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A Fast Graph Matrix Partitioning Algorithm for Solving the Water Distribution System Equations

J. Deuerlein[1,3], S. Elhay[2] and A. R. Simpson[3], M.ASCE

Abstract:

In this paper a method which determines the steady-state hydraulics of a water distribution system, the Graph Matrix Partitioning Algorithm (GMPA), is presented. This method extends the technique of separating the linear and nonlinear parts of the problem and using the more time consuming nonlinear solver only on the nonlinear parts of the problem and faster linear techniques on the linear parts of the problem. The previously developed Forest-Core Partitioning Algorithm (FCPA) used this approach to separate the network graph's external forest from its looped core but did not address the fact that within the core of a network graph there may be many internal trees - nodes in series - for which a more economical linear process can be used. This extension of the separation process can significantly reduce the dimension of the nonlinear problem that must be solved: GMPA applied to eight case study networks with between 900 and 20,000 pipes show reductions to between 5% and 55% of the core dimension (after FCPA). The separation of the problem into its nonlinear and linear parts involves no approximations, such as lumping or skeletonization, and the resulting solution is precisely the solution that would have been obtained by the slower technique of solving the entire network with a nonlinear solver. The new method is applied after the network has been separated into an external forest and core by the FCPA method. The GMPA identifies all the nodes in the core which are in series (the internal forest) and then iterates alternately on the remaining core (the (nonlinear) global step) and the internal forest (the (linear) local step). In this paper, it is formally shown that the smaller set of nonlinear equations in the GMPA corresponds to the network equations of a particular topological subgraph of the original graph. Using algebraic manipulations, the size of the linearized system to be solved is reduced to the number of nodes in the core having degree greater than two. For pipe models of real world applications that are derived from GIS datasets, this can mean a dramatic reduction of the size of the nonlinear problem that has to be solved. The main contributions of the paper are (i) the derivation and presentation of formal proofs for the new method and (ii) demonstrating how significant the reduction in the dimension of the nonlinear problem can be for suitable networks. The method is illustrated on a simple example.

Keywords: Water Distribution Systems, Hydraulic Network Simulation, Graph Matrix Decomposition, Forest-Core Partitioning

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INTRODUCTION

The calculation of steady-state hydraulic solutions of networks of elements has a long history. Early on in 1936, Cross presented a method for the iterative solution of the linear mass balance and nonlinear equations representing conservation of energy. Later, different methods for the simultaneous solution of the equations were proposed by Epp and Fowler (1970). In this context, the relation to graph theory was also established (Kesavan and Chandrashekar 1972, Gupta and Prasad 2000). The most prevalent methods are the nodal method, the loop flow correction method (e.g. Nielsen 1989) and the Global Gradient Algorithm (GGA) (Todini and Pilati 1988). The equivalent formulation of the steady-state equations as a minimization problem that is derived from a variational principle has the advantage that both the existence and uniqueness of solutions can be proven and powerful techniques of convex optimization can be applied to the problem (Birkhoff and Diaz 1956, Birkhoff 1963, Carpentier and Cohen 1993, Cherry 1951, Collins et al. 1978), even in the case where feedback devices have to be considered in addition to pipes, valves and pumps (Deuerlein et al. 2009). An overview of existing methods has been published recently (Todini and Rossman 2013).

Although remarkable speed increases have been achieved by hydraulic solvers, optimization algorithms, in particular evolutionary algorithms that require a huge number of repeated network solution calculations in which the network topology does not change still suffer from long computing times. In addition, speed of computation is important when algorithms are used online for real-time analysis. As a result, a speed-up of existing hydraulic simulation methods is required.

One application of hydraulic online simulation relates to water network security. Online simulation is used for monitoring of the current state of the network. As well, in the case of a contamination event, additional functions such as source identification and development of mitigation measures are based on the knowledge of the actual hydraulic state of the system. These functions again require the simulation of a number of different scenarios with various levels of detail. Reducing the size of the most time-consuming nonlinear problem will help.

Various commercial and non-commercial software programs are available for calculating the hydraulic steady-state condition of a network. Many of them are based on the Global Gradient Algorithm (Todini and Pilati 1988), which, for instance, is implemented in EPANET (Rossman 2000). In this paper, a graph matrix partitioning method is presented that solves the global set of equations but significantly reduces the size of the linearized system of equations that is solved in each iteration. The method is based on a decomposition concept of the network graph (Deuerlein 2008). In that publication it was shown that, in demand-driven analysis, the hydraulic steady-state equations of the forest can be solved independently from the core. The concept of forest-core partitioning was further developed by Simpson et al. (2014). The core can be further subdivided into bridge components and looped blocks. A non-linear solver is required only for the blocks. The content of this manuscript addresses a faster solution method for the significantly smaller set of the non-linear block equations that is based on the partitioning of the nodes of the blocks into supernodes (degree > 2) and internal nodes (degree = 2).

The GMPA begins by using the FCPA to separate the network core from the external forest (the forest which consists of all the trees which have leaf nodes (nodes with index 1)) and then identifying, within the core, all those nodes which have index 2 (the internal forest). The GMPA exploits the fact that the solution for the internal forest can, like the external forest) be achieved by a much faster linear solver. As a result, only the smaller part of the core (which remains when the internal trees have been partitioned for separate treatment), needs a nonlinear solver.

Giustolisi et al. (2010, 2012) presented the Enhanced Global Gradient Algorithm (EGGA), which uses heuristically based transformation matrices to exploit the presence of nodes with index 2. However, (i) GMPA includes a preliminary forest-core partitioning step (Simpson et al. 2014), (ii) allows the inclusion of control devices by appropriate selection of internal tree chords, (iii) and is presented with a formal derivation and rigorous proof of the method.

One possible approach for reducing calculation time is the aggregation of existing models that are usually derived from comprehensive Geographical Information Systems (GIS) datasets. In this case, a similar simplified system is created by merging of links of the same diameter and eliminating the nodes connecting these links. However, the resulting model is only an approximation of the original system and the connection to the original reference data taken from a GIS may be lost.

The structure of the paper is as follows: After a brief review of the fundamental equations, some basics on the subgraph concept are presented. In the main part of this paper, the equivalence of the GMPA that is based on the topological subgraph to the full set of equations is stated. A proof is then given. It is shown that the new method subdivides the solution procedure into global and local steps. In the global step, a nonlinear system of equations has to be solved for the topological subgraph. The solution of the local step consists of linear algebraic calculations applying the global solution to determine the flows and heads of the internal trees. An example system is used to illustrate the method. Eight case-study networks, with between 900 and 20,000 pipes, and which were considered in Simpson et al (2014), are used to illustrate the significant reduction achieved by GMPA in the size of the nonlinear problem that must be solved.

BACKGROUND

Mathematical Modeling of Water Distribution Systems

As mentioned in an earlier paper (Deuerlein et al., 2009), an alternative formulation to solving the pipe network equations by direct methods may be based on the minimization methods of nonlinear optimization. Birkhoff and Diaz (1956) and later Birkhoff (1963) have shown that the calculation of the looped electrical circuit systems with consideration of the first and second laws of Kirchhoff is equivalent to the minimization of a convex function. These principles also can be applied to solving the pipe network equations, which are modelled by analogous equations. Using the work of Cherry (1951) and Millar (1951) for the calculation of electrical networks, Collins et al.(1978) applied the minimization of the Content and Co-Content functions to the calculation of the steady-state for hydraulic networks. The problem of minimizing the Content function can be stated as follows:

$$\begin{aligned} &\min_{\mathbf{q}\in R^m} C(\mathbf{q}) \\ &s.t.\ \ \mathbf{A}^T\mathbf{q} = -\mathbf{Q} \end{aligned} \quad (1)$$

Here, $C(\mathbf{q})$ denotes the system content, *m* is the number of pipes, $\mathbf{q} \in \mathbb{R}^m$ is the vector of pipe flows, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the link-node incidence matrix of pipes and demand nodes (number *n*) with $A_{j,i} = 1$ if node i is the final node of link j, $A_{j,i} = -1$ if node i is the first node of link j and $A_{j,i} = 0$, otherwise. $\mathbf{Q} \in \mathbb{R}^n$ is the vector of given nodal demands (for withdrawals $Q_i < 0$). The equality constraints consist of the continuity equations of the demand nodes. The total system content is composed of the sum of the single contents of the network elements. The content of pipe j is given by

$$C_j = \int_0^{q_j} \left(r_j |x_j|^{\alpha-1} + K_j |x_j| \right) x_j dx_j. \quad (2)$$

Here, q_j is the flow for pipe j, $r_j = r_j(x_j)$ is the pipe resistance which depends on flow for the Darcy-Weisbach head loss and α the exponent of the hydraulic equation (usually for the Darcy-Weisbach or Hazen-Williams formulations where $\alpha \geq 1$). The second term refers to the local minor loss of valves and fittings. If all the content functions are strictly convex (which is guaranteed by the strict monotonicity of the head loss equation) and norm-coercive ($|C_j(q_j)| \longrightarrow \infty$ if $\|q_j\| \rightarrow \infty$) then the total system content is a strictly convex and norm-coercive function of $\mathbf{q}$. As a result, there exists a unique solution if, in addition, the constraints are affine. The proof of existence is a very important result and can be found for example in Piller (1995). In this case, the Lagrangian of Eq. (1) gives a necessary and sufficient condition for the solution:

$$\begin{bmatrix} \mathbf{F} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_\mathrm{R}\mathbf{H}_\mathrm{R} \\ -\mathbf{Q} \end{bmatrix} \tag{3}$$

where $\mathbf{F} \in \mathbb{R}^{m\times m}$ is a diagonal matrix and the product of $\mathbf{Fq}$ are the link head losses. The elements of the diagonal matrix $\mathbf{F}_\mathrm{k}$ are defined by $F_{k_{j,j}} = r_j\left|q_{k,j}\right|^{\alpha-1} + K_j\left|q_{k,j}\right|$ where the first term refers to the head losses due to friction along the pipe wall and the second term includes local minor losses of valves and fittings (K_j: minor loss coefficient).

On the right hand side of Eq. (3) $\mathbf{A}_\mathrm{R} \in \mathbb{R}^{m\times nr}$ and $\mathbf{H}_\mathrm{R} \in \mathbb{R}^{nr}$ refer to the incidence matrix of fixed head nodes and vector of fixed heads, respectively in Eq. (3), nr is the number of fixed head nodes. The Lagrange multipliers $\boldsymbol{\lambda} \in \mathbb{R}^{n}$ that correspond to the affine mass balance constraints or nodal flow continuity equations in Eq. (1) are actually the nodal heads at the demand nodes.

The solution to the non-linear system Eq. (3) is often found iteratively by Newton's method

$$\begin{bmatrix} \mathbf{D}_k & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{k+1} - \mathbf{q}_k \\ \mathbf{H}_{k+1} - \mathbf{H}_k \end{bmatrix} = -\begin{bmatrix} \mathbf{F}_k\mathbf{q}_k + \mathbf{A}\mathbf{H}_k + \mathbf{A}_R\mathbf{H}_R \\ \mathbf{A}^T\mathbf{q}_k + \mathbf{Q} \end{bmatrix}, \tag{4}$$

provided $\mathbf{D}_k$, which is the derivative of the head losses $\mathbf{Fq}$, is invertible. The vector $\mathbf{q}_{k+1} - \mathbf{q}_k$ includes the unknown changes in link flows, $\mathbf{H}_{k+1} - \mathbf{H}_k$ is the vector of unknown nodal head changes, $\mathbf{H}_\mathrm{R}$ is the vector of known heads and $\mathbf{Q}$ is the vector of known nodal demands or input flows.

The GGA of Todini and Pilati (1988), a two-stage implementation of Eq. (4), finds the unknown nodal heads, $\mathbf{H}_{k+1}$ (Lagrangian multipliers $\boldsymbol{\lambda}$ in Eq. (3)) in one-step and then finds the pipe flows, $\mathbf{q}_{k+1}$, by a simple update process:

$$\mathbf{A}^T\mathbf{D}_k^{-1}\mathbf{A}\cdot\mathbf{H}_{k+1} = \mathbf{Q} + \mathbf{A}^T\mathbf{q}_k - \mathbf{A}^T\mathbf{D}_k^{-1}\mathbf{F}_k\cdot q_k - \mathbf{A}^T\mathbf{D}_k^{-1}\mathbf{A}_R\cdot\mathbf{H}_R$$

$$\mathbf{q}_{k+1} = \mathbf{q}_k - \mathbf{D}_k^{-1}[\mathbf{F}_k\mathbf{q}_k + \mathbf{A}\mathbf{H}_{k+1} + \mathbf{A}_R\mathbf{H}_R] \tag{5}$$

The second part of Eq. (5) is solved simply because the matrix $\mathbf{D}_k^{-1}$ is diagonal. The first part, which is more difficult to solve, includes the solution of a linear system of size *n*. In all iterations after the first, the calculated flows will satisfy the continuity equations independently of the starting point (initial flow vector) of the algorithm (see Elhay et al., 2014).

Recall that the size of $\mathbf{D}_k$ is the total number of links in the graph of the full network. For constant r_j the derivative matrix is $D_{k_{j,j}} = \alpha r_j \left|q_{k,j}\right|^{\alpha-1} + 2K_j\left|q_{k,j}\right|$ but in the case of the Darcy-Weisbach formula, r_j is also a function of $q_{k,j}$ and the calculation of the derivative is more complicated (see Simpson and Elhay 2011 for details).

THE GRAPH MATRIX PARTITIONED ALGORITHM

Decomposition to form the reduced topological subgraph system

The first step of the GMPA consists of the identification of the "forest" of the network graph. The forest consists of trees that are connected to bridge components or looped blocks at their root node. Figure 1 shows a simplified example network graph that has typical topological properties of a real water supply system. Branched trees leading to the end-user demands are connected to a looped distribution system at the root nodes. From graph theory, it is known that the incidence matrix of a tree (without its root node) is square and invertible. Therefore, in demand driven analysis, the flows of the forest pipes can be calculated (see Simpson et al. 2014 for details of calculation method) by using just the (linear) continuity equations as a preliminary step before the iterative solution procedure is applied to the "core" of the network.

In what follows it is assumed that the forest has already been virtually separated from the network graph and the demand at the root node of each removed tree has been increased by the total demand of the tree. The separation does not involve any approximation or lumping. At the end of the process, the solution obtained is precisely the solution for the whole (original) network. The resulting graph $G_C \subseteq G = (V, E)$ is called the "core" of the original network graph. It can be shown that G_C is an (induced) subgraph of the original graph $G_C \subseteq G = (V, E)$. More details on the separation of the forest and core decomposition can be found in the literature (Diestel 2010; Deuerlein 2008). From Figure 1, it can be seen that the root nodes (filled with black) of all the trees have degree three or greater (degree: number of connected links at node). After separating each of the trees, most of the root nodes have degree two in the core graph (see Figure 2). This property is used for the next step where the core graph G_C is further simplified by the identification of “supernodes” and “superlinks.”

Supernodes have the property that they connect at least three links of the original core graph (nodes E and I in Figure 3). A superlink can be understood as virtual link that represents the series of links between the two supernodes. The nodes inside the path that connects the supernodes are called “internal tree nodes” (index I, nodes F, G and H in Figure 3). The series of links connecting the two supernodes with the last link removed is called the internal tree (E - F - G -H), and finally, the link (H-I) is called the internal co-tree chord. The reason for the terminology “internal tree” and “internal co-tree chord will become clear later.

The (topological) subgraph $G_S = (V_S, E_S)$ consists of the vertex set of supernodes V_S with $|V_S| = n_S$ and the arc set of superlinks E_S with $|E_S| = m_S$. Note that G_S refers to the topological subgraph of the core of the original network graph that results from link contractions that are iteratively applied to links that have at least one node of degree 2 in the core subgraph. The resulting topological subgraph is maximal in the sense that all the nodes with degree 2 are separated. The graph G_S is smaller than the full core graph for the network if the network core has nodes in series. By means of graph theory it can be shown that G_S is actually a topological minor of G and G_C: $G_S \preccurlyeq G_C \preccurlyeq G$. The minor relation $\preccurlyeq$ is a partial ordering of finite graphs, i. e. reflexive, antisymmetric and transitive (see Diestel 2010, p. 20). In contrast, the full core graph G_C is called a subdivision of graph G_S. It follows that the two graphs G_S and G_C are homeomorphic in the topological sense, which means that they have the same topological properties; for instance, the number of loops remains the same for both G_C and G_S. Please note that the term ‘topological’ used here refers to the fact, that in contrast to induced subgraphs, where the link set of the subgraph is a subset of the original link set, here each superlink of the topological subgraph replaced by a set of pipes in series each with internal vertices of degree 2. As a consequence, the links of the original graph appear as superlinks in the subgraph only in the case where both end nodes have degree > 2 in the original core graph.

Hydraulic Steady-State Equations of the GMPA

In this section, the GMPA method for demand driven hydraulic steady-state calculation of general pressurized pipe systems which subdivides the solution process into a global step (for the graph G_S)

and a local step (for the internal trees) steps is presented. It will be shown that the dimension of the system of equations that has to be solved during the iterations can be reduced to the number of supernodes of the graph G_S that was introduced in the last section. For complex real-world water supply systems, the size of the system to be solved can be reduced significantly for such systems. The full derivation of the method from the basic network equations will be given in the next section. In this context, the linear transformation mapping between full graph and topological subgraph will also be discussed.

The global solution step is formulated as follows. The vertex set of G_S consists of a subset of n_S nodes of the original graph, the supernodes. The set of links between supernodes of the original graph may be treated as–a set of m_S superlinks. Now assume that the heads of the supernodes are known (as boundary conditions of the local subsystem) the series of pipes in-between can be understood as a pseudo loop representing a subsystem that consists of two reservoirs that are connected by a series of pipes. The heads and flows of these isolated subsystems can be efficiently calculated by the loop method since the subsystem always consists of exactly one pseudo loop regardless of the number of intermediate links in series. The superlink is the pseudo-link that represents the known pressure difference between the supernodes and closes the pseudo-loop (see Figure 3). A well-known approach for solving such a system is to separate the links of the loop into a spanning tree and a chord link. Therefore, in the previous section, the term "internal tree" has been introduced for the internal spanning tree that consists of "internal tree branches" and the term "internal tree chord" is used for the link that is chosen arbitrarily as chord of the local pseudo-loop. The union of internal trees is called the internal forest. Please note that in our case where the external forest has been separated by the FCPA before the GMPA is applied and the method is applied to the network core, the internal tree is always a path. However, the method can still be applied where the local systems have tree like structure (for example if the trees of the global forest are not separated).

The superlinks in the GMPA have an important property: it is known (see Elhay et al., 2014) that the continuity equations are satisfied after the first iteration of the GGA or the Reformulated Co-Tree Flows Method (RCTM) (see Elhay et al., 2014) and so the flow corrections of all links that are represented by a common superlink are identical. Thus, if the flow correction for one link (for example an internal co-tree chord belonging to the internal tree of the superlink) is known from the global step calculation, the other flows of the internal tree branches of that superlink can be calculated locally (local step). As mentioned before, in the global step a superlink replaces at least one link, the internal co-tree chord (index *C*), and the appropriate number of internal tree branches (index *T*). The total number of internal co-tree chords is equal to the number of superlinks and the number of internal tree branches is $m_T = m_{2core} - m_S$. The flows of the other internal tree branches are a linear function of the both flow of the last link and the given demands of the internal tree nodes. This is a very important feature of the GMPA and is the fundamental reason why a significant reduction in computation can be achieved. The system of equations of the global step (graph G_S) can be formulated in terms of the unknown superlink flows corrections, $\left[\mathbf{q}_{C,k+1} - \mathbf{q}_{C,k}\right]$, at iteration $k+1$ and the heads corrections, $\left[\mathbf{H}_{S,k+1} - \mathbf{H}_{S,k}\right]$, at iteration $k+1$:

$$\begin{bmatrix} \mathbf{D}_{S,k} & \mathbf{A}_S \\ \mathbf{A}_S^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{C,k+1} - \mathbf{q}_{C,k,} \\ \mathbf{H}_{S,k+1} - \mathbf{H}_{S,k} \end{bmatrix} = -\begin{bmatrix} \mathbf{h}_{S,k} + \mathbf{A}_S \mathbf{H}_{S,k} + \mathbf{A}_{S,R} \mathbf{H}_R \\ \mathbf{A}_S^T \mathbf{q}_{C,k} + \overline{\mathbf{Q}}_S \end{bmatrix} \tag{6}$$

with

$$\overline{\mathbf{Q}}_S = -\mathbf{A}_{S,T}^T \left(\mathbf{A}_{I,T}^T\right)^{-1} \mathbf{Q}_I + \mathbf{Q}_S \tag{6a}$$

$$\mathbf{h}_{S,k} = \left[\mathbf{F}_{C,k} + \mathbf{P}\mathbf{F}_{T,k}\mathbf{P}^T\right] \mathbf{q}_{C,k} - \mathbf{P}\mathbf{F}_{T,k} \left(\mathbf{A}_{I,T}^T\right)^{-1} \mathbf{Q}_I \tag{6b}$$

Eq. (6) forms the Newton iteration system for the topological subgraph G_S. The incidence matrix for the topological subgraph is $\mathbf{A}_S \in \mathbb{R}^{m_S \times n_S}$ and the diagonal matrix $\mathbf{D}_S \in \mathbb{R}^{m_S \times m_S}$ has the head loss derivatives of the superlinks. Submatrices $\mathbf{A}_{S,T} \in \mathbb{R}^{m_T \times n_S}$ and $\mathbf{A}_{I,T} \in \mathbb{R}^{m_T \times m_T}$ are blocks of the (permuted) original incidence matrix, $\mathbf{P} \in \mathbb{R}^{m_S \times m_T}$ is the incidence matrix for the internal forest of nodes with index 2 (the nodes which are in series). Matrices $\mathbf{Q}_I \in \mathbb{R}^{n_I}$ and $\mathbf{Q}_S \in \mathbb{R}^{n_S}$ are the blocks of the decomposed demand vector that refer to internal nodes (Index *I*) and supernodes (index *S*), respectively. Vector $\mathbf{q}_{C,k} \in \mathbb{R}^{m_S}$ includes the flows of internal co-tree chords according to Figure 3 and $\mathbf{F}_{C,k} \in \mathbb{R}^{m_S \times m_S}$ and $\mathbf{F}_{T,k} \in \mathbb{R}^{m_T \times m_T}$ are two parts of the diagonal matrix $\mathbf{F}_k$ of Eq. (5) that are decomposed into blocks for internal tree branches and internal co-tree chords. The matrices and the objects which they model will be explained in more detail in the next section. In what follows we will denote $\left(\mathbf{A}_{I,T}^{-1}\right)^{T}$ (which $=\left(\mathbf{A}_{I,T}^{T}\right)^{-1}$) by $\mathbf{A}_{I,T}^{-T}$.

The solution procedure here deals with a smaller set of equations than for the GGA if the core of the network has some nodes in series. The calculation of the head losses of the superlinks and the respective matrices $\mathbf{D}_{S,k}$ and $\mathbf{F}_{S,k}$ is carried out locally. Matrix $\mathbf{P} \in \mathbb{R}^{m_S \times m_T}$ represents the linear transformation between the links of G_C and G_S. Multiplication of a topological matrix on the left by **P** collapses the branches of an internal tree onto a single superlink whereas right multiplication of a matrix by **P** expands a superlink to the individual branches of the corresponding internal tree. The derivation of **P** is given in the next section. Once the global unknowns $\mathbf{q}_{C,k+1}$ and $\mathbf{H}_{S,k+1}$ are known the local step of calculating the flows of internal tree branches and heads of internal tree nodes is straightforward. This results in two steps of the GMPA:

The Global Step: Calculation of heads of supernodes and flows of internal co-tree chords representing the superlinks

$$\mathbf{A}_S^T \mathbf{D}_{S,k}^{-1} \mathbf{A}_S \mathbf{H}_{S,k+1} = \overline{\mathbf{Q}}_S + \mathbf{A}_S^T \mathbf{q}_{C,k} - \mathbf{A}_S^T \mathbf{D}_{S,k}^{-1}\left[\mathbf{h}_{S,k} + \mathbf{A}_{S,R}\mathbf{H}_R\right] \tag{7}$$

$$\mathbf{q}_{C,k+1} = \mathbf{q}_{C,k} - \mathbf{D}_{S,k}^{-1}\left[\mathbf{h}_{S,k} + \mathbf{A}_S \mathbf{H}_{S,k+1} + \mathbf{A}_{S,R}\mathbf{H}_R\right] \tag{8}$$

The Local Step: Calculation of heads of the internal tree nodes and flows of the internal tree branches

$$\mathbf{q}_{T,k+1} = \mathbf{P}^T \mathbf{q}_{C,k} - \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I \tag{9}$$

$$\mathbf{H}_{I,k+1} = -\mathbf{A}_{I,T}^{-1}\left[\mathbf{A}_{S,T}\mathbf{H}_{S,k+1} + \mathbf{A}_{R,T}\mathbf{H}_R + \mathbf{D}_{T,k}\mathbf{q}_{T,k+1}\right] \tag{10}$$

With the approach presented here, the number of unknowns of the linear system of equations that has to be solved at any iteration of the GMPA (Eq. (7)) is reduced to the number of supernodes n_S. Since $\mathbf{D}_{S,k}$ is a diagonal matrix, Eq. (8) consists of algebraic calculations. The same applies to the local step since matrix $\mathbf{A}_{I,T}$ has an analytical inverse, which will be shown later.

DERIVATION AND PROOF OF THE GRAPH MATRIX PARTITIONING ALGORITHM

In what follows, the equivalence of the modified equations for the local and global steps (Eq. (7) to Eq. (10)) and the original GGA equations (Eq.**Error! Reference source not found.**)) will be shown.

The basic idea is that the incidence matrix of the network graph can be partitioned into four blocks where the upper right square block has full rank and is invertible.

Partitioning of the Incidence Matrix

In the new GMPA method, internal nodes are removed by an algebraic elimination process that is applied to the full system of hydraulic network equations. It is important to note that the GMPA method does not include any approximation. The elimination is based on permuting the rows and the columns of the graph incidence matrix **A** and the diagonal head loss derivative matrix **D** of the original system. Consider the partitioning of both the **A** and **D** matrices as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{S,T} & \mathbf{A}_{I,T} \\ \mathbf{A}_{S,C} & \mathbf{A}_{I,C} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_T & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_C \end{bmatrix} \tag{11}$$

where

T: **T**ree (branches of internal trees) (see #3 in Figure 3)

C: **Co-tree** (**C**hords) (chords of internal trees) (see #4 Figure 3)

S: **S**upernodes (E and I in Figure 3)

I: **I**nternal nodes (F, G and H in Figure 3)

In what follows it will be assumed that the necessary permutations have already been performed: the matrix **A** is in the correct permuted form and **D** is diagonal. It is shown in the Appendix A how to permute **A** in this way while preserving the diagonal structure of **D**. The columns of **A** refer to the nodes of the graph *G*. The first column of the block matrix in Eq. (11) belongs to the supernodes (first index *S*) and the second column belongs to internal tree nodes (first index *I*). In a similar way the internal tree branches (second index *T*) are partitioned from the internal co-tree chords (second index *C*) in matrices **A** and **D**. An important result of the reordering is that matrix $\mathbf{A}_{I,T}$ is always square and invertible. Furthermore, it will be shown that its exact inverse can be computed quickly and simply. This fact can be used for eliminating both the flows in the internal trees as well as the heads of the internal tree nodes. The method can be understood as partitioning the solution process into one global step (analysis of the network consisting of superlinks and supernodes only) and several local steps (where the flows of the internal tree links are determined).

System of equations

Substitution of the matrices **A** and **D** from Eq. (11) into Eq. (5) results in the following system of linearized equations:

$$\begin{bmatrix} \mathbf{D}_{T,k} & \mathbf{0} & \mathbf{A}_{S,T} & \mathbf{A}_{I,T} \\ \mathbf{0} & \mathbf{D}_{C,k} & \mathbf{A}_{S,C} & \mathbf{A}_{I,C} \\ \mathbf{A}_{S,T}^T & \mathbf{A}_{S,C}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{I,T}^T & \mathbf{A}_{I,C}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{T,k+1} - \mathbf{q}_{T,k} \\ \mathbf{q}_{C,k+1} - \mathbf{q}_{C,k} \\ \mathbf{H}_{S,k+1} - \mathbf{H}_{S,k} \\ \mathbf{H}_{I,k+1} - \mathbf{H}_{I,k} \end{bmatrix} = \begin{bmatrix} d\mathbf{E}_T \\ d\mathbf{E}_C \\ d\mathbf{Q}_S \\ d\mathbf{Q}_I \end{bmatrix}_k \tag{12}$$

where

$d\mathbf{E}_T = -\left(\mathbf{F}_{T,k}\mathbf{q}_{T,k} + \mathbf{A}_{S,T}\mathbf{H}_{S,k} + \mathbf{A}_{I,T}\mathbf{H}_{I,k} + \mathbf{A}_{R,T}\mathbf{H}_R\right),$

$d\mathbf{E}_C = -\left(\mathbf{F}_{C,k}\mathbf{q}_{C,k} + \mathbf{A}_{S,C}\mathbf{H}_{S,k} + \mathbf{A}_{I,C}\mathbf{H}_{I,k} + \mathbf{A}_{R,C}\mathbf{H}_R\right),$

$d\mathbf{Q}_S = -\left(\mathbf{A}_{S,T}^T\mathbf{q}_{T,k} + \mathbf{A}_{S,C}^T\,\mathbf{q}_{C,k} + \mathbf{Q}_S\right),$

$d\mathbf{Q}_I = -\left(\mathbf{A}_{I,T}^T\mathbf{q}_{I,k} + \mathbf{A}_{I,C}^T\,\mathbf{q}_{C,k} + \mathbf{Q}_I\right),$

Further developments are based on the following:

Lemma 1:

a) Matrix $\mathbf{A}_{I,T}$ is square and invertible.

b) $\mathbf{A}_{I,T}$ is block diagonal with lower triangular diagonal blocks, all the elements of which are in {-1,0,1}, and has an exact inverse which is easily computed.

Proof:

a) Let m_S be the number of links (internal tree branches and an internal co-tree chord) represented by a superlink. Then, the number of nodes of the internal tree and internal co-tree is m_S+1. If the last link (for example link HI in Figure 3) is separated then the resulting structure is an internal tree.

$\mathbf{A}_{I,T}$ includes for each superlink the series of internal links except for the last link. The resulting subgraph is always a tree (i.e. it is connected and has no loops). The nodes of $\mathbf{A}_{I,T}$ are the interior nodes. The root of the tree is the initial node of the superlink, which is not part of $\mathbf{A}_{I,T}$. It is well known from graph theory (Diestel 2010) that the incidence matrix of a tree without its root node is always square and invertible. It follows that the matrix $\mathbf{A}_{I,T}$ consisting of diagonal blocks of simple tree incidence matrices is always invertible (see solid links in Figure 3).

b) The second part of the proof can be found in Branin (1963). It is shown there that the inverse of the tree incidence matrix is equal to the transpose of the node to datum path matrix of the tree, which can be determined, for instance, by use of basic graph algorithms (depth first search or breadth first search) or by a simple forward substitution process. □

Definition 1:

Matrix $\mathbf{P} = -\mathbf{A}_{I,C}\,\mathbf{A}_{I,T}^{-1}$ is denoted as the internal tree matrix.

The positive direction of a superlink is determined by the direction of the internal co-tree chord. If the initial node of the internal co-tree chord link is a supernode then it will be the initial node of the superlink. In the opposite case where the last node of the internal co-tree chord is a supernode node then it will be the last node of the superlink. The directions of the branches of the internal tree are determined by the sign of the links in the internal tree matrix. A positive sign means that the direction of the link coincides with the direction of the superlink and vice versa.

Using Lemma 1, the unknown flows of the branches of the internal tree can be eliminated from the system of linear equations by solving the fourth row of the Global Linear System [GLS] (Eq. (12)) for $\mathbf{q}_{T,k+1}$:

$$\mathbf{q}_{T,k+1} = \mathbf{P}^T \mathbf{q}_{C,k+1} - \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I \qquad (13)$$

Again, using Lemma 1, the unknown heads of the internal nodes can be eliminated from Eq. (12) by substituting $\mathbf{q}_{T,k+1}$ with use of Eq. (13) and solving the first row of Eq. (12) for $\mathbf{H}_{I,k+1}$:

$$\mathbf{H}_{I,k+1} = \mathbf{A}_{I,T}^{-1}\left[\mathbf{h}_{T,k} - \mathbf{A}_{S,T}\mathbf{H}_{S,k+1} - \mathbf{A}_{R,T}\mathbf{H}_R\right] \qquad (14)$$

where

$$\mathbf{h}_{T,k} = \left[\mathbf{D}_{T,k} - \mathbf{F}_{T,k}\right]\mathbf{q}_{T,k} - \mathbf{D}_{T,k}\left[\mathbf{P}^T\mathbf{q}_{C,k+1} - \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I\right]$$

Eq. (13) and Eq. (14) together can now be used for elimination of $\mathbf{q}_{T,k+1}$ and $\mathbf{H}_{I,k+1}$ from the second and third rows of Eq. (12). The resulting system of equations includes only the unknown heads of the supernodes $\mathbf{H}_{S,k+1}$ and the flows of the internal co-tree chords $\mathbf{q}_{C,k+1}$. With only a few algebraic manipulations we get:

$$\begin{bmatrix} \left[\mathbf{D}_C + \mathbf{P}\mathbf{D}_T\mathbf{P}^T\right] & \left[\mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T}\right] \\ \left[\mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T}\right]^T & \mathbf{0} \end{bmatrix}_k \begin{bmatrix} \mathbf{q}_C \\ \mathbf{H}_S \end{bmatrix}_{k+1} = \begin{bmatrix} \bar{\mathbf{E}}_C \\ -\bar{\mathbf{Q}}_S \end{bmatrix}_k \qquad (15)$$

with

$$\bar{\mathbf{E}}_{C,k} = \mathbf{P}[\mathbf{D}_T - \mathbf{F}_T]_k \mathbf{q}_{T,k} + \mathbf{P}\mathbf{D}_{T,k}\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I - \mathbf{P}\mathbf{A}_{R,T}\mathbf{H}_R + [\mathbf{D}_C - \mathbf{F}_C]_k \mathbf{q}_{C,k} - \mathbf{A}_{R,C}\mathbf{H}_R$$

$$\bar{\mathbf{Q}}_{S,k} = -\mathbf{A}_{S,T}^T \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I + \mathbf{Q}_S$$

Network equations of the topological subgraph G_S

In order to prove the analogy of the network equations of the graph G_S with the original equations the topological incidence matrices of G_S are derived by linear transformations based on the internal tree matrix given in Definition 1.

Observation 1:

a) Matrix $\mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T}$ is the incidence matrix of graph G_S consisting of superlinks (instead of the original links) and supernodes (a subset of the original nodes): $\mathbf{A}_S = \mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T}$

b) Matrix $\mathbf{A}_{R,C} + \mathbf{P}\mathbf{A}_{R,T}$ is the incidence matrix of the fixed head nodes of graph G_S consisting of superlinks (instead of the original links) and supernodes (a subset of the original fixed head nodes): $\mathbf{A}_{S,R} = \mathbf{A}_{R,C} + \mathbf{P}\mathbf{A}_{R,T}$.

c) Matrix $\mathbf{D}_{C,k} + \mathbf{P}\mathbf{D}_{T,k}\mathbf{P}^{\mathrm{T}}$ is a diagonal matrix and corresponds to the derivatives of the hydraulic head losses of the superlinks $\mathbf{D}_{S,k} = \mathbf{D}_{C,k} + \mathbf{P}\mathbf{D}_{T,k}\mathbf{P}^{\mathrm{T}}$.

Proof:

a)

$$\mathbf{A}_S = \mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T} = \mathbf{A}_{S,C} - \mathbf{A}_{I,C}\mathbf{A}_{I,T}^{-1}\mathbf{A}_{S,T}$$

Let us consider the rows of matrix $\mathbf{A}_{S,C}$ and distinguish between the following two cases when:

1.) The internal tree together with internal co-tree consist of more than one link:

For each internal tree together with its corresponding internal co-tree that consists of more than one link, the corresponding row of the matrix $\mathbf{A}_{S,C}$ contains exactly one entry different from zero. This entry belongs to the internal co-tree chord. The connection of the first supernode with the second supernode is accomplished by $\mathbf{P}\mathbf{A}_{S,T}$. The rows of matrix $\mathbf{P}$ correspond to superlinks, the columns correspond to the original links (co-tree branches only). For each internal tree branch *j* that is part of the superlink i the matrix entry P_{ij} is different from zero (-1 or +1). By definition, the mapping from internal tree branches to superlinks is unique. The sign of $P_{i,j}$ indicates the direction of the branch relative to the superlink. Using the same argument that was used above for showing that matrix $\mathbf{A}_{S,C}$ has exactly one element in row *i* that is different from zero it can be proven that the corresponding row in matrix $\mathbf{A}_{S,T}$ has also exactly one element different from zero and is therefore incident with the supernode of the internal tree. Pre-multiplication of the supernode - internal tree incidence matrix ($\mathbf{A}_{S,T}$) by the internal tree matrix (**P**) select the root node of the internal tree as the second node of the superlink.

2.) The superlink includes one link only:

For each superlink containing only one link (the internal co-tree chord), the corresponding rows of matrix $\mathbf{A}_{S,C}$ have two elements in the same row that are different from zero. The internal tree matrix is the zero matrix. A simplification is not possible and the superlink incidence matrix and the link incidence matrix are identical.

b) The proof of part b) is analoguous to that for part a).

c) The first term $\mathbf{D}_{C,k}$ includes the head loss derivatives of the internal co-tree chords. By calculation of the product of matrices $\mathbf{P}\mathbf{D}_{T,k}\mathbf{P}^{\mathrm{T}}$, the head loss derivatives of the internal tree branches are summed up. The first multiplication by **P** on the left replaces the -1 and +1 entries in the internal tree matrix by the head loss derivatives of the links. The second multiplication by the transpose of the internal tree matrix on the right of $\mathbf{D}_{T,k}$ computes the sum of the head loss derivatives for each internal tree. The result of the multiplication is again a diagonal matrix where each element on the main diagonal includes the sum of link head loss derivatives of the corresponding internal tree. It can be seen that the resulting matrix is diagonal due to the fact that the sets of internal tree branches are disjoint. That means that the multiplication delivers a value of zero, where the links are not in the same internal tree.

□

With the results of Observation 1 then Eq. (15) can be rewritten as:

$$\begin{bmatrix} \mathbf{D}_S & \mathbf{A}_S \\ \mathbf{A}_S^T & \mathbf{0} \end{bmatrix}_k \begin{bmatrix} \mathbf{q}_C \\ \mathbf{H}_S \end{bmatrix}_{k+1} = \begin{bmatrix} \bar{\mathbf{E}}_C \\ -\bar{\mathbf{Q}}_S \end{bmatrix}_k \tag{16}$$

Matrix $\mathbf{D}_{S,k}$ is a diagonal matrix and invertible if $D_{S_{i,i}} > 0 \quad \forall i = 1, \dots, m_S$. Consequently, Eq. (16) can be decomposed by using Schur's theorem analogously to the simplification of Eq. (5):

$$\mathbf{q}_{C,k+1} = \mathbf{D}_{S,k}^{-1}\left[\mathbf{P}[\mathbf{D}_T - \mathbf{F}_T]_k\mathbf{q}_{T,k} + [\mathbf{D}_C - \mathbf{F}_C]_k\mathbf{q}_{C,k} + \mathbf{P}\mathbf{D}_{T,k}\left[\mathbf{A}_{I,T}^T\right]^{-1}\mathbf{Q}_I - \mathbf{A}_S\mathbf{H}_{S,k+1} - \mathbf{A}_{S,R}\mathbf{H}_R\right] \quad (17)$$

With $\mathbf{q}_{T,k} = \mathbf{P}^T\mathbf{q}_{C,k} - \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I$ (Eq. (13)) it follows that:

$$\mathbf{q}_{C,k+1} = \mathbf{q}_{C,k} - \mathbf{D}_{S,k}^{-1}\left[\mathbf{F}_{C,k}\mathbf{q}_{C,k} + \mathbf{P}\mathbf{F}_{T,k}\mathbf{P}^T\mathbf{q}_{C,k} - \mathbf{P}\mathbf{F}_{T,k}\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I + \mathbf{A}_S\mathbf{H}_{S,k+1} + \mathbf{A}_{S,R}\mathbf{H}_R\right] \quad (18)$$

With the same arguments that were used for the elimination of $\mathbf{q}_{k+1}$ in Eq. (5) now Eq. (18) can be used for replacing the unknown flows $\mathbf{q}_{C,k+1}$ in the second row of Eq. (16):

$$\mathbf{A}_S^T\mathbf{D}_{S,k}^{-1}\mathbf{A}_S\mathbf{H}_{S,k+1} = \overline{\mathbf{Q}}_S + \mathbf{A}_S^T\mathbf{q}_{C,k} - \mathbf{A}_S^T\mathbf{D}_{S,k}^{-1}\left[\mathbf{F}_{S,k}\mathbf{q}_{C,k} - \mathbf{P}\mathbf{F}_{T,k}\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I + \mathbf{A}_{S,R}\mathbf{H}_R\right] \quad (19)$$

where $\mathbf{F}_{S,k} = \mathbf{F}_{C,k} + \mathbf{P}\mathbf{F}_{T,k}\mathbf{P}^T$ is analogous to Observation 1 c). By consideration of the fact that the term $\mathbf{F}_{S,k}\mathbf{q}_{C,k} - \mathbf{P}\mathbf{F}_{T,k}\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I$ is equal to the vector of head losses $\mathbf{h}_{S,k}$ of the superlinks at iteration *k*, Eq. (18) and Eq. (19) can be further simplified which finally leads to the main result of this paper. In other words, the results of the solution of the reduced set of equations of the topological graph G_S paper (Eq. (7) to Eq. (10)) are identical to the solution of the full set of equations for the core (Eq. (5)). This completes the proof.

Consideration of control devices

Real water supply systems typically include, in addition to the pipes of the network, a number of control devices and pumps for system operations. The devices are normally modelled as links and can be subdivided into devices with given local headloss functions (TCV: throttle control valves) or feedback control devices. The latter are used to control the flow and/or pressure by continuous adjustment of the hydraulic resistance. Flow control is used either for allowing only unidirectional flow (CHV) or maximum flow (FCV), whereas pressure control tries to maintain the downstream (PRV) or upstream (PSV) head at a given set value. In EPANET control device models are based on a set of heuristics that are described in the manual.

A simple way to tackle one-way devices in the GMPA approach is to set the initial and end nodes to supernodes by definition. Then, the superlink consists of one link for the device only. The commonly used heuristics for control devices can then be applied directly. However, the GMPA approach also allows for consideration of flow control within a superlink. The only additional requirement is that the control device link with the flow inequality condition is chosen as the internal tree chord. In this case the flow of the device link is identical with the superlink flow in the global system. Therefore, the linear inequality can be transferred to the global step as inequality condition for the superlink flow.

If there is more than one device with inequality flow constraints included in the superlink then three cases have to be considered locally. 1.) There are contradictions between the constraints. There is no solution to the original system or the reduced system. 2.) The constraints are redundant. One device can be arbitrarily chosen. 3.) There are constraints that can never be active. Those that can be activated have to be identified. Note that the flows of any link can be always expressed by superposition of the internal tree chord flow and a constant local flow:

$$\mathbf{q}_T = \mathbf{P}^T\mathbf{q}_C - \mathbf{A}_{I,T}^{-T}\mathbf{Q}_I$$

Therefore, the constraints (inequality or equality for gate valves) can be transferred from any link to the superlink flow.

Example: Assume that there is an inequality flow constraint representing a check valve in tree link *j* that belongs to superlink i, then: $\mathbf{q}_{T,i,j} \geq 0$, which is equivalent to $[\mathbf{P}^T]_{i,j}\mathbf{q}_{C,i} - \left[\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I\right]_{i,j} \geq 0$. $[\mathbf{P}^T]_{i,j}$ is +1 or -1 depending on the direction of flow in link *j* with respect to the direction of superlink i. The inequality condition for the superlink can now be written as $\mathbf{q}_{C,i} \geq [\mathbf{P}^T]_{i,j}\left[\mathbf{A}_{I,T}^{-T}\mathbf{Q}_I\right]_j$. That means that only the right hand side of the inequality has changed As a consequence, even if the control link changes within the superlink (for example by closing different gate valves) there is no need to re-calculate the decomposition.

EXAMPLE

The system shown in Figure 1 is now used to illustrate the GMPA method. After forest – core – partitioning the core network (shown in Figure 4) consists of one reservoir (*R*) and eight demand nodes (*a* through *h*). There exist two nodes *a* and *b* that connect more than two pipes (degree > 2). Therefore node *a* and node *b* are the supernodes of the network graph. The supernodes are connected by three superlinks. The pipe and node properties of the core are given in Table 1 and Table 2.

The first column in Table 1 refers to the pipe ID, L is the length in meters, D is the inner diameter of the pipe in millimeters, C denotes the Hazen-Williams coefficient and $\mathbf{q}_0$ is the initial guess of the flows. The first column in Table 2 shows the identifier of the node and Q is the demand. A negative sign means withdrawal. The last column shows the elevation of the node. With the parameters shown in Table 1 and Table 2 including the internal tree flows as initial flow distribution q_0, the flow iterates of the new GMPA algorithm are as shown in Table 4. Note that the last four rows (for links 8, 9 10 and 1) refer to the internal tree chords representing the superlinks. The flows of the upper six internal tree branches (links 2, 3, 4, 5, 6 and 7) are a linear function of the flows of the chords (Eq. (9)). The reordered and partitioned incidence matrix of the original network graph is in compliance with Eq. (11) and is:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{S,T} & \mathbf{A}_{I,T} \\ \mathbf{A}_{S,C} & \mathbf{A}_{I,C} \end{bmatrix} = \begin{array}{c} \text{Nodes}\;\; \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \left[\begin{array}{cc:cccccc} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ \hdashline 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \text{Links} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 1 \end{array}$$

The resulting topological matrices of graph G_S (taken from the 6 by 6 block of the upper right corner of the **A** matrix above and inverting the matrix analytically, Figure 5) are:

$$\mathbf{A}_{I,T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{P} = -\mathbf{A}_{I,C}\,\mathbf{A}_{I,T}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_S = \mathbf{A}_{S,C} + \mathbf{P}\mathbf{A}_{S,T} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{A}_{S,R} = \mathbf{A}_{R,C} + \mathbf{P}\mathbf{A}_{R,T} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\overline{\mathbf{Q}}_S = -\mathbf{A}_{S,T}^T\left(\mathbf{A}_{I,T}^T\right)^{-1}\mathbf{Q}_I + \mathbf{Q}_S = \begin{bmatrix} \mathrm{Q_c} + \mathrm{Q}_d + \mathrm{Q}_e + \mathrm{Q_f} + \mathrm{Q}_g + \mathrm{Q_h} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{Q_a} \\ \mathrm{Q_b} \end{bmatrix} = \begin{bmatrix} -340 \\ -20 \end{bmatrix} \quad \left[\frac{m^3}{h}\right]$$

The demands of the supernodes and the internal nodes are included in $\overline{\mathbf{Q}}_S$. The reallocation of the interior demand nodes is given by Eq. (6a) and depends on the choice of the internal tree chord. In the example, the last link (connected to node b) of every superlink is chosen as internal tree chord. As a consequence, the demands are all allocated to the upstream node, a. However, in general, the choice of another internal link as internal co-tree link can be useful, especially if control devices have to be considered. In such a case, by the application of Eq. (6a), parts of the interior node demands are allocated to the upstream supernode, and the other parts are allocated downstream.

The global/local factoring brings certain advantages when considering flow constraints associated with control devices For example, suppose that a check valve is placed on pipe 6 allowing flow only from node f to node g. Suppose also that a check valve is placed on pipe 7 allowing only flow from node h to node g. Pipe 6, for example, could be selected as the internal tree chord of the superlink that consists of link 5, 6, 7, 10. Then, the demand allocation is as follows:

$$\bar{\mathbf{Q}}_S = -\mathbf{A}_{S,T}^T\left(\mathbf{A}_{I,T}^T\right)^{-1}\mathbf{Q}_I + \mathbf{Q}_S = \begin{bmatrix} \mathrm{Q_c} + \mathrm{Q}_d + \mathrm{Q}_e + \mathrm{Q_f} \\ \mathrm{Q}_g + \mathrm{Q_h} \end{bmatrix} + \begin{bmatrix} \mathrm{Q_a} \\ \mathrm{Q_b} \end{bmatrix} = \begin{bmatrix} -190 \\ -170 \end{bmatrix} \quad \left[\frac{m^3}{h}\right]$$

When considering the constraints of the two check valves it is sufficient to consider the local subsystem of the superlink. Furthermore, the flows of the superlink can be expressed as a linear function of the internal tree chord flow q_6 and the internal tree flows (which result from the interior node demands):

$\mathrm{q}_5 = -Q_f + q_6$, $\mathrm{q}_6 = q_6$, $\mathrm{q}_7 = Q_g + q_6$, $\mathrm{q}_{10} = Q_g + Q_h + q_6$. The two check valves are modelled by the following linear inequalities: $\mathrm{q}_6 \geq 0$, $\mathrm{q}_7 \leq 0 \iff Q_g + q_6 \leq 0 \iff q_6 \leq -Q_g$. It follows that $q_6 \in \left[0; -Q_g\right]$. Note that the demand (withdrawal at node g) has a negative sign. Therefore $-Q_g$ is positive. This means that the two check valves in the original network (and the local system of the superlink) have been modelled, in a preliminary step, by lower and upper bound constraints for the flow of the superlink. The global solution can be calculated by any existing technique which models control devices (see, for example Rossman (2000)). The flows are checked in case they exceed the upper or lower bounds. If they do, the flow in the superlink is set to the corresponding boundary value (in the example this is link 6 and its flow would be set to 0 for the lower bound, or $-Q_g$ for the upper bound.)

One advantage of the GMPA is that the local subsystems can be checked independently. Infeasible conditions can be detected before the iteration starts. In the example, such an infeasible condition occurs if the check valve of link 6 is replaced by an FCV and the direction of the CHV at link 7 allows flow only from node g to node h. Thus, the conditions $q_6 \le q_{Set}, q_7 \ge 0 \iff Q_g + q_6 \ge 0 \iff q_6 \ge -Q_g$ apply. The constraint is now $-Q_g \le q_6 \le q_{Set}$. If the maximum flow q_{Set} for q_6 is smaller than the withdrawal at node g there is no feasible solution in demand driven analysis. In the state of the art package EPANET similar configurations often result in failure to converge or in unrealistic results because of the singularity of the Jacobian matrix. GMPA analysis of the local subsystems can detect infeasible sets of constraints *a priori*.

The reduction of the size of the problem by application of FCPA and GMPA is shown in Table 3. The size of the final system dealt with in the GMPA is only 8% of the original network.

CASE STUDIES

Nine case-study networks, with between 900 and 64,000 pipes, and eight of which were considered in Simpson et al. (2014), illustrate the significant reduction achieved by GMPA in the size of the nonlinear problem that must be solved. Table 5 shows, for these networks, the basic dimensions of the nonlinear part of the problem before the forest-core partitioning, after the forest-core partitioning, and after the graph matrix partitioning step of the GMPA: the resulting problems have dimension between 5% and 55%, with a mean 27% (a 73% reduction in dimension), of the core dimension that results from application of the forest-core partitioning.

The data for case study networks N_1, N_3, N_4 and N_7, which are slight modifications of networks in the public domain, are available as Supplemental Data Files. The other four networks considered in this paper are not available because of security concerns.

CONCLUSIONS

The Graph Matrix Partitioning Algorithm (GMPA) developed in this paper, when applied to networks that have many core pipes in series, can save a significant computational effort by reducing the size of the problem for the most time consuming part of the calculation. The FCPA step, applied before the main GMPA step, separates the forest from the core, thereby reducing the dimension of the non-linear part of the problem when a forest is present. The GMPA step, the separation or partitioning of the topological subgraph from the core, further separates linear and nonlinear parts of the problem and further reduces the dimension of the non-linear part of the problem when there are pipes in series within the core. These two steps, which separate out the linear and non-linear parts of the problem and then deal with them using appropriate linear and non-linear methods, save much time that is otherwise wasted applying non-linear techniques to problems which have significant linear components.

The permuted system is decomposed into a global solution for the maximal topological subgraph G_S and a local solution for the internal trees with nodes of degree two. The iterative solution of the smaller system of equations of G_S in combination with the linear local steps gives identical results to those obtained from the standard solution to the full network system. *There are no approximations*. The equivalence of the GMPA and the original set of equations has been proven.

Real world applications have shown reductions from 5,000 unknown in the system of equations to 150 or even 230,000 unknowns to 10,000 unknowns. On average, the number of unknowns in the nonlinear (core) part of the problem for the systems tested was less than 20% of the original size. The case studies reported in Table 5 show that GPMA achieves a reduction to between 5% and 55% of the core dimension that results from application of the FCPA. In the GMPA method, all the important properties of the system matrix of the original system are retained (sparse, symmetric, Stieltjes matrix) and the same efficient factorization methods can be applied to the reduced system.

Last, but not least, the linear mapping between topological subgraph and original graph can be used for other applications. The aggregation methods for simplification of large networks can be replaced by an adaptive modelling based on the topological subgraph. As a consequence, the process of updating existing models from GIS is much easier since the one to one mapping between GIS features and network model features is always valid and not lost by aggregation.

The focus of this paper is on the derivation of, and the theoretical basis for, the GMPA method. The efficiency of a production implementation of method depends on many factors, including, for example, the architecture of the computing platform and the coding language. The global and local steps can be handled separately: by separate program codes or even on separate architectures. In the case where a coarse grid approximation to the solution is all that is required, the user can simply solve the global step: the method can be used in the same way as common aggregation techniques that calculate an approximate solution Where the complete solution is required the local and global steps of the GMPA can be implemented by different codes or on even on different architectures. Where separate architectures are used for the global and local steps the data exchange required is minimal, a significant advantage of the GMPA. Thus, an existing nonlinear solver code can be used by adding to it a separate package to handle the local step in the algorithm. The GMPA is also well suited to handle the inclusion of control devices and parts of the method are well suited to parallelization.

Further research in the application of GMPA to the solution of pressure dependent models would be a useful contribution to the field. Another contribution to the field would be a comprehensive study of just how large would be the time savings available by use of the GMPA. Study of these issues is well beyond the scope of this paper.

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APPENDIX A: Calculation of permutation matrices

The system in Eq. (12), in which the matrix, **A**, on the left has its rows and columns permuted in the required order, as shown in Eq. (11), and the matrix, **D**, is diagonal, can be obtained from the original system (Eq. (5)**Error! Reference source not found.**) as follows. Suppose the original system is

$$\begin{bmatrix} \hat{\mathbf{D}} & \hat{\mathbf{A}} \\ \hat{\mathbf{A}}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\hat{\mathbf{q}} \\ \Delta\hat{\mathbf{H}} \end{bmatrix} = \begin{bmatrix} d\hat{\mathbf{E}} \\ d\hat{\mathbf{Q}} \end{bmatrix} \tag{A.1}$$

and that the $m \times n$ incidence matrix $\hat{\mathbf{A}}$ (m: total number of links, n: total number of nodes) has its rows and columns in an order different from that which is required. Suppose $\mathbf{R} \in \mathbb{R}^{m\times m}$ and $\mathbf{S} \in \mathbb{R}^{n\times n}$ are (orthogonal) permutation matrices which are such that $\mathbf{R}\hat{\mathbf{A}}\mathbf{S}$ has its rows and columns in the required order. Then, multiplying (A.1) on the left by

$$\begin{bmatrix} \mathbf{R} & \\ & \mathbf{S}^T \end{bmatrix}$$

and inserting the product

$$\begin{bmatrix} \mathbf{R}^T & \\ & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \\ & \mathbf{S}^T \end{bmatrix} = \mathbf{I}$$

between the matrix and vector on the left gives

$$\begin{bmatrix} \mathbf{R} & \\ & \mathbf{S}^T \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{D}} & \widehat{\mathbf{A}} \\ \widehat{\mathbf{A}}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & \\ & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \\ & \boldsymbol{S}^T \end{bmatrix} \begin{bmatrix} \Delta\widehat{\mathbf{q}} \\ \Delta\widehat{\mathbf{H}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \\ & \boldsymbol{S}^T \end{bmatrix} \begin{bmatrix} d\widehat{\mathbf{E}} \\ d\widehat{\mathbf{Q}} \end{bmatrix}, \qquad \text{(A.2)}$$

Denoting

$\mathbf{A} = \mathbf{R}\widehat{\mathbf{A}}\mathbf{S}, \mathbf{D} = \mathbf{R}\widehat{\mathbf{D}}\mathbf{R}^T, \Delta\mathbf{q} = \mathbf{R}\Delta\widehat{\mathbf{q}}, \Delta\mathbf{H} = \mathbf{S}^T\Delta\widehat{\mathbf{H}}, \boldsymbol{a}_1 = \mathbf{R}\widehat{\boldsymbol{a}}_1, \boldsymbol{a}_2 = \mathbf{S}^T\widehat{\boldsymbol{a}}_2$

means that (A.1) can be rewritten as

$$\begin{bmatrix} \mathbf{D} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q} \\ \Delta\mathbf{H} \end{bmatrix} = \begin{bmatrix} d\mathbf{E} \\ d\mathbf{Q} \end{bmatrix}, \qquad \text{(A.3)}$$

the system in Eq. (12) with consideration of Eq. (11) in which **A** has its rows and columns permuted as required. By multiplication of the incidence matrix with the matrix **S** the columns that belong to supernodes are separated from the columns that belong to inner nodes. Matrix **S** can be easily determined by traversing the network nodes and selecting first all the n_S nodes that have degree > 2 followed by the rest of the nodes.

Multiplication by matrix **R** separates the internal tree branches of the superlinks from the internal co-tree chords.

REFERENCES

Branin, F. Jr. (1963):"The Inverse of the Incidence Matrix of a Tree and the Formulation of the Algebraic-First-Order Differential Equations of an RLC Network", *IEEE Transac. on Circuit Theory*, Vol. 10, No. 4, pp. 543 – 544. doi: 10.1109/TCT.1963.1082204

Birkhoff,, G. and Diaz, J.-B. (1956): "Non-linear network problems", *Quarterly of Applied Mathematics*, Vol. 13, No. 4, pp. 431 – 443.

Birkhoff, G. (1962): "A variational principle for nonlinear networks", *Quarterly of Applied Mathematics*, Vol. 21, No. 2, pp. 160 – 162.

Carpentier, P. and Cohen, G. (1993): "Applied Mathematics in Water Supply Network Management", *Automatica*, Vol. 29, No. 5, pp. 1215 – 1250. doi: 10.1016/0005-1098(93)90048-X

Cherry, C. (1951): "Some general theorems for non-linear systems possessing reactance," *The Philosophical Magazine*, Vol. 42, No. 7, pp. 1161 – 1177.

Collins, M.; Cooper, L.; Helgarson, R.; Kennington, J. L. and LeBlanc, L. (1978): "Solving the pipe network analysis problem using optimization techniques", *Management Science*, Vol. 24, No. 7, pp. 747-760. doi: 10.1287/mnsc.24.7.747

Cross, H. (1936): "Analysis of flow in networks of conduits or conductors", *Bulletin No. 286*, Engineering Experiment Station, Univ. of Illinois, Urbana, IL.

Deuerlein, J. W. (2008): "Decomposition model of a general water supply network graph", *Journal of Hydraulic. Engineering*, Vol. 134, No. 6, pp. 822 - 832. doi: 0.1061/(ASCE)0733-9429(2008)134:6(822)

Deuerlein, J. W.; Simpson, A. R. and Dempe, S. (2009): "Modeling the Behavior of Flow Regulating Devices in Water Distribution Systems Using Constrained Nonlinear Programming", *Journal of Hydraulic Engineering*., Vol. 135, No. 11, pp. 970-982. doi: 0.1061/(ASCE)HY.1943-7900.0000108

Diestel, R. (2010): "Graph theory," 4th Edition, GTM 173, Springer Verlag, Heidelberg, New York.

Epp, R. and Fowler, A. (1970): "Efficient code for steady-state flows in networks", *Journal of the Hydraulics Division*, Vol. 96, No. HY1, pp. 43-56.

Elhay, S., Simpson, A., Deuerlein, J., Alexander, B., and Schilders, W. (2014). ”Reformulated Co-Tree Flows Method Competitive with the Global Gradient Algorithm for Solving Water Distribution System Equations.” J. Water Resour. Plann. Manage., 140(12), 04014040.

Giustolisi, O. (2010): “Considering Actual Pipe Connections in Water Distribution Network Analysis”, *Journal of Hydraulic Engineering*, Vol. 136, No. 11, 2010, pp. 889 – 900. doi: 10.1061/(ASCE)HY.1943-7900.0000266

Giustolisi, O.; Laucelli, D.; Berardi, L. and Savic, D. (2012) “Computationally Efficient Modeling Method for Large Water Network Analysis”, *Journal of Hydraulic Engineering*, Vol. 138, No. 4, pp. 313 – 326. doi: 0.1061/(ASCE)HY.1943-7900.0000517

Gupta, R. and Prasad, T. D. (2000): “Extended use of linear graph theory for analysis of pipe networks”, *Journal of Hydraulic Engineering*, Vol. 126, No. 1, pp. 56–62. doi: 10.1061/(ASCE)0733-9429(2000)126:1(56)

Kesavan, H. K.; Chandrashekar, M. (1972): “Graph-theoretic models for pipe network analysis”, *Journal of the Hydraulics Division*, Vol. 98, No. HY2, 1972, pp. 345–363.

Millar, W. (1951): “Some general theorems for non-linear systems possessing resistance”, *The Philosophical Magazine*, Vol. 42, No. 7, pp. 1150 – 1160.

Nielsen, H. B. (1989): “Methods for analyzing pipe networks”, *Journal of Hydraulic Engineering*, Vol. 115, No. 2, 1989, pp. 139-157. doi: 10.1061/(ASCE)0733-9429(1989)115:2(139)

Piller, O. (1995). "Modeling the behavior of a network - Hydraulic analysis and sampling procedures for parameter estimation," Applied Mathematics thesis from the University of Bordeaux, 288 pages, Talence, France.

Rossman, L. A. (2000): “EPANET 2 Users manual”, EPA/600/R-00/057, Environmental Protection Agency, Cincinnati, Ohio, USA.

Schilders, W. (2009): “Solution of indefinite linear systems using an LQ decomposition for the linear constraints”, *Linear Algebra and its Applications*, Vol. 431, No. 3 – 4, pp. 381 – 395. doi: 10.1016/j.laa.2009.02.036

Simpson, A. R. and Elhay, S. (2011): “The Jacobian for solving water distribution system equations with the Darcy-Weisbach head loss model,” *Journal of Hydraulic Engineering* Vol. 137, No. 6, pp. 696-700. doi: 10.1061/(ASCE)HY.1943-7900.0000341

Simpson, A. R., Elhay, S., and Alexander, B. (2014). "A Forest-Core Partitioning Algorithm for Speeding up the Analysis of Water Distribution Systems." *J. Water Resour. Plann. Manage.*, 140(4), 435–443. doi: 10.1061/(ASCE)WR.1943-5452.0000336

Todini, E. and Rossman, L. A. (2013): “Unified Framework for Deriving Simultaneous Equation Algorithms for Water Distribution Networks,” *Journal of Hydraulic Engineering*, Vol. 139, No. 5, pp. 511 – 526. doi: 10.1061/(ASCE)HY.1943-7900.0000703

Todini, E. and Pilati, S. (1988): “A gradient algorithm for the analysis of pipe networks,” *Computer Applications in Water Supply*, Research Studies Press, Letchworth, Hertfordshire, UK.

Table 1: Pipe properties of core of example network (after application of the FCPA)

Pipes	L [m]	D [mm]	C [-]	q_0 [m^3/h]
1	1000	300	100	0
2	1000	200	100	70
3	1500	150	100	40
4	1200	200	100	50
5	800	200	100	210
6	800	150	100	150
7	800	150	100	80
8	1200	100	100	0
9	1000	100	100	0
10	800	100	100	0

Table 2: Node properties of core of example network (after application of the FCPA)

Nodes	$Q[m^3/h]$	H_0 [m]
a	-10	1
b	-20	1
c	-30	1
d	-40	1
e	-50	1
f	-60	1
g	-70	1
h	-80	1
R	360	150

Table 3: Reduction in size of the nonlinear problem by application of the GMPA method

Step	number of nodes	number of links	reduction of number of nodes [%]
Original network Fig. 1	38	39	-
After forest-core partitioning. Fig. 2	9	10	76
After topological partitioning (GMPA). Fig 3	3	4	92

Table 4: Flow iterates of the GMPA (m^3/s)

k =	0	1	2	3	4	5
2	0.01944	0.01859	0.02382	0.02508	0.02519	0.02519
3	0.01111	0.01026	0.01548	0.01675	0.01685	0.01685
4	0.01389	0.03868	0.02632	0.02286	0.02251	0.02250
5	0.05833	0.03995	0.04709	0.04928	0.04953	0.04953
6	0.04167	0.02328	0.03042	0.03262	0.03286	0.03286
7	0.02222	0.00384	0.01098	0.01317	0.01342	0.01342
8	0.00000	0.02479	0.01243	0.00897	0.00862	0.00861
9	0.00000	-0.00086	0.00437	0.00563	0.00574	0.00574
10	0.00000	-0.01838	-0.01124	-0.00905	-0.00880	-0.00880
1	0.00000	0.10000	0.10000	0.10000	0.10000	0.10000

Table 5: Network dimensions before forest partitioning (m pipes, n nodes), after forest partitioning ($\tilde{m}$ pipes, $\tilde{n}$ nodes), and after graph matrix partitioning ($\hat{m}$ pipes, $\hat{n}$ nodes) together with partitioned dimensions as percentages

DI	m	n	$\tilde{m}$	$\tilde{n}$	$\hat{m}$	$\hat{n}$	$\frac{\tilde{n}}{n}\%$	$\frac{\hat{n}}{\tilde{n}}\%$	$\frac{\hat{n}}{n}\%$
N1	934	848	573	487	246	160	57	33	19
N2	1118	1039	797	718	235	156	69	22	15
N3	1976	1770	1153	947	547	341	54	36	19
N4	2465	1890	2036	1461	1379	804	77	55	43
N5	2508	2443	1806	1741	187	122	71	7	5
N6	8584	8392	6734	6542	542	350	78	5	4
N7	14830	12523	11898	9591	6405	4098	77	43	33
N8	19647	17971	15233	13557	4878	3202	75	24	18
N9	63829	59157	51845	47173	14228	9556	79	20	16
						Means	71%	27%	19%

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Captions of Figures:

Figure 1: Example network graph.

Figure 2: The core of the example network graph (after forest-core partitioning)

Figure 3: Links and nodes of a superlink.

Figure 4: Labelled core of the example network graph.

Figure 5: Topological minor (topological subgraph) of the core of the example

Figure 1
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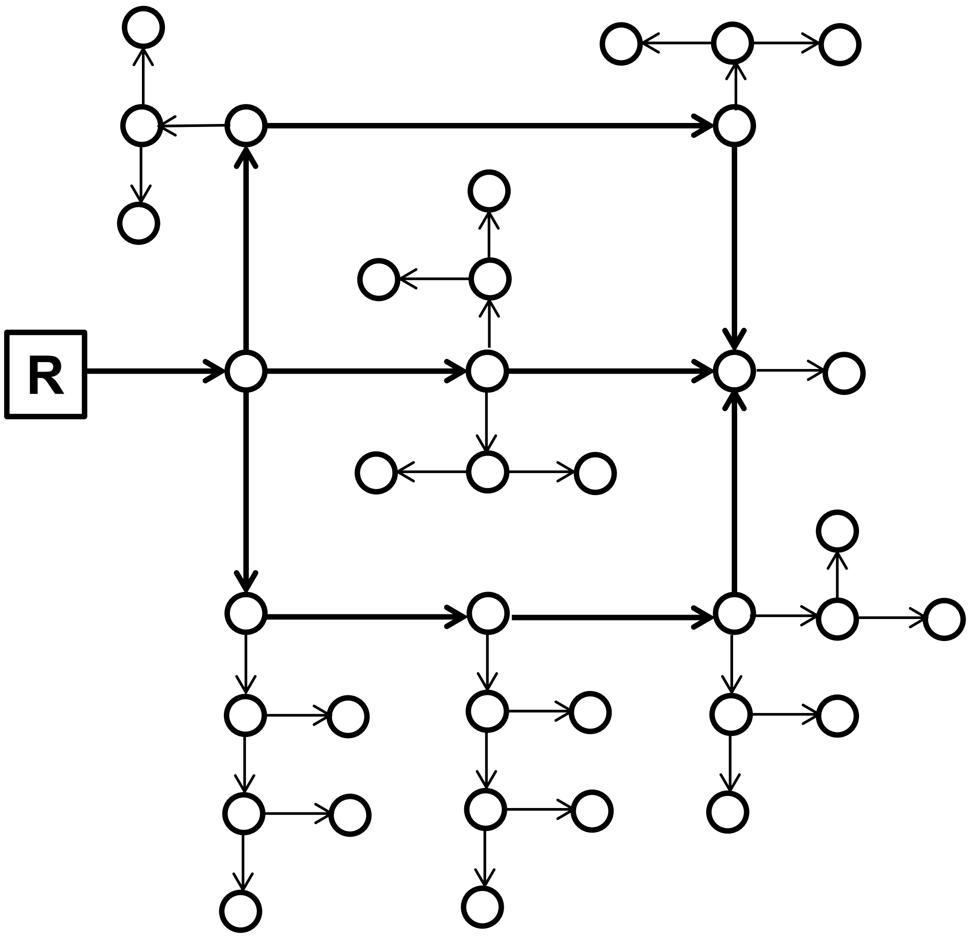

Figure 2
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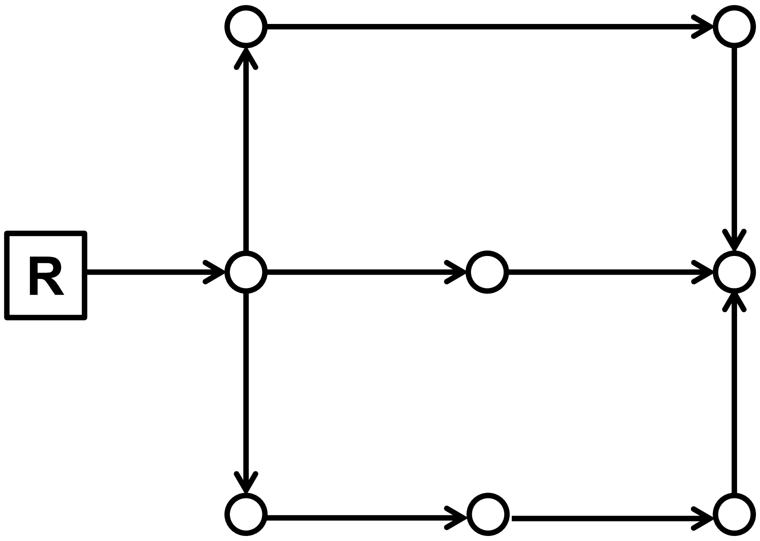

Figure 3
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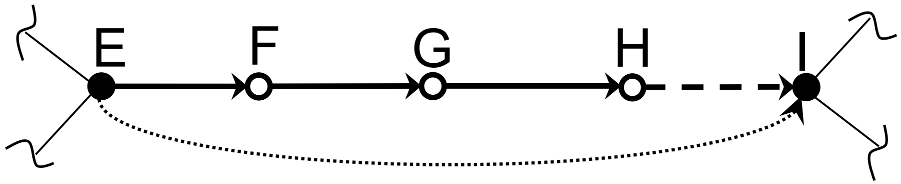

1.) ● supernodes (E, I)
2.) ○ internal tree nodes (F, G, H)
3.) ⟶ internal tree branches (EF, FG, GH) (E to H is the internal tree)
4.) - → internal co-tree chord (HI)
5.) ····→ superlink (EI)

Figure 4
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