}$$

This corresponds to the well known analogue

$$-F_\sigma\, g_{\sigma NN} \simeq M_N \tag{4.99}$$

of the Goldberger-Trieman relation (2.51).

Now consider contributions to Eqs. (4.96) and (4.99) from the gluonic anomaly and the quark mass term in Eq. (3.55) for θ^μ_μ. In general, there are two independent low-energy constants F_{G^2} and $F_{\bar{q}q}$, defined by

$$\begin{aligned} \beta(\alpha_s)/(4\alpha_s)\langle\sigma|G^a_{\mu\nu}G^{a\mu\nu}|\text{vac}\rangle &= -m_\sigma^2 F_{G^2} \\ \{1+\gamma_m(\alpha_s)\}\sum_{q=u,d,s} m_q\langle\sigma|\bar{q}q|\text{vac}\rangle &= -m_\sigma^2 F_{\bar{q}q}\,. \end{aligned} \tag{4.100}$$

Evidently, both constants can contribute to

$$F_\sigma = F_{G^2} + F_{\bar{q}q} \neq 0 \tag{4.101}$$

and hence to

$$M_N \simeq F_{G^2}\, g_{\sigma NN} + F_{\bar{q}q}\, g_{\sigma NN} \tag{4.102}$$

in the chiral-scale limit (3.56). That is because m_σ^2 is $O(m_K^2) = O(m_q)$ in χPT_σ.

In χPT_σ, we can relate the ratio $F_{G^2}/F_{\bar{q}q}$ to the mass dimension

$$d_\text{mass} = 3 - \gamma_m\big(\alpha_\text{IR}\big)\,, \tag{4.103}$$

and the πN sigma term

$$\Sigma_{ud} = \langle N|\big(m_u\bar{u}u + m_d\bar{d}d\big)|N\rangle_\text{conn}\,. \tag{4.104}$$

The explicit formula is found by noting that the state $|\sigma\rangle$ in Eq. (4.100) approximates an $SU(3)_V$ singlet

$$\langle\sigma|\bar{u}u|\text{vac}\rangle \approx \langle\sigma|\bar{d}d|\text{vac}\rangle \approx \langle\sigma|\bar{s}s|\text{vac}\rangle\,, \tag{4.105}$$

so for LO σ-pole amplitudes, we find

$$F_{\bar{q}q}\, g_{\sigma NN} \approx \big\{1+\gamma_m(\alpha_\text{IR})\big\}\big(1+m_s/2\hat{m}\big)\Sigma_{ud}\,, \tag{4.106}$$

and hence

$$\frac{M_N}{\big(1+m_s/2\hat{m}\big)\Sigma_{ud}} \approx \big\{1+\gamma_m(\alpha_\text{IR})\big\}\big(1+F_{G^2}/F_{\bar{q}q}\big)\,. \tag{4.107}$$

Note that in χPT_σ, the range of estimates of Σ_{ud} from lattice QCD [152, 153]

$$30\text{ MeV} \lesssim \Sigma_{ud} \lesssim 75\text{ MeV} \tag{4.108}$$

constrain the right-hand side of Eq. (4.107):

$$1 \lesssim \{1+\gamma_m(\alpha_{\mathrm{IR}})\}\big(1+F_{G^2}/F_{\bar{q}q}\big) \lesssim 2.5\,, \tag{4.109}$$

Since $0 \leq 1+\gamma_m < 3$, Eq. (4.109) would appear to support $\chi\mathrm{PT}_\sigma$ in that the ratio $F_G^2/F_{\bar{q}q}$ is *not* large, indicating that $\langle\bar{q}q\rangle_{\mathrm{vac}}$ sets the scale of the chiral-scale expansion.

We note however, that lattice methods have not yet been able to isolate the $f_0(500)$ resonance [154, 155]. The difficulty is due to the reliance on phase-shift analyses of lattice data, which as we have seen in Chapter 2 is complicated by the f_0's broad width. That may explain why lattice estimates ~ 50 MeV [152, 153] for the strange sigma term[1]

$$\Sigma_s = \langle N|m_s\bar{s}s|N\rangle_{\mathrm{conn}} \tag{4.110}$$

contradict the $\chi\mathrm{PT}_\sigma$ expectation that Σ_s be rather large. It is likely that the $SU(3)_V$ singlet f_0 plays an important role in this matrix element. It may also explain why it is hard to obtain conclusive results for the three-flavour β-function. So we do not believe that current lattice results provide evidence either for or against $\chi\mathrm{PT}_\sigma$.

4.4.2 Determining F_σ

To make quantitative predictions from $\chi\mathrm{PT}_\sigma$, we need (like F_π in $\chi\mathrm{PT}_3$) an estimate for the low-energy constant F_σ. The simplest way to do this is to compare NN-scattering with the relation (4.99)

$$-F_\sigma\, g_{\sigma NN} \approx M_N\,. \tag{4.111}$$

The best analysis to date is due to Calle Cordon and Ruiz Arriola [156, 157], who combine the 'one boson exchange' potential with large-N_c arguments. They find from the NN-scattering data a mean value $g_{\sigma NN} \approx 9$ and hence $F_\sigma \approx -100\,\mathrm{MeV}$, but with an uncertainty which is either model dependent or very large ($\approx$ 70%). That

[1] We note that the distinction between 'sea' and 'valence' quarks is a parton concept which is only valid in the ultraviolet region (2.52) where the relevant scale is Λ_{QCD}.

accounts for the large uncertainty in

$$1\tfrac{1}{2} \lessapprox |2+(1-c_1)\beta'| \lessapprox 6 \tag{4.112}$$

when we compare Eq. (4.94) with $|g_{\sigma\pi\pi}| = 3.31^{+0.35}_{-0.15}\,\text{GeV}$ [18] and $m_\sigma \approx 441\,\text{MeV}$.

4.4.3 The Scale of the Chiral-Scale Expansion

For any effective field theory, the validity of the perturbative expansion depends crucially on a clear separation of scales between the effective and heavy degrees of freedom. We have seen in Chapter 2 that for χPT_3 expansions, the series diverges whenever the f_0 is present with $O(m_K)$ momenta because this state is not part of the NG sector. This is to be compared with χPT_σ (Fig. 4.2), where the rules for counting powers of m_K are changed: $f_0 = \sigma$ pole amplitudes are promoted to LO (NLO in χPT_3). This fixes the LO problem for amplitudes involving 0^{++} channels and $O(m_K)$ extrapolations in momenta. At the same time, χPT_σ *preserves* the LO success of χPT_3 elsewhere.

We can test the convergence of our chiral-scale expansion by adding σ-loop diagrams to the standard analysis [13, 48, 49] for χPT_3. These involve the (as yet) undetermined constants $\beta', \gamma_m, c_{1\ldots4}$: for example, corrections to $g_{\sigma\pi\pi}$ involve the $\sigma\sigma\sigma$ and $\sigma\sigma\pi\pi$ vertices derived from Eq. (4.33). However, when we apply the dimensional arguments of Manohar and Georgi [13, 48, 49] to our scheme, we find that there are *two* χPT_σ scales $\chi_\pi = 4\pi F_\pi$ and $\chi_\sigma = 4\pi F_\sigma$, which are numerically similar ($F_\sigma \sim F_\pi$).

The calculation is a straightforward analysis of $\pi\pi$-scattering at NLO, which in χPT_σ involves one-loop graphs of the type shown in Fig. 4.6. Approximate chiral and scale symmetry ensures that any quadratic divergences arising from the loop amplitudes simply renormalize the low-energy constants $c_{1\ldots4}$ in $\mathcal{L}$. There are also logarithmic corrections to consider, with each diagram in Fig. 4.6 contributing a

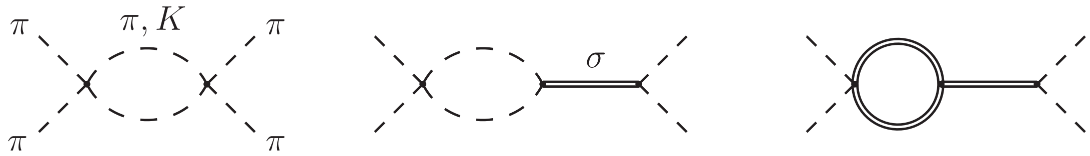


Figure 4.6: Example of next-to-leading graphs for $\pi\pi$-scattering in the chiral-scale expansion of $\chi\mathrm{PT}_\sigma$. Each vertex is generated by $\mathcal{L}$ in leading order. There are additional diagrams (not shown) involving the self-energy of the σ propagator, and internal σ-lines which connect one external π-leg to another. Similar diagrams are found for the t- and u-channels.

factor of order[1]

$$\frac{p^4}{F_\pi^4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right), \quad \frac{p^4}{F_\pi^2 F_\sigma^2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right), \quad \frac{p^4}{F_\sigma^4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right), \tag{4.113}$$

where Λ is some ultraviolet cutoff which preserves chiral symmetry, μ is the renormalization scale, and the 4π are geometric factors associated with loop integrals.

These contributions are to be compared against the amplitudes generated from higher derivative operators (4.59) such as

$$l_1\mathrm{Tr}(\nabla_\mu U\nabla^\nu U^\dagger\nabla_\mu U\nabla_\nu U^\dagger) \quad \text{and} \quad l_4'\mathrm{Tr}(\nabla_\mu U\nabla^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)e^{\beta'\sigma/F_\sigma} \tag{4.114}$$

where the coefficients $l_1 \sim F_\pi^2/\chi_\pi^2$ and $l_4' \sim 1/\chi_\sigma^2$ are suppressed by two independent mass scales χ_π and χ_σ. These operators yields diagrams such as those in Fig. 4.7, with amplitudes of order

$$\frac{p^4}{F_\pi^2\chi_\pi^2} \qquad \text{and} \qquad \frac{p^4}{F_\sigma^2\chi_\sigma^2}. \tag{4.115}$$

By the arguments of Georgi and Manohar [48, 49], changes in μ can be compensated by a redefinition of the couplings (F_π^2/χ_π^2) or $(1/\chi_\sigma^2)$. For example, an $O(1)$ change in μ is accompanied by a change of order $(4\pi)^{-1}$ in (F_π^2/χ_π^2). Thus, we find the following relations

$$4\pi F_\sigma = \chi_\sigma, \qquad \text{and} \qquad 4\pi F_\pi = \chi_\pi, \tag{4.116}$$

[1]In general, one must also take into account powers of m_K^2. To streamline the argument however, we will concentrate on just the momentum dependence.

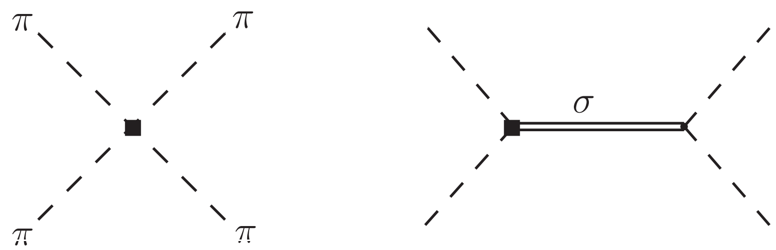


Figure 4.7: Amplitudes generated by next-to-leading Lagrangian operators such as (4.114), with a small square labelling the $O(p^4)$ vertex from $\mathcal{L}_{\rm NLO}$.

which are numerically similar,

$$4\pi F_\sigma \approx 4\pi F_\pi \simeq 1\,\text{GeV}\,, \tag{4.117}$$

and hence χPT_σ possesses a suitable separation of scales between m_σ and the non-NG sector.

It is important to note that the small value of $F_\sigma \ll \chi_{\pi,\sigma}$ implies a σ width

$$\Gamma_{\sigma\pi\pi} \approx \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma} \sim \frac{m_\sigma^3}{16\pi F_\sigma^2} \sim 250\text{ MeV} \tag{4.118}$$

which is numerically misleading: $\Gamma_{\sigma\pi\pi}$ is $O(m_\sigma^3)$ and hence *non-leading* relative to the mass m_σ. So tree diagrams produce the leading order of χPT_σ, as in χPT_2 and χPT_3. Beyond leading order, and in degenerate cases like the K_L–K_S mass difference, methods used to estimate corrections at the Z^0 peak [158, 159, 160, 161, 162, 163] and the ρ resonance [164] may be necessary.

4.5 Electromagnetic Interactions

As in χPT_3, electromagnetic interactions can be studied in χPT_σ by setting the external fields l_μ and r_μ to the values in Eq. (2.34). In the absence of semi-leptonic weak interactions, the covariant derivative becomes

$$\nabla_\mu U = \partial_\mu U + ieA_\mu[Q,U]\,. \tag{4.119}$$

4.5.1 The Electromagnetic Trace Anomaly

It has been shown [19, 20] that there is an anomalous Ward identity which relates the three-point vertex

$$\Delta_{\alpha\beta}(p,-p) = \int d^4x \int d^4y\, e^{ip\cdot x}\, T\langle J_\alpha(x) J_\beta(0) \theta^\mu_\mu(y)\rangle_{\text{vac}} \tag{4.120}$$

to the photon vacuum polarisation

$$\Pi_{\alpha\beta}(p,-p) = i\int d^4x\, e^{ip\cdot x}\, T\langle J_\alpha(x) J_\beta(0)\rangle_{\text{vac}}\,, \tag{4.121}$$

where J_α is the electromagnetic current. The existence (or absence) of an infrared fixed point does not affect the calculation of the anomaly [19, 20],

$$\theta^\mu_\mu\Big|_{\text{strong+e}'\text{mag}} = \theta^\mu_\mu + \frac{R\alpha}{6\pi} F_{\mu\nu}F^{\mu\nu}\,, \tag{4.122}$$

but it does alter the precise relationship between R_{IR} and $g_{\sigma\gamma\gamma}$. The purpose of this section is to derive the correct result (4.11) for χPT_σ.

We begin by defining the effective $\sigma\gamma\gamma$ coupling by the interaction Lagrangian[1]

$$\mathcal{L}_{\sigma\gamma\gamma} = \tfrac{1}{2} g_{\sigma\gamma\gamma}\sigma F_{\mu\nu}F^{\mu\nu}\,. \tag{4.123}$$

As in the pre-QCD calculation, we can expect that the quantities R_{IR} and $g_{\sigma\gamma\gamma}$ are related via the matrix element

$$\langle\gamma_1,\gamma_2|\theta^\mu_\mu(0)|\text{vac}\rangle = (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1)F(s)\,, \tag{4.124}$$

where $s=(k_1+k_2)^2$ is the usual Mandelstam variable. Our task is to calculate both sides of this expression in the limit of zero momentum transfer. The expression for

[1] The *sign* of $\mathcal{L}_{\sigma\gamma\gamma}$ must be chosen to be consistent with the positivity of the form factors which arise in deep-inelastic scattering [109]. In our conventions, the relation $\langle\sigma|\theta^\mu_\mu(0)|0\rangle = -m_\sigma^2 F_\sigma$ is negative, hence the difference in sign between our interaction Lagrangian and that of [20].

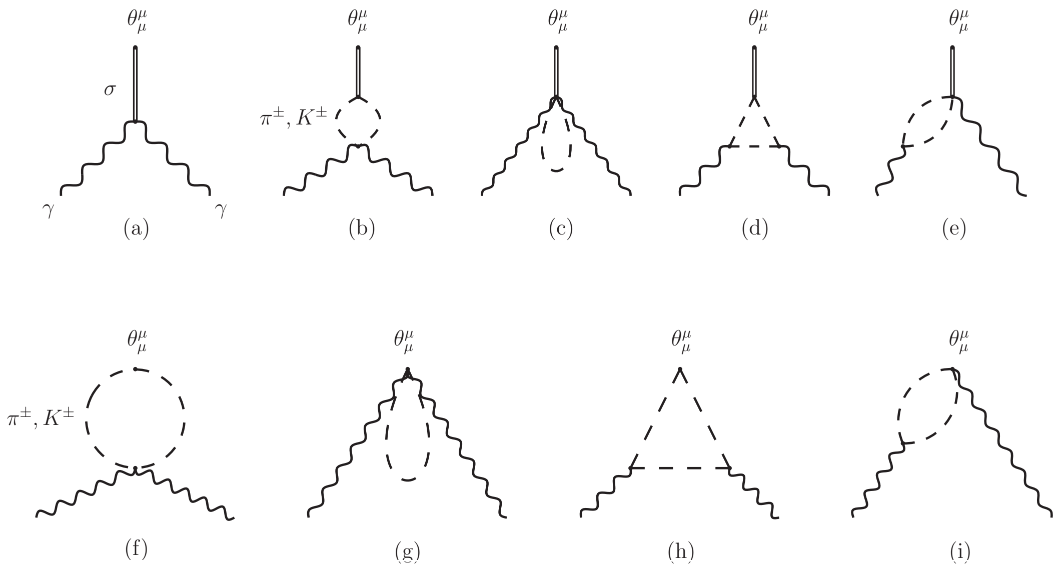


Figure 4.8: Leading order contributions to $\langle\gamma_1,\gamma_2|\theta^\mu_\mu(0)|\text{vac}\rangle$ in χPT_σ. Diagram (a) represents the contact term proportional to $g_{\sigma\gamma\gamma}$, while diagrams (d), (e), (h), and (i) are each accompanied by an additional crossed amplitude (not shown). The vertices for the θ^μ_μ insertions are derived from Eq. (4.55) and given in Appendix A.

the form factor $F(s)$ is known [19, 20],

$$\begin{aligned} F(0) &= -\frac{1}{3}\pi\alpha \int d^4x \int d^4y\,(x\cdot y)T\langle J^\beta(x)J_\beta(0)\theta^\mu_\mu(y)\rangle_{\text{vac}} \\ &= \frac{2R\alpha}{3\pi}, \end{aligned} \tag{4.125}$$

where J_μ is the electromagnetic current. Note that like other observables in χPT_σ, the right-hand side of (4.125) reduces to $2R_{\text{IR}}\alpha/3\pi$ in the infrared regime as a result of expanding in α_s about α_{IR}. The remaining calculation of $\langle\gamma_1,\gamma_2|\theta^\mu_\mu(0)|\text{vac}\rangle$ follows by considering each the following contributions shown in Fig. 4.8:

1. A contact term for $\sigma\to\gamma\gamma$.

2. 12 one-loop diagrams (6 for both $\pi^\pm$ and $K^\pm$) coupled to a σ-pole at zero-momentum. These graphs sum to a finite value in the chiral-scale limit.

3. Similarly, the 12 one-loop diagrams coupled directly to the vacuum via θ^μ_μ sum to a finite answer.

In the first two cases, θ^μ_μ acts on $|\text{vac}\rangle$ to create an on-shell σ, and thus the amplitude due to the σ-pole factorises,

$$\langle\gamma_1,\gamma_2|\sigma(s)\rangle\Big\{\frac{i}{s-m_\sigma^2}\Big\}\langle\sigma(s)|\theta^\mu_\mu(0)|\text{vac}\rangle\,. \tag{4.126}$$

The contact term contributes an amplitude

$$(\epsilon_1\cdot\epsilon_2 k_1\cdot k_2-\epsilon_1\cdot k_2\epsilon_2\cdot k_1)(-2ig_{\sigma\gamma\gamma}) \tag{4.127}$$

to the matrix element $\langle\gamma_1,\gamma_2|\sigma(s)\rangle$, while the loop graphs coupled to σ and θ^μ_μ share a similar Lorentz structure. Loop graphs of this type have been considered in χPT_3 for processes such as $K_S\to\gamma\gamma$ [21, 22], $K_L\to\pi^0\gamma\gamma$ [2], and $\pi^0\pi^0\to\gamma\gamma$ [4, 64]. The techniques employed therein are readily extended to χPT_σ and thus our calculation is of a similar nature. By explicitly calculating the diagrams (b)-(e) from Fig. 4.8 (see Appendix D), we find

$$\mathcal{A}^{\mu\nu}_\phi=\big(g^{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu\big)G_\phi(s), \tag{4.128}$$

where for $\phi=\pi^\pm,K^\pm$, the scalar part is

$$G_\phi(s)=\frac{i\alpha}{\pi F_\sigma}\{m_\phi^2[1-\gamma_m-(1-c_1)\beta']\}\Big\{\frac{1+2I_\phi}{s}\Big\}. \tag{4.129}$$

Here I_ϕ is the integral over Feynman parameters,

$$I_\phi=\int_0^1 dz_1dz_2\theta(1-z_1-z_2)\frac{m_\phi^2-z_1(1-z_1)k_1^2-z_2(1-z_2)k_2^2}{2z_1z_2(k_1\cdot k_2)+z_1(1-z_1)k_1^2+z_2(1-z_2)k_2^2-m_\phi^2+i\epsilon}, \tag{4.130}$$

whose solution is given in [4]. From a completely analogous calculation of the amplitudes in diagrams (f)-(i) we find

$$\mathcal{B}^{\mu\nu}_\phi=\big(g^{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu\big)\tilde{G}_\phi(s), \tag{4.131}$$

where

$$\tilde{G}_\phi(s)=\frac{\alpha}{2\pi}\{2m_\phi^2[1-\gamma_m-(1-c_1)\beta']-4m_\phi^2\}\Big\{\frac{1+2I_\phi}{s}\Big\}. \tag{4.132}$$

The final step is to relate $G_\phi(s)$ and $\tilde{G}_\phi(s)$ to (4.125). This is achieved by Taylor expanding the integrand of I_ϕ for small s,

$$1+2I_\phi = -\frac{s}{12m_\phi^2} + O(s^2), \tag{4.133}$$

and noting that each of the $\pi^\pm$ and $K^\pm$ contributions sum to give

$$\begin{aligned} G_{\pi+K}(0) &= -\frac{i\alpha}{6\pi}\frac{[1-\gamma_m-(1-c_1)\beta']}{F_\sigma}, \\ \tilde{G}_{\pi+K}(0) &= -\frac{\alpha}{12\pi}\{2[1-\gamma_m-(1-c_1)\beta']-4\}. \end{aligned} \tag{4.134}$$

Comparison with Eq. (4.127) yields

$$\begin{aligned} \langle\gamma_1,\gamma_2|\theta^\mu_\mu(0)|\text{vac}\rangle &= (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1) \\ &\quad\times\{\tilde{G}_{\pi+K}(0)+(G_{\pi+K}(0)-2ig_{\sigma\gamma\gamma})(im_\sigma^2)^{-1}(-m_\sigma^2F_\sigma)\} \\ &= (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1)\{\alpha/3\pi+2g_{\sigma\gamma\gamma}\} \\ &= (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1)2R_{\text{IR}}\alpha/3\pi, \end{aligned} \tag{4.135}$$

and hence the desired result

$$g_{\sigma\gamma\gamma} = \frac{\alpha}{3\pi F_\sigma}\left(R_{\text{IR}}-\tfrac{1}{2}\right). \tag{4.136}$$

4.5.2 The Drell-Yan Ratio in the Infrared Limit

A prediction for R_{IR} can be obtained by considering dispersive analyses of data which involve $f_0\to\gamma\gamma$. Currently, the most precise data comes from a measurement of the $\gamma\gamma\to\pi^0\pi^0$ cross-section by the Crystal Ball Collaboration [6]. Unfortunately, the broad width of $f_0(500)$ makes a direct determination of the width $\Gamma_{\gamma\gamma}$ from this experiment (and others) either unfeasible or subject to large model-dependent uncertainties (see “Note on Scalar Mesons Below 2 GeV" in [3]). Instead, one must combine the narrow width formula for the scattering amplitude $\mathcal{A}_{\gamma\gamma}$

$$\Gamma_{\gamma\gamma} = \frac{|\mathcal{A}_{\gamma\gamma}|^2}{32\pi m_\sigma}, \tag{4.137}$$

with precise knowledge from $\pi\pi$-scattering [17, 18] to predict the residue at the f_0-pole in $\gamma\gamma \to \pi^0\pi^0$ [139, 140, 165]. The most precise determination via this method is [140]

$$\Gamma_{\gamma\gamma} = 1.98^{+0.30}_{-0.24}\,\mathrm{keV}. \tag{4.138}$$

As noted in Eq. (4.118), the width $\Gamma_{\sigma\pi\pi}$ is $O(m_\sigma^3)$ and thus NLO. So in the LO of $\chi\mathrm{PT}_\sigma$, the narrow width approximation of Eq. (4.137) is *valid*.

We shall use these results to estimate R_{IR}. The relevant diagrams for $\mathcal{A}_{\gamma\gamma}$ are essentially those shown in (a-e) of Fig. 4.8, but with σ treated as an asymptotic state. Evidently, the results in Eqs. (4.129) and (4.127) provide us with $\mathcal{A}_{\gamma\gamma}$,

$$\mathcal{A}_{\gamma\gamma} = (\epsilon_1\cdot\epsilon_2 k_1\cdot k_2 - \epsilon_1\cdot k_2\epsilon_2\cdot k_1)(G_\pi(s) + G_K(s) - 2i\,g_{\sigma\gamma\gamma}). \tag{4.139}$$

Although this expression contains four unknown parameters γ_m, β', c_1, and $g_{\sigma\gamma\gamma}$, the first three appear only in the contribution from the $\pi^\pm, K^\pm$ loops, in the combination

$$\mathcal{C} = 1 - \gamma_m - (1-c_1)\beta' = \underbrace{3-\gamma_m}_{1\le d_{\mathrm{mass}}<4} - (2 + (1-c_1)\beta'). \tag{4.140}$$

It should be possible to match the Lagrangian vertices which give rise to these terms to dispersive analyses of data on $\gamma\gamma \to \pi^0\pi^0$. Presumably, the dispersive effects from the $\pi^\pm, K^\pm$ loops are damped relative to the contact term involving $g_{\sigma\gamma\gamma}$ and hence may be neglected. It is left as further investigation to establish this quantitatively. To proceed, we note that — provided the sign of $(2 + (1-c_1)\beta')$ is positive — these dispersive contributions are consistent with zero by Eq. (4.112), and so we shall assume $\mathcal{C} \simeq 0$. This allows us to extract $g_{\sigma\gamma\gamma}$ via determinations of $\Gamma_{\gamma\gamma}$. From (4.138), the result is

$$|g_{\sigma\gamma\gamma}| = 0.034 \pm 0.003\,\mathrm{GeV}^{-1}, \tag{4.141}$$

where the uncertainties have been added in quadrature. Within the large uncertainty due to that in F_σ, we find:

$$R_{\mathrm{IR}} \approx 5. \tag{4.142}$$

Note that this constitutes a fully non-perturbative prediction at α_{IR} and should not be compared against the free-field formula for $N_f = 3$

$$R(\alpha_s = 0) = N_c \sum \{\text{quark charges}\}^2 = 2 \tag{4.143}$$

in the ultraviolet asymptotic limit (Fig. 4.9).

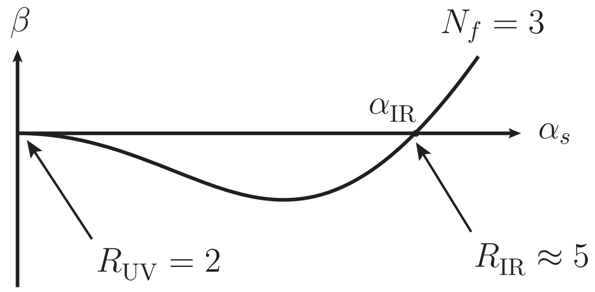


Figure 4.9: Proposed β-function with comparison between determinations of the Drell-Yan ratio in the ultraviolet $R_{\rm UV}$ and infrared $R_{\rm IR}$ regimes. Since each quantity is calculated in the respective *asymptotic* limit, the two predictions are in no way related.

4.6 Weak Interactions

As for the strong interactions, we obtain the weak Lagrangian in LO by adjusting the operator dimensions of Q_8, Q_{27}, and Q_{mw} (from Eqs. (2.60-2.62)) by powers of e^{σ/F_σ},

$$\begin{aligned}\mathcal{L}_{\rm weak} = Q_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27} Q_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} \\ + Q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma} + \text{h.c}\,,\end{aligned} \tag{4.144}$$

noting that Q_8 represents quark-gluon operators [53] with differing dimensions at $\alpha_{\rm IR}$. In χPT$_3$, we saw that although Q_{mw} is a pure isospin-$\frac{1}{2}$ operator, it cannot contribute to $K_S \to \pi\pi$ due to vacuum alignment. In χPT$_\sigma$, the outcome is entirely different. The key point is that the scaling dimension $(3-\gamma_{mw})$ of Q_{mw} is not the same as the dimension $(3-\gamma_m)$ of $\mathcal{L}_{\rm mass}$, so the σ dependence of $Q_{mw} e^{(3-\gamma_{mw})/F_\sigma}$ cannot

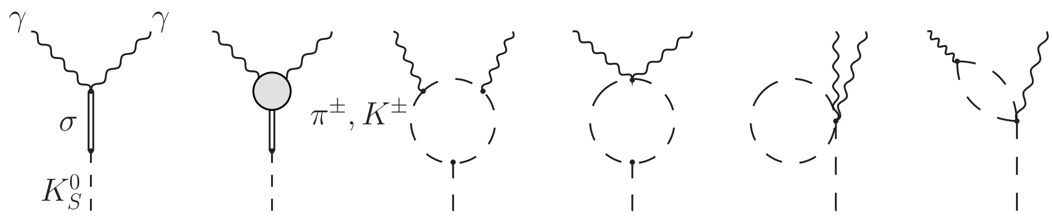


Figure 4.10: Leading order diagrams for $K_S \to \gamma\gamma$ in χPT$_\sigma$, including finite loop graphs [21]. The grey vertex contains $\pi^\pm$, $K^\pm$ loops as in the four χPT$_3$ diagrams to the right. An analogous set of diagrams contributes to $\gamma\gamma \to \pi^0\pi^0$.

be eliminated by a chiral rotation. Instead, after aligning the vacuum, we find

$$\begin{aligned}\mathcal{L}_{\text{weak}}^{\text{align}} = \tilde{Q}_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27}\tilde{Q}_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} \\ + \tilde{Q}_{mw}\{e^{(3-\gamma_{mw})\sigma/F_\sigma} - e^{(3-\gamma_m)\sigma/F_\sigma}\} + \text{h.c.}, \end{aligned} \tag{4.145}$$

where the tilde indicates that the **8** and **2**7 operators are now functions of the rotated field $\tilde{U}$. As before, it is this rotated field which satisfies $\langle\tilde{U}\rangle_{\text{vac}} = I$ and allows us to treat the NG fields ϕ_i as perturbations about the ground state. Consequently, there is a residual interaction $\mathcal{L}_{K_S\sigma} = g_{K_S\sigma} K_S^0 \sigma$ which mixes K_S and σ in *leading order*

$$g_{K_S\sigma} = (\gamma_m - \gamma_{mw})\text{Re}\{(2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M\} F_\pi / 2F_\sigma \tag{4.146}$$

and produces the $\Delta I = 1/2$ amplitude $A_{\sigma\text{-pole}}$ of Fig. 4.3.

An estimate for this coupling can be obtained by comparing Eq. (4.141) with $K_S \to \gamma\gamma$ (Fig. 4.10). Clearly, this vertex can be used to extract $g_{K_S\sigma}$ from $K_S \to \gamma\gamma$, where the scalar part of the amplitude

$$\mathcal{A}_{\mu\nu} = (g_{\mu\nu}s - 2k_{2\mu}k_{1\nu})\mathcal{A}(s) \tag{4.147}$$

receives three contributions

$$\mathcal{A}(s) = \mathcal{A}_\sigma^{\text{tree}} + \mathcal{A}_\sigma^{\text{loop}} + \mathcal{A}_{\pi,K}^{\text{loop}}. \tag{4.148}$$

The explicit expressions are

$$\begin{aligned}
\mathcal{A}_\sigma^{\text{tree}} &= i\,\frac{g_{K_S\sigma}\,g_{\sigma\gamma\gamma}}{s - m_\sigma^2 + i\epsilon}\,, \\
\mathcal{A}_\sigma^{\text{loop}} &= -i\,\frac{\alpha\mathcal{C}}{\pi F_\sigma}\,\frac{g_{K_S\sigma}}{s - m_\sigma^2 + i\epsilon}\left[\frac{m_\pi^2(1+2I_\pi)+m_K^2(1+2I_K)}{s}\right], \\
\mathcal{A}_{\pi,K}^{\text{loop}} &= -i\,\frac{\alpha}{\pi F_\pi^3}(g_8+g_{27})\left[\frac{m_\pi^2(1+2I_\pi)+m_K^2(1+2I_K)}{s}\right].
\end{aligned} \tag{4.149}$$

where $\mathcal{C} = 1-\gamma_m-(1-c_1)\beta'$ as in Eq. (4.140). Note that unlike our determination of R_{IR}, we can fix $g_{K_S\sigma}$ *without* making any assumptions about the value of $\mathcal{C}$. This is because $\mathcal{A}_\sigma^{\text{tree}}+\mathcal{A}_\sigma^{\text{loop}}$ is proportional to the combination

$$G_\pi(s)+G_K(s)-2i\,g_{\sigma\gamma\gamma} \tag{4.150}$$

of Eq. (4.139) which can be determined directly from data on $\Gamma_{\gamma\gamma}$ (Eq. (4.137)). From Eq. (4.138), we find

$$|G_\pi(m_\sigma^2)+G_K(m_\sigma^2)-2i\,g_{\sigma\gamma\gamma}| = 0.068\pm 0.006\ \text{GeV}^{-1}\,, \tag{4.151}$$

and thus estimate

$$|g_{K_S\sigma}| \approx 4.4\times 10^3\,\text{keV}^2 \tag{4.152}$$

to about 30% precision, where we have neglected the contributions involving $g_{8,27}$. Thus, to the extent that $g_{\sigma NN}$ and hence F_σ can be determined, we find

$$\left|A_{\sigma\text{-pole}}\right| \approx 0.34\,\text{keV}. \tag{4.153}$$

This accounts for the large $I=0$ $\pi\pi$ amplitude A_0 [3]

$$|A_0|_{\text{expt.}} = 0.33\,\text{keV} \tag{4.154}$$

compared with A_2. So we conclude that the observed ratio $|A_0/A_2|\simeq 22$ is mostly due to the dilaton-pole diagram of Fig. 4.3, that $g_8=\sum_n g_{8n}$ and g_{27} may have similar magnitudes as simple calculations indicate, and that only g_{27} can be fixed precisely (from $K^+\to\pi^+\pi^0$).

Consequently, the leading order of χPT_σ solves the $\Delta I = 1/2$ problem for kaon decays. The chiral Ward identities which relate the on-shell $K \to 2\pi$ and $K \to \pi$ amplitudes have extra terms due to σ poles, but the no-tadpoles theorem (Eq. (2.69)) is still valid:

$$\langle K|\mathcal{H}_{\text{weak}}|\text{vac}\rangle = O\big(m_s^2 - m_d^2\big)\,,\ K \text{ on shell}\,. \tag{4.155}$$

Chapter 5

Discussion and Directions for Further Research

5.1 Summary of Results

In this thesis, we have explored the conditions under which the low-energy structure of QCD may be considered to be approximately scale and chiral $SU(3)_L \times SU(3)_R$ invariant. We put forth the proposal that such a scenario may conceivably occur in QCD if the strong running coupling α_s for the three light quarks u, d, s runs non-perturbatively to an infrared fixed point α_{IR}. As noted in Chapter 3, the theoretical evidence for this picture is (as yet) unclear due to the technical challenges associated with comparisons between different approaches in defining α_s. What is clear however, is that to avoid the introduction of theoretical artifacts, non-perturbative definitions are required: extrapolations based on fixed-order perturbation theory are afflicted with unphysical features like Landau singularities.

Taken as a working hypothesis, we showed that the absence of the gluonic term in the trace anomaly at α_{IR} does not necessarily imply that QCD becomes scale invariant in the Wigner-Weyl mode, as commonly assumed in discussions on fixed points in quantum field theory. The underlying dynamics must always be taken into consideration. In QCD we know with some degree of confidence that strong gluon fields are responsible for setting the infrared mass scale $\chi_{\text{ch}} \approx 1$ GeV through the formation of a quark condensate $\langle \bar{q}q \rangle_{\text{vac}}$. As argued in Chapter 3, we are unaware of any physical reason why the same should not occur at α_{IR}. Since the operator

$\bar{q}q$ is not a singlet under scale transformations, we concluded that scale and chiral $SU(3)_L \times SU(3)_R$ symmetry is spontaneously broken in the infrared $\alpha_s \to \alpha_{\mathrm{IR}}$ and massless limit $m_q \to 0$. It was then a simple application of Goldstone's theorem to show that in this limit there are nine Nambu-Goldstone bosons: π, K, η and a 0^{++} QCD dilaton σ.

We argued that the small explicit breaking of scale invariance due to the quark mass term $m_q \bar{q}q$ — in particular, that of the strange quark — is responsible for most of the σ's mass. It was then natural for us to identify σ with the lowest QCD resonance $f_0(500)$, whose $O(m_K)$ mass and width are now known rather precisely.

For low-energy amplitudes, we constructed in Chapter 4 a chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$ based on expansions in α_s about α_{IR}. At leading and next-to-leading order in the expansion, we constructed the most general effective Lagrangians for strong interactions of π, K, η, σ consistent with approximate scale and chiral $SU(3)_L \times SU(3)_R$ symmetry. We achieved this by making use of well established techniques which abstract chiral-scale effective operators from those found in the chiral Lagrangians of $\chi\mathrm{PT}_3$. In leading order, we found that compared to $\chi\mathrm{PT}_3$, the $\chi\mathrm{PT}_\sigma$ Lagrangian contains an additional seven low-energy constants $c_{1,\ldots,4}, F_\sigma, \beta', \gamma_m$, of which we were only able to fix $F_\sigma \approx 100$ MeV from analyses of NN-scattering.

We extended our scope to include background fields whose presence, subject to appropriate transformation properties, rendered QCD (and hence $\chi\mathrm{PT}_\sigma$) *formally* invariant under local scale and chiral $SU(3)_L \times SU(3)_R$ symmetry. In this formalism, we obtained expressions for the equations of motion and the effective trace anomaly in the leading order of $\chi\mathrm{PT}_\sigma$. Through a simple extension of the background field method applied by Gasser and Leutwyler [13] to $\chi\mathrm{PT}_3$, we were able to calculate a closed form expression for the one-loop effective action.

The convergence of $\chi\mathrm{PT}_\sigma$ expansions was tested by extending the arguments of Manohar and Georgi [48, 49] to account for the enlarged symmetry group. Since F_σ and F_π are a priori independent of each other, it was found that there are two scales χ_σ and χ_π which govern the rate of convergence. Since $F_\sigma \approx F_\pi$ numerically, we concluded that in $\chi\mathrm{PT}_\sigma$ there is a clear separation of scales between $\{\pi, K, \eta, \sigma = f_0\}$ and its non-Goldstone sector $\{\rho, \omega, \ldots\}$.

Through an analysis of the electromagnetic trace anomaly, we derived a relationship between the effective $\sigma\gamma\gamma$ coupling and the non-perturbative Drell-Yan ratio R_{IR} at α_{IR}.

In the weak interaction sector, we constructed the leading order effective Lagrangian and showed that unlike $\chi\mathrm{PT}_3$, vacuum alignment *does not* remove the weak mass operator Q_M. We showed that as a consequence, K_S mixes with σ through an effective coupling $g_{K_S\sigma}$.

Regarding phenomenological applications of $\chi\mathrm{PT}_\sigma$, we showed that in leading order, a dominant σ-pole in $K_S \to \pi\pi$ accounts for the $\Delta I = 1/2$ rule. Using our result from the electromagnetic trace anomaly, we also obtained the estimate $R_{\mathrm{IR}} \approx 5$.

5.2 Implications of this Work

In this thesis we have presented a serious proposal to explain the $\Delta I = 1/2$ rule in K-decays, a puzzle of almost 60 years standing. While this is certainly our most important result, there is a sense in which it may be considered a special case of our more general idea to include $f_0 = \sigma$ in the Nambu-Goldstone sector $\{\pi, K, \eta, \sigma\}$ so that f_0 amplitudes contribute in the leading order of low-energy expansions with $O(m_K)$ extrapolations. Despite a seemingly radical change to the structure of low-energy QCD, our proposal is in fact rather conservative: we modify $\chi\mathrm{PT}_3$ only where it fails and preserve its successful leading order predictions elsewhere. We have shown that $\chi\mathrm{PT}_\sigma$ is the appropriate model-independent framework to study the low-energy strong, weak, and electromagnetic interactions of these mesons, and it is the generality of the method which we consider to be our most significant achievement.

The proposed replacement for $\chi\mathrm{PT}_3$ relates amplitudes in the physical region $0 < \alpha_s < \alpha_{\mathrm{IR}}$ to high-energy quantities like $\gamma_m(\alpha_{\mathrm{IR}})$ and R_{IR} characteristic of massless QCD at α_{IR}. This may imply that QCD simplifies in that limit and, unlike QED [166, 167], allows $\beta' \neq 0$ at the fixed point.

We also note that in attempts to extend the AdS_5/CFT (or "gauge-gravity") duality to QCD [168, 169], the infrared mass scale χ_{ch} is typically introduced through a hard IR cutoff on the string modes in the radius of AdS_5. This procedure however, implicitly assumes that scale condensates like $\langle \bar{q}q \rangle_{\mathrm{vac}}$ are absent. If our view is correct, the construction and application of gauge-gravity models may be subject to modification.

5.3 Limitations of the Theory

While $\chi\mathrm{PT}_\sigma$ has some clear advantages over $\chi\mathrm{PT}_3$, we must acknowledge that there are new challenges associated with the proposed theory.

The large number of low-energy constants in $\chi\mathrm{PT}_\sigma$ place practical limitations on the theory's predictive power. This problem is particularly acute at next-to-leading order, where the effective Lagrangian (4.59) contains $O(40)$ parameters whose values are unknown a priori. The scant data on processes such as $f_0 f_0$-scattering means that the empirical determination of many of these couplings is likely to remain out of reach for the foreseeable future.

A related problem is one of computational complexity. At one-loop order, the number of graphs to calculate is considerably larger in $\chi\mathrm{PT}_\sigma$ compared with $\chi\mathrm{PT}_3$. Take for instance the classic example of $\pi\pi$-scattering in the s-channel. In $\chi\mathrm{PT}_3$ there is a single one-loop graph to calculate. On the other hand, in $\chi\mathrm{PT}_\sigma$ there are $O(20)$ diagrams which must be evaluated! Clearly, this limits the degree of precision one may hope to achieve in observables of interest.

Another limitation is that while $\chi\mathrm{PT}_\sigma$ works for non-leptonic K-decays, our analysis does not shed any light on the failure of heavy baryon χPT [170] (or standard χPT [171]) to simultaneously explain the S- and P-wave data of the hyperon sector.

5.4 Directions for Future Research

Despite the aforementioned limitations, there is a rich phenomenological vista to be explored in $\chi\mathrm{PT}_\sigma$. Some potentially fruitful areas of investigation include the following.

1. An investigation of other reactions where f_0 is present would be worthwhile. In particular, it would be interesting to examine the decays $K_L \to \pi^0\gamma\gamma$, $K^+ \to \pi^+\pi^- e^+\nu_e$, and $\eta \to \pi\pi\pi$, where the leading order predictions from $\chi\mathrm{PT}_3$ are

known to fall short of the experimental data by a factor of 2 or more in the rate [172, 173].

2. With octet dominance not required in $\chi\mathrm{PT}_\sigma$ (see Sec. 4.6), a number of CP violating observables associated with the decay modes of K-mesons need to be re-examined. This provides an important testing ground on the internal consistency of the theory and furthermore, offers the exciting possibility to make progress beyond the predictions of $\chi\mathrm{PT}_3$. Reactions where the σ-pole is important include $K_L \to \pi\pi$, $K_L \to 3\pi$, and rare K decays with two leptons in the final state.

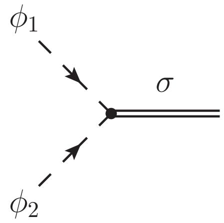
3. A related line of research would be to examine the predictions of $\chi\mathrm{PT}_\sigma$ for the CP violation parameters ϵ and ϵ'. This involves the calculation of quantities such as the K_L-K_S mass difference $\Delta m = m_L - m_S$, the phase shifts δ_I from $\pi\pi$-scattering with $O(m_K)$ momentum, and the isospin amplitudes A_I from $K \to \pi\pi$. The $\chi\mathrm{PT}_\sigma$ predictions for these quantities will differ from those of $\chi\mathrm{PT}_3$, and it would be worthwhile establishing whether progress can be made in calculating ϵ and ϵ' from the new effective field theory.

4. The existing calculations [174, 175, 176, 177] of the scalar exchange contribution to hadronic light-by-light scattering require the use of models whose precise connection to f_0 is unclear. Can $\chi\mathrm{PT}_\sigma$ do better in this regard? Presumably so, since the model dependence in this picture is strictly limited to the form factors $\mathcal{F}_{\sigma\gamma\gamma^*}$ and $\mathcal{F}_{\sigma\gamma^*\gamma^*}$, not on the underlying dynamics of the f_0 resonance.

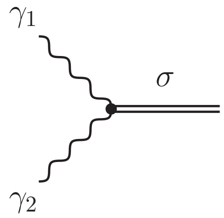
5. By analogy with π, K, η, it should be possible (in principle) to test our hypothesised Nambu-Goldstone interpretation of f_0 on the lattice. Naturally, the technical complications associated with f_0's broad width must be overcome (extrapolations based on phase shift analyses will be model-dependent at best). This raises the following question. Can the Roy equations, analytically continued to *complex* momenta, be implemented in lattice simulations? An answer in the affirmative should allow one to isolate the elusive f_0, and irrespective of the view presented in this thesis, test its effect in $K_S \to \pi\pi$.

Appendix A: Feynman Rules

Here we provide for reference, the Feynman rules for the $\sigma\phi\phi$ and $\sigma\gamma\gamma$ vertices obtained from $\chi\mathrm{PT}_\sigma$ and used extensively in Chapter 4. Given an interaction vertex $\mathcal{L}_I$, our convention is to define the Feynman rule by $i\langle\mathcal{L}_I\rangle$, where the brackets imply that the fields are to be omitted. The relevant expressions are shown in Fig. 1.



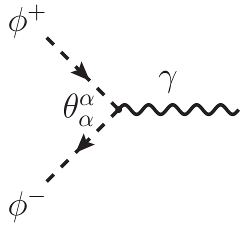
$$= -i[(2 + (1 - c_1)\beta')p_1 \cdot p_2 + (3 - \gamma_m)m_\phi^2]/2F_\sigma$$



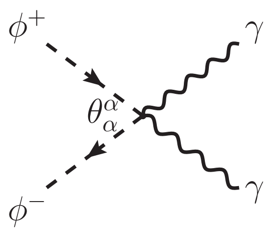
$$= -2ig_{\sigma\gamma\gamma}(g^{\mu\nu}k_1 \cdot k_2 - k_2^\mu k_1^\nu)$$

Figure 1: Feynman rules for vertices involving σ with all momenta flowing inwards. In our notation, $\gamma_i = \gamma(\epsilon_i, k_i)$ represents a photon with polarization ϵ_i and momentum k_i, while ϕ_i represents $\pi^\pm$ or $K^\pm$ with momentum p_i. The flow of electric charge is indicated by the arrows.

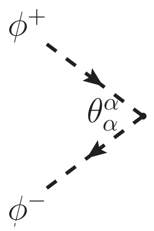
Furthermore, we provide the amplitudes for the θ_μ^μ insertions which arose during our treatment of the electromagnetic trace anomaly in Chapter 4. The explicit expressions are given in Fig. 2.



$$= -e(p_+ - p_-)_\mu(1 - c_1)\beta'$$



$$= 2e^2 g^{\mu\nu}(1 - c_1)\beta'$$



$$= -[(p_+ \cdot p_-)(1 - c_1)\beta' - m_\phi^2(1 + \gamma_m)]$$

Figure 2: Amplitudes for insertions of the effective θ^μ_μ operator in amplitudes involving photons γ and charged $\phi = \pi, K$ mesons. The flow of momentum and electric charge is indicated by the arrows.

Appendix B: Equations of Motion

We obtain the equations of motion for the U and X fields[1] by the standard method, i.e we look for stationary points of the action integral

$$S[U, U^\dagger, X] = \int \mathrm{d}^4x\, \mathcal{L}[U, U^\dagger, X]\,. \tag{1}$$

For reference, we reproduce the LO expression for $\mathcal{L}$,

$$\mathcal{L}_{\mathrm{LO}}^{d=4} = z_1 \mathcal{K} X^2 + z_2 \tfrac{1}{2}(D_\mu X)^2 + z_3 X^4\,, \tag{2}$$

$$\mathcal{L}_{\mathrm{LO}}^{d>4} = z_4 \mathcal{K} X^{2+\beta'} + z_5 \tfrac{1}{2}(D_\mu X)^2 X^{\beta'} + z_6 X^{4+\beta'}\,, \tag{3}$$

$$\mathcal{L}_{\mathrm{LO}}^{d<4} = z_7 \mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) X^{3-\gamma_m}\,, \tag{4}$$

where $\mathcal{K} = \frac{1}{4} F_\pi^2 \mathrm{Tr}(\nabla_\mu U \nabla^\mu U^\dagger)$, $X = F_\sigma e^{\sigma/F_\sigma}$, and $\chi = \frac{1}{2} F_\pi^2 B_0(s + ip)$. Under variations in U we get

$$\begin{aligned}\delta_U S = \int \mathrm{d}^4x \Big\{ &-\frac{F_\pi^2}{4} \mathrm{Tr}(\nabla_\mu \delta U \nabla^\mu U^\dagger + \nabla_\mu U \nabla^\mu \delta U^\dagger)(z_1 X^2 + z_4 X^{2+\beta'}) \\ &+ z_7 \mathrm{Tr}(\chi \delta U^\dagger + \delta U \chi^\dagger) X^{3-\gamma_m} \Big\}\,,\end{aligned} \tag{5}$$

[1] The equations of motion for $\chi\mathrm{PT}_3$ are derived in [29].

where the derivatives ∇_μ can be shifted via integration by parts (taking care to include the X contribution) to get,

$$\begin{aligned}\therefore \delta_U S = \int \mathrm{d}^4 x \Big\{ & \frac{F_\pi^2}{4}\mathrm{Tr}(-\delta U(\nabla_\mu \nabla^\mu U^\dagger) - (\nabla_\mu \nabla^\mu U)\delta U^\dagger)(z_1 X^2 + z_4 X^{2+\beta'}) \\ & + z_7 \mathrm{Tr}(\chi \delta U^\dagger + \delta U \chi^\dagger) X^{3-\gamma_m} \\ & - \frac{F_\pi^2}{4}\mathrm{Tr}(\delta U \nabla^\mu U^\dagger + (\nabla^\mu U)\delta U^\dagger)(2z_1 X + (2+\beta') z_4 X^{1+\beta'})\partial_\mu X \Big\}. \end{aligned} \tag{6}$$

For infinitesimal variations, $\delta U = i\Delta^i \lambda^i U$, where $\Delta^i(x)$ are real functions and λ^i are the Gell-Mann matrices. Combined with $\delta U^\dagger = -U^\dagger \delta U U^\dagger$, the condition $\delta_U S = 0$ yields for $i = 1, \ldots, 8$,

$$\begin{aligned} 0 = \mathrm{Tr}\Big(\lambda^i \big\{ & [(\nabla^\mu \nabla_\mu U)U^\dagger - U(\nabla^\mu \nabla_\mu U^\dagger)][z_1 X^2 + z_4 X^{2+\beta'}] \\ & + [(\nabla^\mu U)U^\dagger - U(\nabla^\mu U^\dagger)][2z_1 X + (2+\beta') z_4 X^{1+\beta'}]\partial_\mu X \big\} \\ & + \frac{4}{F_\pi^2} z_7 [U\chi^\dagger - \chi U^\dagger] X^{3-\gamma_m} \Big). \end{aligned} \tag{7}$$

Since any 3×3 matrix A can be decomposed as

$$A = \tfrac{1}{3}\mathrm{Tr}(A) I_{3\times 3} + \tfrac{1}{2}\sum_{i=1}^{8} \mathrm{Tr}(\lambda^i A)\lambda^i, \tag{8}$$

the above can be expressed as an operator equation

$$\begin{aligned} 0 = {} & [(\nabla^\mu \nabla_\mu U)U^\dagger - U(\nabla^\mu \nabla_\mu U^\dagger)][z_1 X^2 + z_4 X^{2+\beta'}] \\ & + \frac{4}{F_\pi^2} z_7 [U\chi^\dagger - \chi U^\dagger + \frac{1}{3}\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)] X^{3-\gamma_m} \\ & + [(\nabla^\mu U)U^\dagger - U(\nabla^\mu U^\dagger)][2z_1 X + (2+\beta') z_4 X^{1+\beta'}]\partial_\mu X. \end{aligned} \tag{9}$$

Similarly, for the X equations of motion we find the variations

$$\frac{\partial \mathcal{L}}{\partial (D_\mu X)} = z_2 D^\mu X + z_5 (D^\mu X) X^{\beta'} , \tag{10}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X} &= 2z_1 \mathcal{K} X + 4z_3 X^3 + (2+\beta') z_4 \mathcal{K} X^{1+\beta'} + \beta' z_5 \tfrac{1}{2} (D_\mu X)^2 X^{\beta'-1} \\ &\quad + (4+\beta') z_6 X^{3+\beta'} + (3-\gamma_m) z_7 \mathrm{Tr}(MU^\dagger + UM^\dagger) X^{2-\gamma_m} , \end{aligned} \tag{11}$$

from which one immediately obtains

$$\begin{aligned} 0 &= (\partial^\mu \partial_\mu X)[z_2 + z_5 X^{\beta'}] + \beta' z_5 \tfrac{1}{2} (D_\mu X)^2 X^{\beta'-1} - \mathcal{K}[2z_1 X + (2+\beta') z_4 X^{1+\beta'}] \\ &\quad - 4z_3 X^3 - (4+\beta') z_6 X^{3+\beta'} - (3-\gamma_m) z_7 \mathrm{Tr}(MU^\dagger + UM^\dagger) X^{2-\gamma_m} . \end{aligned} \tag{12}$$

Appendix C: Tadpole Cancellation in the Background Field Method

To ensure that the background fields $\bar{U}$ and $\bar{X}$ are stable solutions to the classical field equations, it is necessary that terms linear in the fluctuation fields ξ and ρ vanish (no tadpoles). First we consider the $O(\xi)$ terms in the notation of Sec. 4.3.5, to wit

$$S\Big|_{\xi} = i\int \mathrm{d}^4x \left\{\frac{F_\pi^2}{4} L_\xi (z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'}) + \mathrm{Tr}(\xi\Sigma^-)\bar{X}^{3-\gamma_m}\right\}, \tag{13}$$

where we reproduce the expressions

$$L_\xi = \mathrm{Tr}(\nabla_\mu\bar{U}^\dagger\nabla^\mu(u\xi u)) - \mathrm{Tr}(\nabla_\mu(u^\dagger\xi u^\dagger)\nabla^\mu\bar{U}), \tag{14}$$

$$\Sigma^\pm = u\chi^\dagger u \pm u^\dagger\chi u^\dagger. \tag{15}$$

We can show that the right-hand side of (13) vanishes by first using integration by parts to shift the derivative in L_ξ,

$$\begin{aligned}\int \mathrm{d}^4x\, L_\xi\bar{X}^d = -\int \mathrm{d}^4x \Big\{&\mathrm{Tr}\big((\nabla^\mu\nabla_\mu\bar{U}^\dagger)(u\xi u) - (u^\dagger\xi u^\dagger)\nabla^\mu\nabla_\mu\bar{U}\big)\bar{X}^d \\ &+ d\,\mathrm{Tr}\big((\nabla^\mu\bar{U}^\dagger)(u\xi u) - (u^\dagger\xi u^\dagger)\nabla^\mu\bar{U}\big)\bar{X}^{d-1}\partial_\mu\bar{X}\Big\}.\end{aligned} \tag{16}$$

By inserting appropriate factors of $u^\dagger u = I$, we obtain the desired result,

$$
\begin{aligned}
S\Big|_\xi = i\frac{F_\pi^2}{4}\int \mathrm{d}^4x \Big\{ & \mathrm{Tr}\big(u\xi u^\dagger[(\nabla^\mu\nabla_\mu\bar{U})\bar{U}^\dagger - \bar{U}(\nabla^\mu\nabla_\mu\bar{U}^\dagger)]\big)(z_1\bar{X}^2 + z_4\bar{X}^{2+\beta'}) \\
& + \mathrm{Tr}\big(u\xi u^\dagger[(\nabla^\mu\bar{U})\bar{U}^\dagger - \bar{U}(\nabla^\mu\bar{U}^\dagger)]\big)(2z_1\bar{X} + (2+\beta')z_4\bar{X}^{1+\beta'})\partial_\mu\bar{X} \\
& + \frac{4}{F_\pi^2}z_7\mathrm{Tr}(u\xi u^\dagger[\bar{U}\chi^\dagger - \chi\bar{U}^\dagger])\Big\} \\
& = 0\,,
\end{aligned}
\tag{17}
$$

since $\bar{U}$ satisfies the classical equations of motion (Eq. (4.49)). Similarly, the $O(\rho)$ terms

$$
\begin{aligned}
S\Big|_\rho = \int \mathrm{d}^4x \Big\{ & \rho\mathcal{K}(2z_1\bar{X}^2 + (2+\beta')z_4\bar{X}^{2+\beta'}) + z_2[\rho(D_\mu\bar{X})^2 + (D_\mu\rho D^\mu\bar{X})\bar{X}] \\
& + 4z_3\rho\bar{X}^4 + (4+\beta')z_6\rho\bar{X}^{4+\beta'} + (3-\gamma_m)\rho\,\mathrm{Tr}(\chi\bar{U}^\dagger + \bar{U}\chi^\dagger)\bar{X}^{3-\gamma_m} \\
& + z_5[(2+\beta')\rho\tfrac{1}{2}(D_\mu\bar{X})^2\bar{X}^{\beta'} + (D_\mu\rho D^\mu\bar{X})\bar{X}^{1+\beta'}]\Big\}\,,
\end{aligned}
\tag{18}
$$

can be shown to vanish through a judicious use of integration by parts

$$
\int \mathrm{d}^4x\,[\rho(D_\mu\bar{X})^2 + (D_\mu\rho D^\mu\bar{X})\bar{X}] = -\int \mathrm{d}^4x\,\rho\bar{X}D^\mu D_\mu\bar{X}\,, \tag{19}
$$

$$
\begin{aligned}
& \int \mathrm{d}^4x\,[(2+\beta')\rho\tfrac{1}{2}(D_\mu\bar{X})^2\bar{X}^{\beta'} + (D_\mu\rho D^\mu\bar{X})\bar{X}^{1+\beta'}] \\
& = -\int \mathrm{d}^4x\,\rho[\beta'\tfrac{1}{2}(D_\mu\bar{X})^2\bar{X}^{\beta'} + \bar{X}^{1+\beta'}D^\mu D_\mu\bar{X}]\,.
\end{aligned}
\tag{20}
$$

Making use of the X equations of motion, we get

$$
\begin{aligned}
S\Big|_{\rho} = \int \mathrm{d}^4x\,\rho \Big\{ & -\bar{X}(D^{\mu}D_{\mu}\bar{X})(z_2 + z_5\bar{X}^{\beta'}) - \beta' z_5 \tfrac{1}{2}(D_{\mu}\bar{X})^2\bar{X}^{\beta'} \\
& + \mathcal{K}(2z_1\bar{X}^2 + (2+\beta')z_4\bar{X}^{2+\beta'}) + 4z_3\rho\bar{X}^4 \\
& + (4+\beta')z_6\rho\bar{X}^{4+\beta'} + (3-\gamma_m)\rho\,\mathrm{Tr}(\chi\bar{U}^{\dagger} + \bar{U}\chi^{\dagger})\bar{X}^{3-\gamma_m}\Big\} \\
& = 0\,.
\end{aligned}
\tag{21}
$$

Appendix D: Cancellation of UV Divergences in the $\gamma\gamma$ Channel

In this appendix we show how the ultraviolet divergences are cancelled in the $\gamma\gamma$ channel with charged π, K loops, leaving a finite answer. Our case example is the calculation of diagrams for the matrix element $\langle\gamma_1,\gamma_2|\theta^{\mu}_{\mu}|\text{vac}\rangle$ in χPT_σ. We recall these contributions were part of our derivation for the effective $\sigma\gamma\gamma$ coupling. As noted in Chapter 4, the one-loop amplitudes in Fig. 3 share a similar Lorentz structure. In particular, one can combine into a single expression the diagrams

$$\text{(b) with (c)}\,, \quad \text{(d) with (e)}\,, \quad \text{(f) with (g)}\,, \quad \text{(h) with (i)}\,.$$

This can be seen as follows. We label the loop momenta for graphs (f-i) as shown in Fig. 4, and introduce the notation

$$\lambda_1 = (1-c_1)\beta'(\alpha_{\text{IR}}) \qquad \text{and} \qquad \lambda_2 = 1+\gamma_m(\alpha_{\text{IR}})\,. \tag{22}$$

The explicit expressions for the amplitudes are then given by

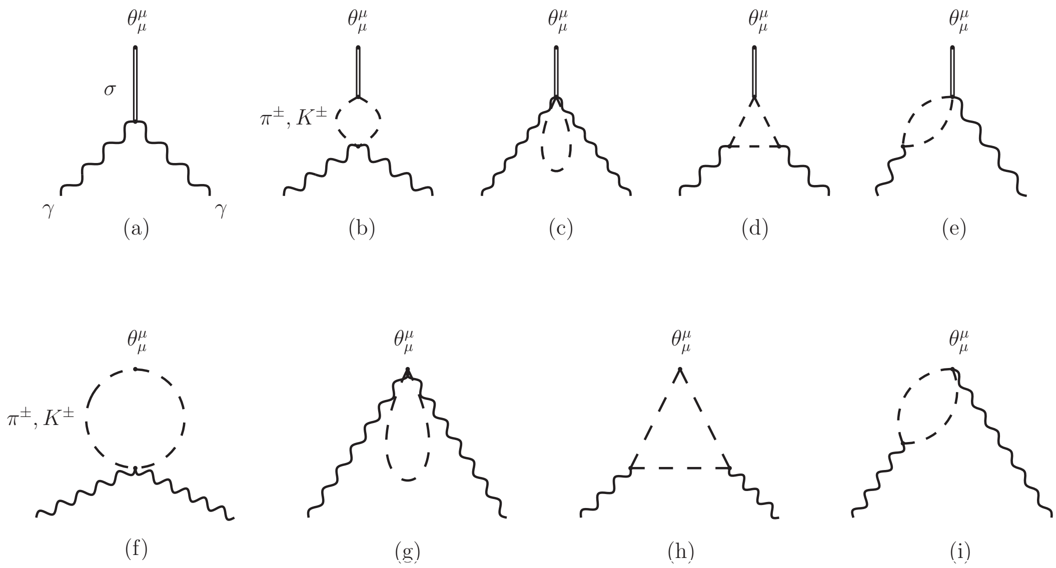


Figure 3: Leading order contributions to $\langle\gamma_1,\gamma_2|\theta^\mu_\mu(0)|\text{vac}\rangle$ in χPT_σ. Diagram (a) represents the contact term proportional to $g_{\sigma\gamma\gamma}$, while diagrams (d), (e), (h), and (i) are each accompanied by an additional crossed amplitude (not shown).

$$
\begin{aligned}
L^{\mu\nu}_f &= (-ie^2 g^{\mu\nu})\int \frac{d^4\ell}{(2\pi)^4}\frac{\left[2\lambda_1\ell\cdot(\ell+k_1+k_2)+2\lambda_2 m_\phi^2\right]}{[\ell^2-m_\phi^2+i\epsilon][(\ell+k_1+k_2)^2-m_\phi^2+i\epsilon]},\\
L^{\mu\nu}_g &= (2ie^2 g^{\mu\nu})\int \frac{d^4\ell}{(2\pi)^4}\frac{\lambda_1}{\ell^2-m_\phi^2+i\epsilon},\\
L^{\mu\nu}_{h_1+h_2} &= (ie^2)\int \frac{d^4\ell}{(2\pi)^4}\Bigg\{\frac{\left[\lambda_1(\ell-k_1)\cdot(\ell+k_2)+\lambda_2 m_\phi^2\right](2\ell-k_1)^\mu(2\ell+k_2)^\nu}{[\ell^2-m_\phi^2+i\epsilon][(\ell+k_2)^2-m_\phi^2+i\epsilon][(\ell-k_1)^2-m_\phi^2+i\epsilon]}\\
&\qquad +(k_1,\mu)\longleftrightarrow(k_2,\nu)\Bigg\},\\
L^{\mu\nu}_{i_1+i_2} &= (-ie^2)\int \frac{d^4\ell}{(2\pi)^4}\Bigg\{\frac{\lambda_1(2\ell-k_1)^\mu(2\ell-k_1)^\nu}{[\ell^2-m_\phi^2+i\epsilon][(\ell-k_1)^2-m_\phi^2+i\epsilon]}+\{k_1\longleftrightarrow k_2\}\Bigg\}. \qquad (23)
\end{aligned}
$$

To combine the various amplitudes we make use of the shift symmetry in dimensional regularization. For example, to combine (f) with (g), we first shift the mo-

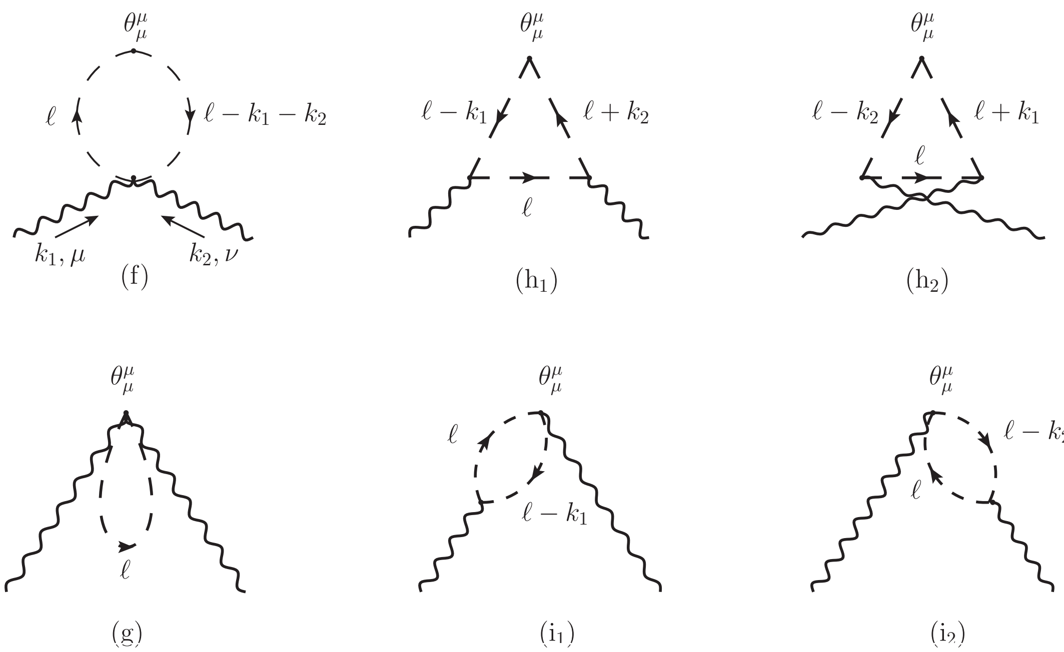


Figure 4: Flow of loop momenta for one-loop graphs (f-i) in Fig. 3.

mentum $\ell \to \ell - k_2$ and rewrite the resulting numerator in (f) as[1],

$$2\lambda_1(\ell - k_2)\cdot(\ell + k_1) = \lambda_1\big\{[(\ell + k_1)^2 - m_\phi^2] + [(\ell - k_2)^2 - m_\phi^2] + 2m_\phi^2\big\}. \tag{24}$$

We then shift the momentum in (g) so that[2]

$$\begin{aligned} L_g^{\mu\nu} &= (ie^2 g^{\mu\nu})\lambda_1 \int_\mu \frac{d^n\ell}{(2\pi)^n}\left\{\frac{1}{(\ell+k_1)^2 - m_\phi^2 + i\epsilon} + \frac{1}{(\ell-k_2)^2 - m_\phi^2 + i\epsilon}\right\} \\ &= (ie^2 g^{\mu\nu})\lambda_1 \int_\mu \frac{d^n\ell}{(2\pi)^n}\left\{\frac{[(\ell+k_1)^2 - m_\phi^2 + i\epsilon] + [(\ell-k_2)^2 - m_\phi^2 + i\epsilon]}{[(\ell+k_1)^2 - m_\phi^2 + i\epsilon][(\ell-k_2)^2 - m_\phi^2 + i\epsilon]}\right\}, \end{aligned} \tag{25}$$

[1]For simplicity, we consider on-shell photons: $k_1^2 = 0 = k_2^2$. Keeping them off-shell does not affect the final result.

[2]In our notation for dimensionally regularized integrals, $\int_\mu \equiv \mu^{4-n} \int$.

from which we immediately deduce that the divergent terms cancel between (f) and (g). The combined result is,

$$L_{f+g}^{\mu\nu} = (ie^2)2m_\phi^2(\lambda_1 - \lambda_2)\int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{g^{\mu\nu}}{[(\ell+k_1)^2 - m_\phi^2 + i\epsilon][(\ell-k_2)^2 - m_\phi^2 + i\epsilon]}. \tag{26}$$

To combine (h) with (i), we first factor out 1/2 from the amplitude and rewrite the terms in the numerator,

$$2\lambda_1(\ell - k_1)\cdot(\ell + k_2) = \lambda_1\{[(\ell-k_1)^2 - m_\phi^2] + [(\ell+k_2)^2 - m_\phi^2] + 2m_\phi^2\}, \tag{27}$$

with a similar expression for (h_2). We then halve each ($i_{1,2}$) term and shift appropriately, i.e.,

$$\begin{aligned}
L_{i_1}^{\mu\nu} &= -\frac{ie^2}{2}\int_\mu \frac{d^n\ell}{(2\pi)^n}\left\{\frac{\lambda_1(2\ell-k_1)^\mu(2\ell-k_1)^\nu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell-k_1)^2 - m_\phi^2 + i\epsilon]} + \{\ell \to \ell + k_1\}\right\},\\
L_{i_2}^{\mu\nu} &= -\frac{ie^2}{2}\int_\mu \frac{d^n\ell}{(2\pi)^n}\left\{\frac{\lambda_1(2\ell-k_2)^\mu(2\ell-k_2)^\nu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell-k_2)^2 - m_\phi^2 + i\epsilon]} + \{\ell \to \ell + k_2\}\right\}.
\end{aligned} \tag{28}$$

Combining the terms with common denominators yields the integrals,

$$\begin{aligned}
\mathcal{I}_{h_1+i_1} &= -(k_1+k_2)^\nu \int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{(2\ell-k_1)^\mu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell-k_1)^2 - m_\phi^2 + i\epsilon]},\\
\mathcal{I}_{h_1+i_2} &= (k_1+k_2)^\mu \int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{(2\ell+k_2)^\mu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell+k_2)^2 - m_\phi^2 + i\epsilon]},\\
\mathcal{I}_{h_2+i_1} &= (k_1+k_2)^\nu \int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{(2\ell+k_1)^\mu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell+k_1)^2 - m_\phi^2 + i\epsilon]},\\
\mathcal{I}_{h_2+i_2} &= -(k_1+k_2)^\mu \int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{(2\ell-k_2)^\mu}{[\ell^2 - m_\phi^2 + i\epsilon][(\ell-k_2)^2 - m_\phi^2 + i\epsilon]},
\end{aligned} \tag{29}$$

all of which are odd in ℓ (and hence vanish) under the shift symmetry. Thus the only surviving terms are combined to form,

$$\begin{aligned} L^{\mu\nu}_{b+d} =& -(ie^2)2m^2_\phi(\lambda_1-\lambda_2) \\ &\times \int_\mu \frac{d^n\ell}{(2\pi)^n}\frac{(2\ell+k_1)^\mu(2\ell-k_2)^\nu}{[\ell^2-m^2_\phi+i\epsilon][(\ell+k_1)^2-m^2_\phi+i\epsilon][(\ell-k_2)^2-m^2_\phi+i\epsilon]}, \end{aligned} \tag{30}$$

and the complete matrix element reads,

$$\begin{aligned} L^{\mu\nu}_{\phi} =& (ie^2)2m^2_\phi(\lambda_1-\lambda_2) \\ &\times \underbrace{\int \frac{d^4\ell}{(2\pi)^4}\frac{g^{\mu\nu}(\ell^2-m^2_\phi+i\epsilon)-(2\ell+k_1)^\mu(2\ell-k_2)^\nu}{[\ell^2-m^2_\phi+i\epsilon][(\ell+k_1)^2-m^2_\phi+i\epsilon][(\ell-k_2)^2-m^2_\phi+i\epsilon]}}_{I^{\mu\nu}}. \end{aligned} \tag{31}$$

The (finite) integral $I^{\mu\nu}$ can now be calculated with standard techniques,

$$I^{\mu\nu} = \left[\frac{-i}{16\pi^2}\right]\frac{(g^{\mu\nu}(k_1\cdot k_2)-k_2^\mu k_1^\nu)}{k_1\cdot k_2}(1+2I_\phi). \tag{32}$$

An analogous approach for the diagrams (b-e) in Fig. 3 yields the expression

$$N^{\mu\nu}_\phi = -\left(\frac{e^2}{F_\sigma}\right)2m^2_\phi(\lambda_1-\lambda_2+2)I^{\mu\nu}, \tag{33}$$

which when combined with the PCDC relation leads to a cancellation of the terms proportional to $(1-\gamma_m+(1-c_1)\beta')$.

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