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IEEE Transactions on Fuzzy Systems, 2014; 22(5):1113-1126

DOI: http://dx.doi.org/10.1109/TFUZZ.2013.2282167

Fuzzy $n-$ellipsoid Numbers and Representations of Uncertain Multichannel Digital Information

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***Abstract*—In this paper, we present the definition of fuzzy $n-$ellipsoid numbers which are a special kind $n-$dimensional fuzzy numbers, not only are more objective and more rational in expressing uncertain multichannel digital information than fuzzy $n-$cell numbers but also keep the convenience being used in applications and researches of theory, and obtain two representation results. Then, for the sake of the application of fuzzy $n-$ellipsoid numbers, we define some special kind of fuzzy $n-$ellipsoid numbers, investigate their properties, set up a specific iterative algorithm of their membership function value, and prove the convergence of the iterative algorithm. And then we establish an algorithmic version of constructing fuzzy $n-$ellipsoid numbers to express a object which is characterized by a group of uncertain multichannel digital information, and also give practical examples to show the application and rationality of the proposed techniques.**

***Index Terms*—Fuzzy numbers, fuzzy $n-$cell numbers, fuzzy $n-$ellipsoid numbers, expression of uncertain multichannel information, membership function value.**

I. INTRODUCTION

THE concept of fuzzy numbers was introduced by Chang and Zadeh [6] in 1972 with the consideration of the properties of probability functions. Since then both the numbers and the problems in relation to them have been widely studied, see for example, [2], [15], [17], [19], [25], [27] and the references therein. In addition, Dubois and Prade studied operations of $1-$dimensional fuzzy numbers in [9], and discussed differential calculus of fuzzy number valued mappings in [10]; Goetschel and Voxman also researched elementary calculus for $1-$dimensional fuzzy numbers in [11]; Diamond and Kloeden very more systematically studied fuzzy number space in [8]. With the development of theories and applications of fuzzy numbers, this concept becomes more and more important, see for example, [7], [16], [21], [22], [26], [23], [35]. In addition, Butnariu studied methods of solving optimization problems and linear equations in the space of fuzzy vectors in [4]; Campin, Candeal and Indurin using fuzzy numbers to represented binary relations in [5]; Huang and Wu studied the problem of approximation of fuzzy number valued functions by regular fuzzy neural networks in [13]; Adam and Pawel discussed a class of sequencing problems with uncertain parameters which is modeled by the usage of fuzzy intervals in [1]; Hosseini, Qanadli, Barman, Mazinani, Ellis and Dehmeshki presented an automatic approach to learn and tune Gaussian interval type-2 membership functions in [12]; Pagola, Lopez-Molina, Fernandez, Barrenechea and Bustince constructed an interval type-2 fuzzy set with different fuzzy sets such that the length of the (membership) interval represents the uncertainty of the expert with respect to the choice of the membership function in [20].

It is well known that in a precise or certain environment, multi-channel digital signals can be represented by elements of multi-dimensional Euclidean space, i.e., crisp multi-dimensional vectors. If however we wish to study multi channel digital signals in an imprecise environment, then the signals themselves are imprecise, and it becomes unwise to use crisp multidimensional vectors to represent them. We know that $n-$dimension fuzzy number is a good mean to express uncertain multichannel digital information. However, for general n-dimension fuzzy numbers, due to their structural complexity, they can not be used conveniently in some fields of applications and some researches of theory [28], [33]. In [3], Bandemer and Näther, from n 1-dimension fuzzy numbers $u_1, u_2, \cdots, u_n$, defined a special kind $n-$dimensional fuzzy numbers whose membership function value $u(x)$ at $x = (x_1, x_2, \cdots, x_n)$ is defined by $u(x) = \min_{i=1,2\cdots,n} u_i(x_i)$, and Inuiguchi, Ramík and Tanino called it vector of non-interactive 1-dimension

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This work was partially supported partially by the Nature Science Foundation of China (61273077), the Nature Science Foundation of Zhejiang Province, China (LY12A01001), the National Key Basic Research Program, China (2012CB215202), and 111 Project (B12018).

fuzzy numbers in [14]. In 2002, we also carefully studied a special type of $n-$dimensional fuzzy numbers whose cut (or level) sets are all hyperrectangles and called them fuzzy $n-$cell numbers in [33]. From the both definitions, it is obvious that vectors of n non-interactive 1-dimension fuzzy numbers are all fuzzy $n-$cell numbers. On the other hand, in [28] we showed that fuzzy $n-$cell numbers and $n-$dimensional fuzzy vectors (the Cartesian products of n 1$-$dimensional fuzzy numbers, them are called vectors of fuzzy quantities by Mare$\check{s}$ in [18]) can be represented each other, and obtained the representations of joint membership function and the edge membership functions of a fuzzy $n-$cell number by each other, i.e., the relational expressions of joint membership function and the edge membership functions (Theorem 3.2 [28]). Under the relational expressions, fuzzy $n-$cell numbers can be also regarded as vectors of n non-interactive 1-dimension fuzzy numbers. In [29], [30], we studied the problem of using fuzzy $n-$cell numbers to represent imprecise or uncertain multi-channel digital signals, and established constructing methods of such fuzzy $n-$ell numbers. It has been demonstrated that the fuzzy $n-$cell number is used much more conveniently than general $n-$dimensional fuzzy numbers in theoretical investigations in [28], [33], [34] and in some fields of application in [29], [30], [31], [32].

But using fuzzy $n-$cell number to express uncertain multichannel digital information has some defects or weaknesses. This can be seen from its constructer or the relationship formula $u(x_1, x_2, \cdots, x_n) = \min\{u_1(x_1), u_2(x_2), \cdots, u_n(x_n)\}$ (see [28]), it means that the membership degree only depends on the minimal of the membership degrees of its all components (1-D fuzzy numbers), i.e., almost, only some one factor in the all n factors of the repressed object decide the membership degree. This is obviously not rational. For example, for fuzzy 2$-$cell number $u = (u_1, u_2)$, where u_1, u_2 are triangular model fuzzy numbers (see [30]) $u_1 = (0, 1, 2)$, $u_2 = (1, 2, 3)$, we have $u(0.5, x_2) = u(0.5, 1.5) = 0.5$ for any $x_2 \in [1.5, 2.5]$. Generally speaking, this is not rational since $u(0.5, x_2)$ should be bigger than $u(0.5, 1.5)$ for any x_2 with $1.5 < x_2 < 2.5$. In addition, this can be also seen from its application (see Example 2 and 4).

Therefore, we need to find another special kind $n-$dimensional fuzzy numbers which can overcome the defects and weaknesses, and keep the convenience being used in applications and researches of theory. In [14], Inuiguchi, Ramík and Tanino proposed oblique fuzzy vectors as a special model of interactive fuzzy numbers and show that they are tractable to a certain extent especially in calculations of fuzzy linear functions. In [3], Bandemer and Näther introduced a special kind $n-$dimensional fuzzy numbers—-bean fuzzy vectors, which are defined by $u(x) = \max\{1-(x-x^0)^T B(x-x^0), 0\}$, where B is a positive definite $n \times n$-matrix, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in R$. In [24], M. Sato and Y. Sato used the $n-$dimensional fuzzy numbers whose membership functions are all symmetrical conical to cluster fuzzy data. In this paper, from the perspective of fuzzy subsets of R^n (but not n interactive 1-dimensional fuzzy numbers), we define a special kind $n-$dimensional fuzzy numbers—-fuzzy $n-$ellipsoid numbers which are not same with Inuiguchi's interactive fuzzy numbers, and whose scope is very more wider than bean fuzzy vectors and the fuzzy numbers of conical membership functions. Such special type of $n-$dimensional fuzzy numbers not only are more objective and more rational in expressing uncertain multichannel digital information than fuzzy $n-$cell numbers but also keep the convenience being used in applications and researches of theory. The next, we investigate the properties of fuzzy $n-$cell numbers, and obtain two representation theorems of them. Then, for the sake of the application of fuzzy $n-$ellipsoid numbers, we also define some special kind of fuzzy $n-$ellipsoid numbers, investigate their properties, set up a specific iterative algorithm of their membership function value, and prove the convergence of the iterative algorithm. And then we establish an algorithmic version of constructing fuzzy $n-$ellipsoid numbers to express a object which is characterized by a group of uncertain multichannel digital information, and also give practical examples to show the application and rationality of the proposed techniques.

II. Basic definitions and notations

Let n be a natural number. A fuzzy subset (in short, a fuzzy set) of R^n (the $n-$dimensional Euclidean space) is a function $u: R^n \to [0, 1]$. For each such fuzzy set u, we denote by $[u]^r = \{x \in R^n : u(x) \geq r\}$ for any $r \in (0, 1]$, its $r-$level set. By supp$u$ we denote the support of u, i.e., the $\{x \in R^n : u(x) > 0\}$. By $[u]^0$ we denote the closure of the supp$u$, i.e., $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$.

If u is a normal and fuzzy convex fuzzy set of R^n, $u(x)$ is upper semi-continuous, $[u]^0$ is compact, then we call u a $n-$dimensional fuzzy number, and denote the collection of all $n-$dimensional fuzzy numbers by E^n.

It is known that if $u \in E^n$, then for each $r \in [0, 1]$, $[u]^r$ is a compact set in R^n.

If $u \in E^n$, and for each $r \in [0, 1]$, $[u]^r$ is a hyper rectangle, i.e., $[u]^r$ can be represented by $\prod_{i=1}^n [\underline{u_i}(r), \overline{u_i}(r)]$, where $\underline{u_i}(r), \overline{u_i}(r) \in R$ with $\underline{u_i}(r) \leq \overline{u_i}(r) \in R$ when $r \in [0, 1]$, $i = 1, 2, \ldots, n$, then we call u a fuzzy $n-$cell number. And we denote the collection of all fuzzy $n-$cell numbers by $L(E^n)$.

Let $u_i \in E\ (= E^1)$, $i = 1, 2, \ldots, n$. We call the ordered 1-dimensional fuzzy number class $u_1, u_2, \ldots, u_n$ (i.e., the Cartesian product of one-dimensional fuzzy numbers $u_1, u_2, \ldots, u_n$) a $n-$dimensional fuzzy vector, denote it as $(u_1, u_2, \ldots, u_n)$, and call the collection of all $n-$ dimensional fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}^{n}$) $n-$dimensional fuzzy vector space, and denote it as $(E)^n$.

By Theorem 3.1 in [28], we know that fuzzy $n-$cell numbers and $n-$dimensional fuzzy vectors can represent each other, and the representation is unique, so $L(E^n)$ and $(E)^n$ may be regarded as identity.

For any $a \in R^n$, define an $n-$dimensional fuzzy number $\hat{a}$ by $\hat{a}(x) = \begin{cases} 1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \neq a \end{cases}$ for any $x \in R^n$.

The addition, multiplication and scalar product on space $K(R^n)$ (the collection of non-empty compact subsets of R^n) are defined by $A+B = \{a+b \mid a \in A, b \in B\}$, $AB = \{ab \mid a \in A, b \in B\}$ and $\lambda A = \{\lambda a \mid a \in A\}$ for any $A, B \in K(R^n)$, $\lambda \in R$.

The addition, multiplication and scalar product on E^n are defined by

$$(u+v)(x) = \sup_{y+z=x} \min(u(y), v(z))$$

$$(uv)(x) = \sup_{yz=x} \min(u(y), v(z))$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if} \quad \lambda \neq 0 \\ \hat{0}(x) & \text{if} \quad \lambda = 0 \end{cases}$$

for $u, v \in E^n$ and $\lambda \in R$.

It is well known ([8]) that for $u, v \in E^n$ and $k \in R$, $[u+v]^r = [u]^r + [v]^r$, $[uv]^r = [u]^r[v]^r$ and $[ku]^r = k[u]^r$ for any $r \in [0, 1]$.

III. Fuzzy $n-$ellipsoid numbers

Let $a_i, b_i \in R$ with $a_i \leq b_i$, $i = 1, 2, \cdots, n$. We denote

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} = \{(x_1, \cdots, x_n) \in R^n | \sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2} \leq 1\} \quad (1)$$

i.e., the closed $n-$ellipsoid with the center $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \cdots, \frac{a_n+b_n}{2})$ and the coordinates of the vertexes on coordinate axis x_1, x_2, $\cdots$, x_n are in turn a_1, b_1, a_2, b_2, $\cdots$, a_n, b_n.

Remark 1: In the Equation (1), we stipulate that $a_i = b_i$ for some $i = 1, 2, \cdots, n$ can be allowed, and if $a_{i_1} = b_{i_1}$, $a_{i_2} = b_{i_2}$, $\cdots$, $a_{i_k} = b_{i_k}$ $(k \leq n)$, then $\sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2} \leq 1$ indicates $\sum_{a_i \neq b_i, 1 \leq i \leq n} \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2} \leq 1$ and $x_{i_1} = a_{i_1} = b_{i_1}$, $x_{i_2} = a_{i_2} = b_{i_2}$, $\cdots$, $x_{i_k} = a_{i_k} = b_{i_k}$. For example, as $a_1 = b_1$ and $a_3 = b_3$, the inequality $\sum_{i=1}^4 \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2} \leq 1$ indicates $\frac{(x_2 - \frac{b_2+a_2}{2})^2}{(\frac{b_2-a_2}{2})^2} + \frac{(x_4 - \frac{b_4+a_4}{2})^2}{(\frac{b_4-a_4}{2})^2} \leq 1$ and $x_1 = a_1 = b_1$ and $x_3 = a_3 = b_3$.

Definition 1: If $u \in E^n$, and for each $r \in [0, 1]$, $[u]^r$ is a closed $n-$ellipsoid, i.e., exist $\underline{u_i}(r), \overline{u_i}(r) \in R$ with $\underline{u_i}(r) \leq \overline{u_i}(r)$, $\forall r \in [0, 1]$, $i = 1, 2, \ldots, n$ such that

$$[u]^r = \begin{pmatrix} \underline{u_1}(r) & \overline{u_1}(r) \\ \underline{u_2}(r) & \overline{u_2}(r) \\ \vdots & \vdots \\ \underline{u_n}(r) & \overline{u_n}(r) \end{pmatrix} \quad (2)$$

then we call u a fuzzy $n-$ellipsoid number. And we denote the collection of all fuzzy $n-$ellipsoid numbers by $E(E^n)$.

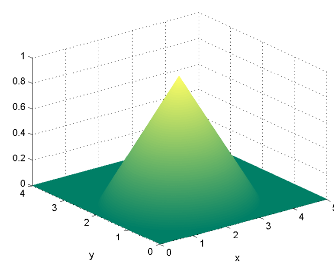

Fig. 1. Fuzzy $2-$ellipsoid number u $(a = 0, b = 4, c = 0, d = 3)$

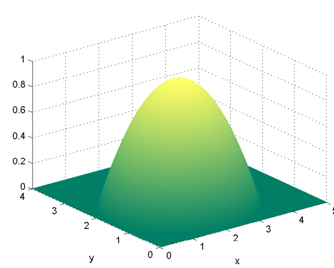

Fig. 2. Fuzzy $2-$ellipsoid number v ($a = 0, b = 4, c = 0, d = 3$)

Example 1. (see Figs. 1 and 2). The fuzzy subset of R^2, $u : R^2 \to [0, 1]$ defined by

$$u(x,y) = \begin{cases} 1 - (\frac{(x - \frac{b+a}{2})^2}{(\frac{b-a}{2})^2} + \frac{(y - \frac{d+c}{2})^2}{(\frac{d-c}{2})^2})^{\frac{1}{2}}, & (x, y) \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ 0, & \text{otherwise} \end{cases}$$

i.e., the value $u(x, y)$ is decided by the side face of the elliptic cone in Fig.1 as $(x, y) \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{(x, y) \in R^2 | \frac{(x - \frac{b+a}{2})^2}{(\frac{b-a}{2})^2} + \frac{(y - \frac{d+c}{2})^2}{(\frac{d-c}{2})^2} \leq 1\}$, and by 0 as $(x, y) \bar{\in} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is a fuzzy $2-$ellipsoid number (see Theorem 3).

The fuzzy subset of R^2, $v : R^2 \to [0, 1]$ defined by

$$v(x,y) = \begin{cases} 1 - \frac{(x - \frac{b+a}{2})^2}{(\frac{b-a}{2})^2} - \frac{(y - \frac{d+c}{2})^2}{(\frac{d-c}{2})^2}, & (x, y) \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ 0, & \text{otherwise} \end{cases}$$

i.e., the value $v(x,y)$ is decided by the values decided by the side face of the elliptic paraboloid in Fig.2 as $(x,y) \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and by 0 as $(x,y) \bar{\in} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is also a fuzzy 2−ellipsoid number (see Theorem 4).

Example 2. In the Example 3.2 of [33], in an uncertain or imprecise environment, we recommended using a fuzzy 2−cell number to express the working state (efficiency) of one person. For example, if one person's producing speed and quali1cation rate at t time are, respectively, about 100 and $0.95(=95\%)$ (i.e., two estimated quantities that are in near 100 and 0.95, respectively), then the fuzzy 2−cell number $u(t)$ (see Fig 3) defined as

$$u(t)(x,y) = \begin{cases} 20y-18, & 0.9 \le y \le 0.95, 200y-90 \le x \le 290-200y \\ -0.1x+11, & 100 \le x \le 110, 1.45-0.005x \le y \le 0.45+0.005x \\ -20y+20, & 0.95 \le y \le 1, 290-200y \le x \le 200y-90 \\ 0.1x-9, & 90 \le x \le 100, 0.45+0.005x \le y \le 1.45-0.005x \\ 0, & \text{otherwise} \end{cases}$$

can be used to express the working state of the person.

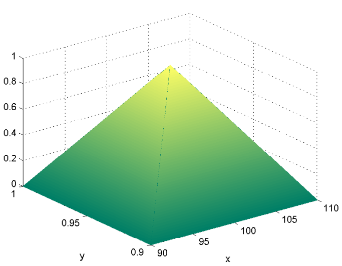

Fig. 3. Fuzzy 2−cell number u

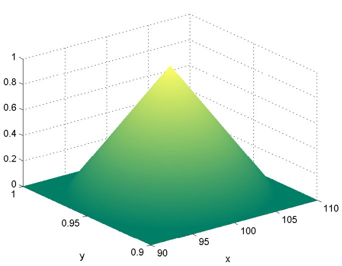

Fig. 4. Fuzzy 2−ellipsoid number v

However, if we defined a fuzzy 2−ellipsoid number v (see Fig 4) or w (see Fig 5) respectively as

$$v(t)(x,y) = \begin{cases} 1-(\frac{(x-100)^2}{(10)^2}+\frac{(y-0.95)^2}{(0.05)^2})^{\frac{1}{2}}, & \frac{(x-100)^2}{(10)^2}+\frac{(y-0.95)^2}{(0.05)^2} \le 1 \\ 0, & \text{otherwise} \end{cases}$$

or

$$w(t)(x,y) = \begin{cases} 1-\frac{(x-100)^2}{(10)^2}+\frac{(y-0.95)^2}{(0.05)^2}, & \frac{(x-100)^2}{(10)^2}+\frac{(y-0.95)^2}{(0.05)^2} \le 1 \\ 0, & \text{otherwise} \end{cases}$$

then using fuzzy 2−ellipsoid number $v(t)$ or $w(t)$ to express the working state (efficiency) of the person is

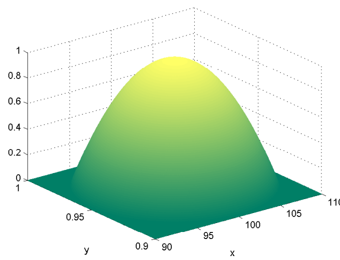

Fig. 5. Fuzzy 2−ellipsoid number w

more rational than using fuzzy 2−cell number $u(t)$. For example, let $(x(t),y(t)) = (95, 0.925)$ (producing speed $x(t) = 95$ and quali1cation rate $y(t) = 0.925$ at t time) and $(\grave{x}(t),\grave{y}(t)) = (100, 0.925)$ (producing speed $\grave{x}(t) = 100$ and quali1cation rate $\grave{y}(t) = 0.925$ at t time). Then the degree of $(\grave{x}(t),\grave{y}(t))$ belonging to "the person's working state" should be bigger than $(x(t),y(t))$. But if the fuzzy 2−cell number $u(t)$ is used to express the person's working state (efficiency), then we have $u(t)(\grave{x},\grave{y}) = u(t)(100, 0.925) = 0.5 = u(t)(95, 0.925) = u(t)(x,y)$ which can not tell us the fact that the degree of $(\grave{x}(t),\grave{y}(t))$ belonging to "the person's working state" should be bigger than $(x(t),y(t))$. However, if the fuzzy 2−ellipsoid number $v(t)$ or $w(t)$ is used to express the person's working state (efficiency), then it is able to tell us the fact since $v(t)(\grave{x},\grave{y}) = v(t)(100, 0.925) = 0.5 > 0.293 = v(t)(95, 0.925) = v(t)(x,y)$ and $w(t)(\grave{x},\grave{y}) = w(t)(100, 0.925) = 0.75 > 0.5 = w(t)(95, 0.925) = w(t)(x,y)$. This demonstrate that using fuzzy n−ellipsoid numbers to express uncertain multichannel digital information are more objective and more rational than using fuzzy n−cell numbers.

Remark 2: In real situations, usually characteristics (or factors) may have different importance. Although this can not be considered in constructing fuzzy cell numbers to represent imprecise or uncertain multi-channel digital signals, it must be considered as we set up aggregation operators in fuzzy cell number space to fuse these digital signals. the more important the characteristic (or factor) is, the bigger the corresponding weight in the aggregation operators should be.

For the sake of the following discussion, we firstly give the following two representation theorems.

Theorem 1: (Representation theorem). If $u \in E(E^n)$, then

(1) $[u]^r$ is a non-empty n−dimension closed ellipsoid, i.e., $[u]^r = \begin{pmatrix} \underline{u_1}(r) & \overline{u_1}(r) \\ \underline{u_2}(r) & \overline{u_2}(r) \\ \vdots & \vdots \\ \underline{u_n}(r) & \overline{u_n}(r) \end{pmatrix}$ for each $r \in [0,1]$;

(2) $[u]^{r_2} \subset [u]^{r_1}$ for $0 \le r_1 \le r_2 \le 1$;

(3) $\bigcap_{m=1}^{\infty} [u]^{r_m} = [u]^r$ for positive non-decreasing

sequence $\{r_m\}$ with $\lim_{m\to\infty} r_m = r$.

Conversely, if A_r, $r \in [0,1]$ satisfies
(a) A_r is a non-empty $n-$dimension closed ellipsoid for each $r \in [0,1]$;
(b) $A_{r_2} \subset A_{r_1}$ for $0 \le r_1 \le r_2 \le 1$;
(c) $\bigcap_{m=1}^{\infty} A_{r_m} = A_r$ for positive non-decreasing sequence $\{r_m\}$ with $\lim_{m\to\infty} r_m = r$,
then there exists a unique fuzzy $n-$ellipsoid number u such that $[u]^r = A_r$ for each $r \in (0,1]$ and $[u]^0 = \overline{\bigcup_{0<r\le 1} A_r} \subset A_0$.

Proof: Let $u \in E(E^n)$. By the definition of fuzzy $n-$ellipsoid numbers, we have $u \in E^n$. So by Proposition 6.1.6 in [8] (Page 38-39), we know that the Conclusions (2) and (3) hold. And the Conclusion (1) can be directly also shown by the definition of fuzzy $n-$ellipsoid numbers.

Conversely, if A_r, $r \in [0,1]$ satisfies the Conditions (a),(b) and (c), then by Inverse proposition of Proposition 6.1.6 in [8] (Page 39), we know that there exists a unique $u \in E^n$ such that $[u]^r = A_r$ for each $r \in (0,1]$ and $[u]^0 = \overline{\bigcup_{0<r\le 1} A_r} \subset A_0$. Noting again the Condition (a), we have $u \in E(E^n)$. The proof of the theorem is completed. ∎

Theorem 2: If $u \in E(E^n)$, then for $i = 1,2,\cdots,n$, $\underline{u_i}(r)$, $\overline{u_i}(r)$ are real-valued functions on $[0,1]$, and satisfy
(1) $\underline{u_i}(r)$ are non-decreasing and left continuous;
(2) $\overline{u_i}(r)$ are non-increasing and left continuous;
(3) $\underline{u_i}(r) \le \overline{u_i}(r)$ (it is equivalent to $\underline{u_i}(1) \le \overline{u_i}(1)$);
(4) $\underline{u_i}(r)$, $\overline{u_i}(r)$ are right continuous at $r = 0$.

Conversely if $a_i(r)$, $b_i(r)$, $i = 1,2,\cdots,n$ are real-valued functions on $[0,1]$ which satisfy the conditions
(a) $a_i(r)$ are non-decreasing and left continuous;
(b) $b_i(r)$ are non-increasing and left continuous;
(c) $a_i(1) \le b_i(1)$ (it is equivalent to $a_i(r) \le b_i(r)$);
(d) $a_i(r)$, $b_i(r)$ are right continuous at $r = 0$,
then there exists a unique $u \in E(E^n)$ such that $[u]^r = \begin{pmatrix} a_1(r) & b_1(r) \\ a_2(r) & b_2(r) \\ \vdots & \vdots \\ a_n(r) & b_n(r) \end{pmatrix}$ for any $r \in [0,1]$.

Proof: The proof is placed in appendix A. ∎

Since for any closed $n-$ellipsoid $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$ and $\lambda \in R$, $\lambda \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 & \lambda b_1 \\ \lambda a_2 & \lambda b_2 \\ \vdots & \vdots \\ \lambda a_n & \lambda b_n \end{pmatrix}$ as $\lambda \ge 0$; $\lambda \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} = \begin{pmatrix} \lambda b_1 & \lambda a_1 \\ \lambda b_2 & \lambda a_2 \\ \vdots & \vdots \\ \lambda b_n & \lambda a_n \end{pmatrix}$ as $\lambda < 0$, and it is still a closed $n-$ellipsoid, by Theorem 1 and $[\lambda u]^r = \lambda [u]^r$ ($\lambda \in R, u \in E^n, r \in [0,1]$), we can obtain the following result about scalar products of fuzzy $n-$ellipsoid numbers.

Proposition 1: Let $\lambda \in R$ and $u \in E(E^n)$. Then $\lambda u \in E(E^n)$, and for any $r \in [0,1]$,

$$\begin{pmatrix} \underline{(ku)_1}(r) & \overline{(ku)_1}(r) \\ \underline{(ku)_2}(r) & \overline{(ku)_2}(r) \\ \vdots & \vdots \\ \underline{(ku)_n}(r) & \overline{(ku)_n}(r) \end{pmatrix} = \begin{cases} \begin{pmatrix} k\underline{u_1}(r) & k\overline{u_1}(r) \\ \vdots & \vdots \\ k\underline{u_n}(r) & k\overline{u_2}(r) \end{pmatrix}, & k \ge 0 \\ \begin{pmatrix} k\overline{u_1}(r) & k\underline{u_1}(r) \\ \vdots & \vdots \\ k\overline{u_n}(r) & k\underline{u_n}(r) \end{pmatrix}, & k < 0 \end{cases}$$

However, due to that the addition ("+") and multiplication ("·") of two closed $n-$ellipsoids are not necessarily closed $n-$ellipsoids, by Theorem 1 and $[u+v]^r = [u]^r + [v]^r, [uv]^r = [u]^r[v]^r$ ($u, v \in E^n, r \in [0,1]$), we know that although $u + v \in E^n$ and $uv \in E^n$ hold, $u + v \in E(E^n)$ and $uv \in E(E^n)$ do not necessarily hold for $u, v \in E(E^n)$.

It is obvious that if $u, v \in E(E^n)$, then $A_r = \begin{pmatrix} \underline{u_1}(r) + \underline{v_1}(r) & \overline{u_1}(r) + \overline{v_1}(r) \\ \underline{u_2}(r) + \underline{v_2}(r) & \overline{u_2}(r) + \overline{v_2}(r) \\ \vdots & \vdots \\ \underline{u_n}(r) + \underline{v_n}(r) & \overline{u_n}(r) + \overline{v_n}(r) \end{pmatrix}$, $r \in [0,1]$ satisfy the Conditions (a),(b) and (c) of Theorem 1, so $A_r, r \in [0,1]$ determine a unique fuzzy $n-$ellipsoid number w. We define the addition "$\oplus$" of u and v by $u \oplus v = w$, which is an approximation of addition "+". Then, we have the following result.

Proposition 2: let $u, v \in E(E^n)$. Then $u \oplus v \in E(E^n)$, and for any $r \in [0,1]$,

$$\begin{pmatrix} \underline{u_1 \oplus v_1}(r) & \overline{u_1 \oplus v_1}(r) \\ \underline{u_2 \oplus v_2}(r) & \overline{u_2 \oplus v_2}(r) \\ \vdots & \vdots \\ \underline{u_n \oplus v_n}(r) & \overline{u_n \oplus v_n}(r) \end{pmatrix} = \begin{pmatrix} \underline{u_1}(r) + \underline{v_1}(r) & \overline{u_1}(r) + \overline{v_1}(r) \\ \underline{u_2}(r) + \underline{v_2}(r) & \overline{u_2}(r) + \overline{v_2}(r) \\ \vdots & \vdots \\ \underline{u_n}(r) + \underline{v_n}(r) & \overline{u_n}(r) + \overline{v_n}(r) \end{pmatrix}$$

In addition, we can also prove the following result.

Proposition 3: let $u, v \in E(E^n)$. If $u + v \in E(E^n)$, then $u + v = u \oplus v$.

About multiplications of fuzzy $n-$ellipsoid numbers, we can make a similar discussion.

IV. MEMBERSHIP FUNCTION VALUE OF CONE FUZZY $n-$ELLIPSOID NUMBER

For the sake of the application of fuzzy $n-$ellipsoid numbers, we study two special kind of fuzzy $n-$ellipsoid numbers in the following.

Theorem 3: Let $a_i, b_i \in R$ with $a_i \le b_i$, $i = 1,2,\cdots,n$. If fuzzy subset u of R^n is defined as

$$u(x_1, x_2, \cdots, x_n) = \begin{cases} 1 - \left(\sum_{i=1}^{n} \frac{(x_i - \frac{b_i + a_i}{2})^2}{(\frac{b_i - a_i}{2})^2}\right)^{\frac{1}{2}}, & (x_1, \cdots, x_n) \in D \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where $D = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$, then $u \in E(E^n)$.

Proof: For each $i = 1,2,\cdots,n$, $a_i \le b_i$ implies $a_i + \frac{b_i - a_i}{2} r \le b_i - \frac{b_i - a_i}{2} r$ for any $r \in [0,1]$. Let $r \in$

$(0,1)$. Then we have

$$
\begin{aligned}
[u]^r &= \{x \in R^n | u(x) \geq r\} \\
&= \{x \in R^n | 1 - (\textstyle\sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2})^{\frac{1}{2}} \geq r\} \\
&= \{x \in R^n | \textstyle\sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}) \leq (1-r)^2\} \\
&= \{x \in R^n | \textstyle\sum_{i=1}^n \frac{(x_i - \frac{(b_i - \frac{b_i-a_i}{2} r) + (a_i + \frac{b_i-a_i}{2} r)}{2})^2}{[\frac{(b_i - \frac{b_i-a_i}{2} r) - (a_i + \frac{b_i-a_i}{2} r)}{2}]^2} \leq 1\} \\
&= \begin{pmatrix} a_1 + \frac{b_1-a_1}{2} r & b_1 - \frac{b_1-a_1}{2} r \\ a_2 + \frac{b_2-a_2}{2} r & b_2 - \frac{b_2-a_2}{2} r \\ \vdots & \vdots \\ a_n + \frac{b_n-a_n}{2} r & b_n - \frac{b_n-a_n}{2} r \end{pmatrix}
\end{aligned}
$$

so $\underline{u_i}(r) = a_i + \frac{b_i-a_i}{2} r$ and $\overline{u_i}(r) = b_i - \frac{b_i-a_i}{2} r$ $(i = 1, 2, \cdots, n)$ hold as $r \in (0,1)$. On the other hand, from the definitions of $[u]^0$ and $[u]^1$, we can obtain $[u]^0 = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$ and $[u]^1 = \{(\frac{b_1+a_1}{2}, \frac{b_2+a_2}{2}, \cdots, \frac{b_n+a_n}{2})\}$, i.e., $\underline{u_i}(r) = a_i + \frac{b_i-a_i}{2} r$ and $\overline{u_i}(r) = b_i - \frac{b_i-a_i}{2} r$ $(i = 1, 2, \cdots, n)$ also hold as $r = 0$ or $r = 1$. It is obvious that $a_i + \frac{b_i-a_i}{2} r$ and $b_i - \frac{b_i-a_i}{2} r$ $(i = 1, 2, \cdots, n)$ satisfy the conditions (a)-(d) of theorem 2, so $u \in E(E^n)$. The proof of the theorem is completed. ■

Theorem 4: Let $a_i, b_i \in R$ with $a_i \leq b_i$, $i = 1, 2, \cdots, n$. If fuzzy subset u of R^n is defined as

$$
\begin{aligned}
& u(x_1, x_2, \cdots, x_n) \\
= & \begin{cases} 1 - \sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}, & (x_1, x_2, \cdots, x_n) \in D \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{4}
$$

where $D = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$, then $u \in E(E^n)$.

Proof: For each $i = 1, 2, \cdots, n$, $a_i \leq b_i$ implies $\frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2} \leq \frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}$ for any $r \in [0,1]$. Let $r \in (0,1)$. Then we have

$$
\begin{aligned}
& [u]^r \\
= & \{x \in R^n | u(x) \geq r\} \\
= & \{x \in R^n | 1 - \textstyle\sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2} \geq r\} \\
= & \{x \in R^n | \textstyle\sum_{i=1}^n \frac{(x_i - \frac{b_i+a_i}{2})^2}{[\frac{b_i-a_i}{2}(\sqrt{1-r})]^2}) \leq 1\} \\
= & \{x \in R^n | \\
& \textstyle\sum_{i=1}^n \frac{(x_i - \frac{\frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2} + \frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}}{2})^2}{(\frac{\frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2} - \frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}}{2})^2} \leq 1\} \\
= & \begin{pmatrix} \frac{a_1+b_1-(b_1-a_1)\sqrt{1-r}}{2} & \frac{a_1+b_1+(b_1-a_1)\sqrt{1-r}}{2} \\ \frac{a_2+b_2-(b_2-a_2)\sqrt{1-r}}{2} & \frac{a_2+b_2+(b_2-a_2)\sqrt{1-r}}{2} \\ \vdots & \vdots \\ \frac{a_n+b_n-(b_n-a_n)\sqrt{1-r}}{2} & \frac{a_n+b_n+(b_n-a_n)\sqrt{1-r}}{2} \end{pmatrix}
\end{aligned}
$$

so $\underline{u_i}(r) = \frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2}$, $\overline{u_i}(r) = \frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}$, $i = 1, 2, \cdots, n$ hold as $r \in (0,1)$. From the definitions of $[u]^0$ and $[u]^1$, we see $[u]^0 = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$ and $[u]^1 = \{(\frac{b_1+a_1}{2}, \cdots, \frac{b_n+a_n}{2})\}$, i.e., $\underline{u_i}(r) = \frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2}$ and $\overline{u_i}(r) = \frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}$ $(i = 1, 2, \cdots, n)$ also hold as $r = 0$ or $r = 1$. It is obvious that $\frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2}$ and $\frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}$ $(i = 1, 2, \cdots, n)$ satisfy the conditions (a)-(d) of theorem 2, so $u \in E(E^n)$. The proof of the theorem is completed. ■

Definition 2: If $u \in E(E^n)$ is defined as Theorem 3, then we call u a symmetric cone fuzzy $n-$ellipsoid number, and denote it as $u = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}_{SCF}$.

Definition 3: If $u \in E(E^n)$ is defined as Theorem 4, then we call u a symmetric paraboloid fuzzy $n-$ellipsoid number, and denote it as $u = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}_{SCP}$.

Remark 3: In [3], Bandemer and Näther defined the concept of $(n-)$bean fuzzy vectors by $u(x) = max\{1 - (x - x^0)^T B (x - x^0), 0\}$, where B is a positive definite $n \times n$-matrix, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in R$. Although the expressions of the membership functions of bean fuzzy vectors and symmetric paraboloid fuzzy $n-$ellipsoid numbers are different, in fact, they are same n-dimension fuzzy numbers. For a such fuzzy number u, using Expression (4) in this paper, we can more conveniently obtain its cut-sets type (Theorem 1) and real functions type (Theorem 2) representations, and can more conveniently construct such fuzzy numbers to represent uncertain multichannel digital information. In addition, the parameters $a_i, b_i \in R$ in Expression (4) are more intuitive geometric meaning. Likewise, the symmetrical conical fuzzy numbers used by M. Sato and Y. Sato in [24] and symmetric cone fuzzy $n-$ellipsoid number here are also similar relationships.

In general, it is difficult even impossible to find out the expression of membership function of a fuzzy $n-$ellipsoid number (defined as Definition 3.1). For example, it is very difficult to work out the expression of membership function of the fuzzy $2-$ellipsoid number u (by Theorem 1 or 2, we know u decided by $[u]^r$, $r \in [0,1]$, is indeed a fuzzy $2-$ellipsoid number) defined by $[u]^r = \begin{pmatrix} (1+r)^2 & 5-r \\ r & 2-r^3 \end{pmatrix}$.

In the following, we study the method of solving membership function value for some special kind of fuzzy $n-$ellipsoid numbers (include the fuzzy $n-$ellipsoid numbers such as the fuzzy $2-$ellipsoid number u decided by $[u]^r$, $r \in [0,1]$, above). Firstly, we give the following definitions.

Definition 4: Let $u{\in}E(E^n)$, $\underline{u_i}(0){=}a_i$, $\overline{u_i}(0){=}b_i$, $i=1,2,\cdots,n$. If $\{(x_1,\cdots,x_n)\in R^n|u(x_1,\cdots,x_n)=1\}$ is a single point set (denoted as $\{(c_1,c_2,\cdots,c_n)\}$, of course, we have $a_i\le c_i\le b_i$, $i=1,2,\cdots,n$ and $\underline{u_i}(r)=a_i+(c_i-a_i)r$, $\overline{u_i}(r)=b_i-(b_i-c_i)r$, $i=1,2,\cdots,n$, then we call u a cone fuzzy $n-$ellipsoid number, and denote it as $u=\begin{pmatrix} a_1 & c_1 & b_1\\ a_2 & c_2 & b_2\\ \vdots & \vdots & \vdots\\ a_n & c_n & b_n\end{pmatrix}_{CF}$.

Definition 5: Let $u{\in}E(E^n)$, $\underline{u_i}(0){=}a_i$, $\overline{u_i}(0){=}b_i$, $i=1,2,\cdots,n$. If $\{(x_1,\cdots,x_n)\in R^n|u(x_1,\cdots,x_n)=1\}$ is a single point set (denoted as $\{(c_1,c_2,\cdots,c_n)\}$), and $\underline{u_i}(r)=c_i-(c_i-a_i)\sqrt{1-r}$, $\overline{u_i}(r)=c_i+(b_i-c_i)\sqrt{1-r}$, $i=1,2,\cdots,n$, then we call u a paraboloid fuzzy $n-$ellipsoid number, and denote it as $u{=}\begin{pmatrix} a_1 & c_1 & b_1\\ a_2 & c_2 & b_2\\ \vdots & \vdots & \vdots\\ a_n & c_n & b_n\end{pmatrix}_{PF}$.

Definition 6: Let $u\in E(E^n)$. If $\{(x_1,x_2,\cdots,x_n)\in R^n|u(x_1,x_2,\cdots,x_n)=1\}$ is a single point set, and for each $i=1,2,\cdots,n$, $\underline{u_i}(r)$ is strictly increasing and continuous on $[0,1]$, $\overline{u_i}(r)$ is strictly decreasing and continuous on $[0,1]$, then we call u a quasi cone fuzzy $n-$ellipsoid number.

Theorem 5: (1) Each symmetric cone fuzzy $n-$ellipsoid number is a cone fuzzy $n-$ellipsoid number;
(2) Each symmetric paraboloid fuzzy $n-$ellipsoid number is a paraboloid fuzzy $n-$ellipsoid number;
(3) For each cone fuzzy $n-$ellipsoid number u, if $\underline{u_i}(0)\neq\underline{u_i}(1)(=\overline{u_i}(1))\neq\overline{u_i}(0)$, $i=1,2,\cdots,n$, then u is a quasi cone fuzzy $n-$ellipsoid number;
(4) For each paraboloid fuzzy $n-$ellipsoid number u, if $\underline{u_i}(0)\neq\underline{u_i}(1)(=\overline{u_i}(1))\neq\overline{u_i}(0)$, $i=1,2,\cdots,n$, then u is a quasi cone fuzzy $n-$ellipsoid number.

Proof: Let u be a symmetric cone fuzzy $n-$ellipsoid number. Then there exist $a_i,b_i\in R$ with $a_i\le b_i$, $i=1,2,\cdots,n$, such that $u=\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\\ \vdots & \vdots\\ a_n & b_n\end{pmatrix}_{SCF}$. From the proof of Theorem 3, we know $\underline{u_i}(r)=a_i+\frac{b_i-a_i}{2}r$ and $\overline{u_i}(r)=b_i-\frac{b_i-a_i}{2}r$, $i=1,2,\cdots,n$. Taking $c_i=\frac{b_i+a_i}{2}$, then we have $\underline{u_i}(r)=a_i+(c_i-a_i)r$ and $\overline{u_i}(r)=b_i-(b_i-c_i)r$, $i=1,2,\cdots,n$. By Definition 4, we see u is a cone fuzzy $n-$ellipsoid number. Thus, we have completed the proof of (1). Similarly, taking $c_i=\frac{b_i+a_i}{2}$, and paying attention to $\underline{u_i}(r)=\frac{a_i+b_i-(b_i-a_i)\sqrt{1-r}}{2}=c_i-(c_i-a_i)\sqrt{1-r}$ and $\overline{u_i}(r)=\frac{a_i+b_i+(b_i-a_i)\sqrt{1-r}}{2}=c_i+(b_i-ci)\sqrt{1-r}$, $i=1,2,\cdots,n$, we can also complete the proof of (2).

Let $u=\begin{pmatrix} a_1 & c_1 & b_1\\ a_2 & c_2 & b_2\\ \vdots & \vdots & \vdots\\ a_n & c_n & b_n\end{pmatrix}_{CF}$ be a cone fuzzy $n-$ellipsoid number. It is obvious that $\underline{u_i}(0)=a_i$, $\underline{u_i}(1)=\overline{u_i}(1)=c_i$ and $\overline{u_i}(0)=b_i$, $i=1,2,\cdots,n$. From $\underline{u_i}(0)\neq\underline{u_i}(1)(=\overline{u_i}(1))\neq\overline{u_i}(0)$ $(i=1,2,\cdots,n)$, we can easily see that $\{(x_1,x_2,\cdots,x_n)\in R^n|u(x_1,x_2,\cdots,x_n)=1\}=\{(c_1,c_2,\cdots,c_n)\}$ is a single point set, $\underline{u_i}(r)=a_i+(c_i-a_i)r$ is strictly increasing on $[0,1]$, and $\overline{u_i}(r)=b_i-(b_i-c_i)r$ is strictly decreasing on $[0,1]$ for $i=1,2,\cdots,n$, so u is a quasi cone fuzzy $n-$ellipsoid number. In the same way, we can also prove (2). The proof of the theorem is completed. ■

For any $A\subset R^n$, we denote $A^{Ker}=\{x\in R^n\mid x$ is a interior point of $A\}$.

Lemma 1: Let $a_i,b_i,\grave{a}_i,\grave{b}_i\in R$ with $a_i<\grave{a}_i<\grave{b}_i<b_i$, $i=1,2,\cdots,n$. Then for any $x^0=(x_1^0,x_2^0,\cdots,x_n^0)\in R^n$,
(1) $\sum_{i=1}^n\frac{(x_i^0-\frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}=1\Longrightarrow\sum_{i=1}^n\frac{(x_i^0-\frac{\grave{b}_i+\grave{a}_i}{2})^2}{(\frac{\grave{b}_i-\grave{a}_i}{2})^2}>1$;
(2) $\sum_{i=1}^n\frac{(x_i^0-\frac{\grave{b}_i+\grave{a}_i}{2})^2}{(\frac{\grave{b}_i-\grave{a}_i}{2})^2}=1\Longrightarrow\sum_{i=1}^n\frac{(x_i^0-\frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}<1$.

Proof: We firstly show (1) by reduction to absurdity. Assume $\sum_{i=1}^n\frac{(x_i^0-\frac{\grave{b}_i+\grave{a}_i}{2})^2}{(\frac{\grave{b}_i-\grave{a}_i}{2})^2}>1$ is not valid, then $\sum_{i=1}^n\frac{(x_i^0-\frac{\grave{b}_i+\grave{a}_i}{2})^2}{(\frac{\grave{b}_i-\grave{a}_i}{2})^2}\le1$, so $x^0\in\begin{pmatrix}\grave{a}_1 & \grave{b}_1\\ \grave{a}_2 & \grave{b}_2\\ \vdots & \vdots\\ \grave{a}_n & \grave{b}_n\end{pmatrix}\subset\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\\ \vdots & \vdots\\ a_n & b_n\end{pmatrix}^{Ker}$. It implies $\sum_{i=1}^n\frac{(x_i^0-\frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}<1$ which is in contradiction to $\sum_{i=1}^n\frac{(x_i^0-\frac{b_i+a_i}{2})^2}{(\frac{b_i-a_i}{2})^2}=1$, so (1) holds. Similarly, we can completed the proof of (2). ■

Theorem 6: Let u be a quasi cone fuzzy $n-$ellipsoid number. Then for any fixed $x^0=(x_1^0,x_2^0,\cdots,x_n^0)\in[u]^0$, there exists a unique $r_0\in[0,1]$ such that

$$\sum_{i=1}^n\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}=1 \tag{5}$$

Proof: The proof is placed in appendix A. ■

Theorem 7: Let u be a quasi cone fuzzy $n-$ellipsoid number. If $x^0=(x_1^0,x_2^0,\cdots,x_n^0)\in[u]^0$, $r_0\in[0,1]$ satisfy Equation (5), then
(1) $\sum_{i=1}^n\frac{(x_i^0-\frac{\underline{u_i}(r)+\overline{u_i}(r)}{2})^2}{(\frac{\underline{u_i}(r)-\overline{u_i}(r)}{2})^2}<1$ for any $r\in[0,r_0)$;
(2) $\sum_{i=1}^n\frac{(x_i^0-\frac{\underline{u_i}(r)+\overline{u_i}(r)}{2})^2}{(\frac{\underline{u_i}(r)-\overline{u_i}(r)}{2})^2}>1$ for any $r\in(r_0,1]$;
(3) $u(x^0)=u(x_1^0,x_2^0,\cdots,x_n^0)=r_0$.

Proof: By Lemma 1 and from the the strictly monotonicity of $\underline{u_i}(r)$ and $\overline{u_i}(r)$, we can easily show that (1) and (2) are valid. In the following, we prove (3) by reduction to absurdity.

Assume conclusion (3) is not valid, i.e., $u(x^0)=u(x_1^0,x_2^0,\cdots,x_n^0)\neq r_0$. On the other hand, from Equation (5), we know $x^0\in\begin{pmatrix}\underline{u_1}(r_0) & \overline{u_1}(r_0)\\ \underline{u_2}(r_0) & \overline{u_2}(r_0)\\ \vdots & \vdots\\ \underline{u_n}(r_0) & \overline{u_n}(r_0)\end{pmatrix}=[u]^{r_0}$, i.e., $u(x^0) \geq r_0$. Thus, we have $u(x^0) > r_0$, so there exist $\grave{r}_0 \in [0,1]$ such that $u(x^0) > \grave{r}_0 > r_0$. From $u(x^0) > \grave{r}_0$, we see $x^0 \in [u]^{\grave{r}_0}$, it implies $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\overline{u_i}(\grave{r}_0)+\underline{u_i}(\grave{r}_0)}{2})^2}{(\frac{\overline{u_i}(\grave{r}_0)-\underline{u_i}(\grave{r}_0)}{2})^2} \leq 1$. However, from $\grave{r}_0 > r_0$ and by conclusion (2), we have $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\overline{u_i}(\grave{r}_0)+\underline{u_i}(\grave{r}_0)}{2})^2}{(\frac{\overline{u_i}(\grave{r}_0)-\underline{u_i}(\grave{r}_0)}{2})^2} > 1$, which is in contradiction to $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\overline{u_i}(\grave{r}_0)+\underline{u_i}(\grave{r}_0)}{2})^2}{(\frac{\overline{u_i}(\grave{r}_0)-\underline{u_i}(\grave{r}_0)}{2})^2} \leq 1$, so conclusion (3) holds. Thus, the proof of the theorem is completed. ∎

Remark 4: By Theorem 6 and 7 and the proof of Theorem 6, for a quasi cone fuzzy $n-$ellipsoid number u and a point $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ in R^n, we can propose the following method to solve the membership function value $u(x^0)$:

Fixing $q \in (0,1)$, denote $\xi_1 = 0$, $\eta_1 = 1$ and $r_1 = (1-q)\xi_1 + q\eta_1 = q$. Then denote $\xi_2 = \xi_1$, $\eta_2 = r_1$ as $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2} > 1$, while, denote $\xi_2 = r_1$, $\eta_2 = \eta_1$ as $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2} \leq 1$, and denote $r_2 = (1-q)\xi_2 + q\eta_2$. And then denote $\xi_3 = \xi_2$, $\eta_3 = r_2$ as $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2} > 1$, while, denote $\xi_3 = r_2$, $\eta_3 = \eta_2$ as $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2} \leq 1$, and denote $r_3 = (1-q)\xi_3 + q\eta_3$. ...... So always go on, we can obtain a convergent sequence $\{r_m\}_{m=1}^{\infty}$, and denote $r_0 = \lim_{m\to\infty} r_m$, then $u(x^o) = r_0$.

For a quasi cone fuzzy $n-$ellipsoid number u and $x^0 = (x_1, \cdots, x_n) \in R^n$, in order to obtain the membership function value $u(x^0)$ as soon as possible, we can take $q = 0.618$ in accordance by the golden section method.

Specific iterative algorithm: Let u be a quasi cone fuzzy $n-$ellipsoid number, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in R^n$, and $\delta \in (0,1)$ be a given real number.

1. If $x^0 \bar{\in} [u]^0$, then $u(x^0) = 0$, and the algorithm end.
2. If $x^0 \in [u]^0$, then take $\xi_1 = 0$, $\eta_1 = 1$.
2.1. Take $r_1 = 0.382\xi_1 + 0.618\eta_1 = 0.618$.
2.1.1. If $\eta_1 - \xi_1 \leq \delta$, then $u(x^0) = r_1$, the algorithm end;
2.1.2. If $\eta_1 - \xi_1 > \delta$, then
2.1.2.1. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2} = 1$, then $u(x^0) = r_1$, and the algorithm end;
2.1.2.2. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2} > 1$, then take $\xi_2 = \xi_1$, $\eta_2 = r_1$;
2.1.2.3. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2} < 1$, then take $\xi_2 = r_1$, $\eta_2 = \eta_1$.
2.2. Take $r_2 = 0.382\xi_2 + 0.618\eta_2$.
2.2.1. If $\eta_2 - \xi_2 \leq \delta$, then $u(x^0) = r_2$, the algorithm end;
2.2.2. If $\eta_2 - \xi_2 > \delta$, then
2.2.2.1. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2} = 1$, then $u(x^0) = r_2$, and the algorithm end;
2.2.2.2. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2} > 1$, then take $\xi_3 = \xi_2$, $\eta_3 = r_2$;
2.2.2.3. if $\sum_{i=1}^{n} \frac{(x_i^0 - \frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2} < 1$, then take $\xi_3 = r_2$, $\eta_3 = \eta_2$.
2.3. Take $r_3 = 0.382\xi_3 + 0.618\eta_4$.
...... and so on until the end.

V. REPRESENTATION OF UNCERTAIN INFORMATION AND APPLICATION

In this section, we establish a method constructing fuzzy $n-$ellipsoid numbers representing uncertain or imprecise multi-channel digital signals

Consider an object (denoted by O) having n characteristics (denoted by $O_1, O_2, \cdots, O_n$). And suppose the following data set about the object are from m sources (such as sensors or samples) in an imprecise or uncertain environment:

$$O: \begin{array}{cccc} O_1 & O_2 & \cdots & O_n \\ o_{11} & o_{21} & \cdots & o_{n1} \\ o_{12} & o_{22} & \cdots & o_{n2} \\ \vdots & \vdots & & \vdots \\ o_{1m} & o_{2m} & \cdots & o_{nm} \end{array} \tag{6}$$

The problem to be solved is how to construct a suitable fuzzy $n-$ellipsoid number to represent the object which possess some imprecise or uncertain attributes.

First: For each $i = 1, 2, \cdots, n$, we work out the means μ_i of the ith character values O_i:

$$\mu_i = \frac{1}{m} \sum_{j=1}^{m} o_{ij} \tag{7}$$

Second: For each $i = 1, 2, \cdots, n$, we work out the left separation degrees $L\sigma_i$ and right separation degrees $R\sigma_i$ of the ith character values O_i, respectively:

$$\begin{array}{l} L\sigma_i = \frac{1}{N_{Li}} \sum_{o_{ij}<\mu_i} (\mu_i - o_{ij}) \\ R\sigma_i = \frac{1}{N_{Ri}} \sum_{o_{ij}>\mu_i} (o_{ij} - \mu_i) \end{array} \tag{8}$$

where N_{Li} and N_{Ri} are the number of the character values o_{ij} $(i = 1, 2, \cdots, n)$ which satisfy $o_{ij} < \mu_i$ and the number of the character values o_{ij} $(i = 1, 2, \cdots, n)$ which satisfy $o_{ij} > \mu_i$, respectively.

Third: For each $i = 1, 2, \cdots, n$, making a domain $[\alpha_i, \beta_i]$ (with $\mu_i \in (\alpha_i, \beta_i)$) of the ith character value O_i according to the practical case, we construct a cone fuzzy $n-$ellipsoid number u as

$$u = \begin{pmatrix} (\underline{\max})_1 & \mu_1 & (\overline{\min})_1 \\ (\underline{\max})_2 & \mu_2 & (\overline{\min})_2 \\ \vdots & \vdots & \vdots \\ (\underline{\max})_n & \mu_n & (\overline{\min})_n \end{pmatrix}_{CF} \tag{9}$$

or a paraboloid fuzzy $n-$ellipsoid number

$$v = \begin{pmatrix} (\underline{\max})_1 & \mu_1 & (\overline{\min})_1 \\ (\underline{\max})_2 & \mu_2 & (\overline{\min})_2 \\ \vdots & \vdots & \vdots \\ (\underline{\max})_n & \mu_n & (\overline{\min})_n \end{pmatrix}_{PF} \quad (10)$$

where $(\underline{\max})_i = \max\{\mu_i - \lambda L\sigma_i, \alpha_i\}$, $(\overline{\min})_i = \min\{\mu_i + \lambda R\sigma_i, \beta_i\}$ $(i = 1, 2, \cdots, n)$, i.e., u and v are respectively the quasi cone fuzzy $n-$ellipsoid numbers decided by n pair functions $\underline{u_i}(r) = \max\{\mu_i - \lambda L\sigma_i, \alpha_i\} + (\mu_i - \max\{\mu_i - \lambda L\sigma_i, \alpha_i\})r$, $\overline{u_i}(r) = \min\{\mu_i + \lambda R\sigma_i, \beta_i\} - (\min\{\mu_i + \lambda R\sigma_i, \beta_i\} - \mu_i)r$, $i = 1, 2, \cdots, n$, and by n pair functions $\underline{u_i}(r) = \mu_i - (\mu_i - \max\{\mu_i - \lambda L\sigma_i, \alpha_i\})\sqrt{1-r}$, $\overline{u_i}(r) = \mu_i + (\min\{\mu_i + \lambda R\sigma_i, \beta_i\} - \mu_i)\sqrt{1-r}$, where λ is a parameter, that may be chosen in interval $[2, 4]$ according to practical case. Then we can use the fuzzy $n-$ellipsoid number u or v to express the object O.

Example 3. In remote sensing classification for land-cover, we use "Korean pine accounts for the main part" to denote forest that mainly contains Korean pines. Because in different "Korean pine accounts for the main part" areas, there are many different factors, such as the difference of the density of Korean pines, of the species and quantity of other plants, of the physiognomy, etc., the values of reflections of the electromagnetic spectrum are also different. Therefore, "Korean pine accounts for the main part" should not be a certain crisp value but a fuzzy set without certain bound. In [30], we used ever fuzzy cell number to express "Korean pine accounts for the main part". In order to make the express more objective and rational, in the following, we construct a fuzzy ellipsoid number to express "Korean pine accounts for the main part".

Suppose that we use 4 wave bands: MSS-4, MSS-5, MSS-6, MSS-7. We take 10 samples, and acquire the following data for some zone of "Korean pine accounts for the main part":

	MSS-4	MSS-5	MSS-6	MSS-7
Sample 1:	15.01	13.30	40.50	19.37
Sample 2:	15.60	12.56	38.81	16.35
Sample 3:	15.82	12.79	37.70	18.16
Sample 4:	14.90	11.70	35.50	14.75
Sample 5:	16.10	13.80	42.10	20.75
Sample 6:	13.80	11.94	32.10	15.54
Sample 7:	15.90	10.98	30.87	14.29
Sample 8:	16.82	13.67	37.64	18.62
Sample 9:	15.50	12.58	36.10	18.02
Sample 10:	15.38	12.48	34.08	17.45

According to Formulas (7) and (8), we can work out:

MSS-4	MSS-5	MSS-6	MSS-7
$\mu_1 = 15.46$	$\mu_2 = 12.58$	$\mu_3 = 36.54$	$\mu_4 = 17.33$
$L\sigma_1 = 0.69$	$L\sigma_2 = 0.67$	$L\sigma_3 = 2.81$	$L\sigma_4 = 2.10$
$R\sigma_1 = 0.49$	$R\sigma_2 = 0.81$	$R\sigma_3 = 2.81$	$R\sigma_4 = 1.40$

Taking $[\alpha_i, \beta_i) = [0, +\infty)$, $i = 1, 2, 3, 4$ (since electromagnetic spectrum values should be in $[0, +\infty)$) and $\lambda = 4$, we can obtain by calculation $\max\{\mu_1 - \lambda L\sigma_1, \alpha_1\} = 12.70$, $\min\{\mu_1 + \lambda R\sigma_1, \beta_1\} = 17.42$, $\max\{\mu_2 - \lambda L\sigma_2, \alpha_2\} = 9.90$, $\min\{\mu_2 + \lambda R\sigma_2, \beta_2\} = 18.82$, $\max\{\mu_3 - \lambda L\sigma_3, \alpha_3\} = 25.30$, $\min\{\mu_3 + \lambda R\sigma_3, \beta_3\} = 47.78$, $\max\{\mu_4 - \lambda L\sigma_4, \alpha_4\} = 8.93$ and $\min\{\mu_4 + \lambda R\sigma_4, \beta_4\} = 22.93$. By Formulas (9) or (10), we construct $u = \begin{pmatrix} 12.70 & 15.46 & 17.42 \\ 9.90 & 12.58 & 18.82 \\ 25.30 & 36.54 & 47.78 \\ 8.93 & 17.33 & 22.93 \end{pmatrix}_{CF}$ or $v = \begin{pmatrix} 12.70 & 15.46 & 17.42 \\ 9.90 & 12.58 & 18.82 \\ 25.30 & 36.54 & 47.78 \\ 8.93 & 17.33 & 22.93 \end{pmatrix}_{PF}$, i.e., u and v are respectively the quasi cone fuzzy $n-$ellipsoid numbers decided by 4 pair functions

$$\underline{u_1}(r) = 12.70 + (15.46 - 12.70)r, \overline{u_1}(r) = 17.42 - (17.42 - 15.46)r$$
$$\underline{u_2}(r) = 9.90 + (12.58 - 9.90)r, \overline{u_2}(r) = 18.82 - (18.82 - 12.58)r$$
$$\underline{u_3}(r) = 25.30 + (36.54 - 25.30)r, \overline{u_3}(r) = 47.78 - (47.78 - 36.54)r$$
$$\underline{u_4}(r) = 8.93 + (17.33 - 8.93)r, \overline{u_4}(r) = 22.93 - (22.93 - 17.33)r$$

and by 4 pair functions

$$\underline{v_1}(r) = 15.46 - (15.46 - 12.70)\sqrt{1-r}$$
$$\overline{v_1}(r) = 15.46 + (17.42 - 15.46)\sqrt{1-r}$$
$$\underline{v_2}(r) = 12.58 - (12.58 - 9.90)\sqrt{1-r}$$
$$\overline{v_2}(r) = 12.58 + (18.82 - 12.58)\sqrt{1-r}$$
$$\underline{v_3}(r) = 36.54 - (36.54 - 25.30)\sqrt{1-r}$$
$$\overline{v_3}(r) = 36.54 + (47.78 - 36.54)\sqrt{1-r}$$
$$\underline{v_4}(r) = 17.33 - (17.33 - 8.93)\sqrt{1-r}$$
$$\overline{v_1}(r) = 17.33 + (22.93 - 17.33)\sqrt{1-r}$$

Then we can use the fuzzy $n-$ellipsoid number u or v to represent "Korean pine accounts for the main part".

Example 4. Only two breeds of wheat are planted in some farming region. One belongs to "High-fertilizer High-yielding Wheat" (denoted by C_1). The other one belongs to "Mid-fertilizer High-yielding Wheat" (denoted by C_2). They mainly have four state characters: Ear Emergence (denoted by x_1), Plant Height (denoted by x_2), Number of Grain per Ear (denoted by x_3), Weight per hundred-grains (denoted by x_4). Suppose that the following two class data come from C_1 and C_2, respectively (from Example 4.2 in [30]):

		x_1	x_2	x_3	x_4
	Sample 1 :	8.8	67.8	40.7	4.2
	Sample 2 :	7.6	81.1	51.6	3.5
	Sample 3 :	9.9	77.5	46.5	3.8
	Sample 4 :	6.2	78.6	46.8	3.6
	Sample 5 :	8.4	68.8	47.2	4.4
	Sample 6 :	8.5	75.6	45.3	4.5
	Sample 7 :	7.9	64.2	44.2	3.8
	Sample 8 :	8.4	77.6	48.8	3.2
	Sample 9 :	9.1	79.5	42.1	3.2
C_1	*Sample* 10 :	8.4	77.8	49.8	4.6
	Sample 11 :	8.6	68.9	50.9	3.4
	Sample 12 :	8.2	78.3	41.1	4.0
	Sample 13 :	6.5	76.8	47.1	4.1
	Sample 14 :	8.6	77.7	45.8	3.6
	Sample 15 :	8.5	76.9	46.9	3.7
	Sample 16 :	8.5	70.3	47.9	3.3
	Sample 17 :	8.4	68.5	46.2	4.3
	Sample 18 :	8.5	78.2	43.2	4.4
	Sample 19 :	8.6	68.4	47.6	4.6
	Sample 20 :	8.2	77.2	43.1	3.9

		x_1	x_2	x_3	x_4
	$Sample\,1:$	7.0	83.8	48.3	3.9
	$Sample\,2:$	6.9	81.3	37.1	3.7
	$Sample\,3:$	5.1	88.9	43.7	3.5
	$Sample\,4:$	8.9	72.2	34.8	3.4
	$Sample\,5:$	6.8	82.5	50.2	3.6
	$Sample\,6:$	7.2	76.8	41.3	3.2
	$Sample\,7:$	8.7	73.3	39.3	4.1
	$Sample\,8:$	5.9	70.3	45.8	4.0
	$Sample\,9:$	8.4	75.1	46.4	3.3
C_2	$Sample\,10:$	7.5	84.8	42.8	3.8
	$Sample\,11:$	6.4	88.1	44.7	3.6
	$Sample\,12:$	6.1	80.7	40.1	3.7
	$Sample\,13:$	6.8	89.5	51.1	3.6
	$Sample\,14:$	6.0	87.5	47.8	3.6
	$Sample\,15:$	7.4	86.9	49.1	3.6
	$Sample\,16:$	7.6	79.4	36.6	3.7
	$Sample\,17:$	6.8	77.9	38.3	3.8
	$Sample\,18:$	6.9	73.1	36.2	3.4
	$Sample\,19:$	6.7	85.8	35.8	3.7
	$Sample\,20:$	6.9	74.6	49.6	3.5

By Formula (7) and (8), for C_1 we can work out

$$\begin{array}{llll} \mu_{11}=8.29 & \mu_{12}=74.49 & \mu_{13}=46.14 & \mu_{14}=3.91 \\ L\sigma_{11}=0.86 & L\sigma_{12}=6.36 & L\sigma_{13}=2.95 & L\sigma_{14}=0.36 \\ R\sigma_{11}=0.37 & R\sigma_{12}=3.41 & R\sigma_{13}=1.89 & R\sigma_{14}=0.43 \end{array}$$

and for C_2 we can work out

$$\begin{array}{llll} \mu_{21}=7.00 & \mu_{22}=80.63 & \mu_{23}=42.95 & \mu_{24}=3.64 \\ L\sigma_{21}=0.56 & L\sigma_{22}=5.89 & L\sigma_{23}=4.72 & L\sigma_{24}=0.16 \\ R\sigma_{21}=0.96 & R\sigma_{22}=4.81 & R\sigma_{23}=4.72 & R\sigma_{24}=0.18 \end{array}$$

By Formula (9) ($[\alpha_{ij}, \beta_{ij}) = [0, +\infty), i = 1, 2, j = 1, 2, 3, 4$ and $\lambda = 4$) we can construct fuzzy 4−ellipsoid number u_1 and u_2 as $u_1 = \begin{pmatrix} 4.85 & 8.29 & 9.77 \\ 49.05 & 74.49 & 88.13 \\ 34.34 & 46.14 & 53.70 \\ 2.47 & 3.91 & 5.63 \end{pmatrix}_{CF}$ and $u_2 = \begin{pmatrix} 4.76 & 7.00 & 10.84 \\ 57.07 & 80.63 & 99.87 \\ 24.07 & 42.95 & 61.83 \\ 3.00 & 3.64 & 4.36 \end{pmatrix}_{CF}$. Then "High-fertilizer High-yielding Wheat" and "Mid-fertilizer High-yielding Wheat" can be respectively represented by u_1 and u_2.

Suppose that a sample of wheat (denoted by O) has the following four state character values: $x_1 = 7.69$, $x_2 = 76.9$, $x_3 = 44.1$, $x_4 = 3.7$, i.e., $O = (7.69, 76.9, 44.1, 3.7)$. Then by the specific iterative algorithm set up in Section 4, we can write a computer program and work out $u_1(O) = u_1(7.69, 76.9, 44.1, 3.7) = 0.6717$ and $u_2(O) = u_2(7.69, 76.9, 44.1, 3.7) = 0.7584$ (where the control parameter δ of absolute errors is 0.0001), so we can know $u_2(O) > u_1(O)$. Therefore, by the maximal membership principle, we can consider that O belongs to C_2, i.e., O comes from "Mid-fertilizer High-yielding Wheat". This identifying result is more objective and rational than using fuzzy $n-$cell numbers (see the Example 4.2 in [30]).

Remark 5: As with "Korean pine accounts for the main part"(see Example 3), "High-fertilizer High-yielding Wheat" and "Mid-fertilizer High-yielding Wheat" should not also be certain crisp values but fuzzy sets without certain bound, so we should use fuzzy numbers to represent them. In Example 3 and Example 4, from $L\sigma \neq R\sigma$, we see the given uncertain multichannel digital information have no symmetry, so using Bandemer's bean fuzzy vectors (i.e., symmetric paraboloid fuzzy $n-$ellipsoid numbers here, see Remark 3) or Sato's conical fuzzy numbers (i.e., symmetric cone fuzzy $n-$ellipsoid numbers here, see Remark 3) to represent these uncertain multichannel digital information will become unwise. It indicates that the fields of application of quasi cone fuzzy $n-$ellipsoid number proposed in this paper are more extensive.

Remark 6: Example 3 and Example 4 show how the fuzzy ellipsoid numbers to express a group of uncertain multichannel digital information are constructed, and Example 4 also tell us that using fuzzy ellipsoid numbers to express uncertain multichannel digital information are more objective and more rational than using fuzzy cell numbers. However, the two examples do not tell more us how to use fuzzy ellipsoid numbers to deal with uncertain multichannel digital information in practical applications except the identification introduced in the last part of Example 4. In fact, after the fuzzy ellipsoid numbers expressing uncertain multichannel digital information are constructed, we can respectively achieve the identification, classification and ranking of uncertain multichannel digital information through the establishment of specific aggregation operators in the fuzzy ellipsoid number space like [29], [30], [32], [31] in the fuzzy cell number space. We can also use fuzzy ellipsoid numbers to cluster uncertain multichannel digital information like M. Sato and Y. Sato use the $n-$dimensional fuzzy numbers whose membership functions are all symmetrical conical to cluster fuzzy data in [24]. Since ellipsoid numbers have more advantages than fuzzy cell numbers (see Example 2 and 4 and the third paragraph of Introduction) and Sato's conical type fuzzy numbers which have the limitation of symmetry in expressing uncertain multichannel digital information, we have reason to believe that using fuzzy ellipsoid numbers to deal with uncertain multichannel digital information can obtain more objective and more rational results than using fuzzy cell numbers or Sato's conical type fuzzy numbers (see Remark 5).

VI. Conclusion

In this paper, the defects or weaknesses were pointed out for using fuzzy cell numbers to express uncertain multichannel digital information (see the paragraph 3 of Section 1 and Example 2), and the concept of fuzzy ellipsoid numbers was proposed, that are also a special kind dimensional fuzzy numbers. Fuzzy ellipsoid numbers not only overcome the defects or weaknesses of fuzzy cell numbers but also keep the convenience being used in applications and researches of theory to some extent.

Using fuzzy ellipsoid numbers to express uncertain multichannel digital information is more objective and more rational than using fuzzy cell numbers. In theory, two representation theorems of fuzzy ellipsoid numbers were obtained, it can provide convenience for applications and researches of theory. Then, some special kind of fuzzy ellipsoid numbers were defined, their properties and relationships were investigated, a specific iterative algorithm of their membership function value was set up in accordance by the golden section method, and the convergence of the iterative algorithm was proved (see Theorem 7 and the proof of Theorem 6). And then we established an algorithmic version of constructing fuzzy ellipsoid numbers to express a object which is characterized by a group of uncertain multichannel digital information, and also given practical examples to show the application and rationality of the proposed techniques. Of course, fuzzy ellipsoid numbers may also have some defects. For example, the general addition and multiplication of fuzzy ellipsoid numbers does not preserve the closeness of the operation (since the addition and multiplication of ellipsoids does not preserve the closeness of the operation), this may cause some difficulties in researches of theory. Another example, generally, it is also difficult to obtain the mathematical formulas (analytical expressions) of fuzzy ellipsoid numbers, this may also bring some inconvenience in the studying of theory and application.

Appendix A
Proofs of Theorem 2 and Theorem 6

A. The proof of Theorem 2:

Let $u\in E(E^n)$. By the definition of fuzzy $n-$ellipsoid numbers, it is obvious that for each $i=1,2,\cdots,n$, $\underline{u_i}(r)$ and $\overline{u_i}(r)$ are real-valued functions on $[0,1]$. First we show the Conclusions (1)-(4) hold. By (2) of Theorem 1, we have that $[u]^{r_2}\subset[u]^{r_1}$. It implies $\underline{u_i}(r_1)\le\underline{u_i}(r_2)$ and $\overline{u_i}(r_2)\le\overline{u_i}(r_1)$ for $0\le r_1\le r_2\le 1$, so $\underline{u_i}(r)$ is non-decreasing, and $\overline{u_i}(r)$ is non-increasing for each $i=1,2,\cdots,n$. Let $r_0\in(0,1]$, and $\{r_m\}$ be a positive non-decreasing sequence with $\lim_{m\to\infty}r_m=r_0$. By Theorem 1, we have

$$\begin{pmatrix} \underline{u_1}(r_0) & \overline{u_1}(r_0)\\ \underline{u_2}(r_0) & \overline{u_2}(r_0)\\ \vdots & \vdots\\ \underline{u_n}(r_0) & \overline{u_n}(r_0) \end{pmatrix} = [u]^{r_0} = \bigcap_{m=1}^{\infty}[u]^{r_m} = \lim_{m\to\infty}\begin{pmatrix} \underline{u_1}(r_m) & \overline{u_1}(r_m)\\ \underline{u_2}(r_m) & \overline{u_2}(r_m)\\ \vdots & \vdots\\ \underline{u_n}(r_m) & \overline{u_n}(r_m) \end{pmatrix} = \begin{pmatrix} \lim_{m\to\infty}\underline{u_1}(r_m) & \lim_{m\to\infty}\overline{u_1}(r_m)\\ \lim_{m\to\infty}\underline{u_2}(r_m) & \lim_{m\to\infty}\overline{u_2}(r_m)\\ \vdots & \vdots\\ \lim_{m\to\infty}\underline{u_n}(r_m) & \lim_{m\to\infty}\overline{u_n}(r_m) \end{pmatrix}$$

so, $\underline{u_i}(r_0)=\lim_{m\to\infty}\underline{u_i}(r_m)$, $\overline{u_i}(r_0)=\lim_{m\to\infty}\overline{u_i}(r_m)$ for all $i=1,2,\cdots,n$. Therefore, conclusions (1) and (2) have been shown. Conclusion (3) can be directly obtained by the definition of fuzzy $n-$ellipsoid numbers. Let $\{r_m\}$ be a positive non-increasing sequence with $\lim_{m\to\infty}r_m=0$. From

$$\begin{pmatrix} \underline{u_1}(0) & \overline{u_1}(0)\\ \underline{u_2}(0) & \overline{u_2}(0)\\ \vdots & \vdots\\ \underline{u_n}(0) & \overline{u_n}(0) \end{pmatrix} = [u]^0 = \overline{\{x\in R^n|u(x)>0\}} = \overline{\bigcup_{m=1}^{\infty}[u]^{r_m}} = \lim_{m\to\infty}\begin{pmatrix} \underline{u_1}(r_m) & \overline{u_1}(r_m)\\ \underline{u_2}(r_m) & \overline{u_2}(r_m)\\ \vdots & \vdots\\ \underline{u_n}(r_m) & \overline{u_n}(r_m) \end{pmatrix} = \begin{pmatrix} \lim_{m\to\infty}\underline{u_1}(r_m) & \lim_{m\to\infty}\overline{u_1}(r_m)\\ \lim_{m\to\infty}\underline{u_2}(r_m) & \lim_{m\to\infty}\overline{u_2}(r_m)\\ \vdots & \vdots\\ \lim_{m\to\infty}\underline{u_n}(r_m) & \lim_{m\to\infty}\overline{u_n}(r_m) \end{pmatrix}$$

so $\underline{u_i}(0)=\lim_{m\to\infty}\underline{u_i}(r_m)$, $\overline{u_i}(0)=lim_{m\to\infty}\overline{u_i}(r_m)$ for all $i=1,2,\cdots,n$. Therefore, conclusion (4) has been also shown.

Conversely, let $a_i(r)$ and $b_i(r)$ $(i=1,2,\cdots,n)$ be real-valued functions on $[0,1]$ and satisfy the conditions (a)-(d). Let $A_r=\begin{pmatrix} a_1(r) & b_1(r)\\ a_2(r) & b_2(r)\\ \vdots & \vdots\\ a_n(r) & b_n(r) \end{pmatrix}$ for any $r\in[0,1]$. From the conditions (a)-(c), it can be easily seen that A_r, $r=1,2,\cdots,n$, satisfy the conditions (a) and (b) of Theorem 1. Let $r\in(0,1]$, and $\{r_m\}$ be a positive non-decreasing sequence with $\lim_{m\to\infty}r_m=r$. From the left continuity of $a_i(r)$ and $b_i(r)$, the non-decrease of $a_i(r)$ and the non-increase of $b_i(r)$ $(i=1,2,\cdots,n)$, we have

$$\bigcap_{m=1}^{\infty}A_{r_m} = \lim_{m\to\infty}\begin{pmatrix} a_1(r_m) & b_1(r_m)\\ a_2(r_m) & b_2(r_m)\\ \vdots & \vdots\\ a_n(r_m) & b_n(r_m) \end{pmatrix} = \begin{pmatrix} \lim_{m\to\infty}a_1(r_m) & \lim_{m\to\infty}b_1(r_m)\\ \lim_{m\to\infty}a_2(r_m) & \lim_{m\to\infty}b_2(r_m)\\ \vdots & \vdots\\ \lim_{m\to\infty}a_n(r_m) & \lim_{m\to\infty}b_n(r_m) \end{pmatrix} = \begin{pmatrix} a_1(r) & b_1(r)\\ a_2(r) & b_2(r)\\ \vdots & \vdots\\ a_n(r) & b_n(r) \end{pmatrix} = A_r$$

so A_r, $r=1,2,\cdots,n$, satisfy also the condition (c) of Theorem 1. By Theorem 1, it can be shown that there exists a unique $u\in E(E^n)$ such that $[u]^r=A_r$ for $r\in(0,1]$. From the right continuity of $a_i(r)$ and $b_i(r)$ at $r=0$, we can also obtained $[u]^0=A_0$, so, $[u]^r=A_r$ hold for all $r\in[0,1]$. The proof is completed.

B. The proof of Theorem 6:

Let u be a quasi cone fuzzy $n-$ellipsoid number, $x^0=(x_1^0,\cdots,x_n^0)\in[u]^0$ and $q\in(0,1)$. Denote $\xi_1=0$,

$\eta_1=1$ and $r_1=(1-q)\xi_1+q\eta_1=q$, we have $\xi_1<r_1<\eta_1$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\xi_1)+\underline{u_i}(\xi_1)}{2})^2}{(\frac{\overline{u_i}(\xi_1)-\underline{u_i}(\xi_1)}{2})^2}\leq 1$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\eta_1)+\underline{u_i}(\eta_1)}{2})^2}{(\frac{\overline{u_i}(\eta_1)-\underline{u_i}(\eta_1)}{2})^2}\geq 1$ since $x^0\in[u]^0$ and $x^0\in R^n=\overline{\{x\in R^n \mid x\in[u]^1\}}$. Then denote $\xi_2=\xi_1,\ \eta_2=r_1$ as $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2}>1$, and $\xi_2=r_1,\ \eta_2=\eta_1$ as $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\underline{u_i}(r_1)+\overline{u_i}(r_1)}{2})^2}{(\frac{\underline{u_i}(r_1)-\overline{u_i}(r_1)}{2})^2}\leq 1$. And denote $r_2=(1-q)\xi_2+q\eta_2$, then we have still $\xi_2<r_2<\eta_2$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\xi_2)+\underline{u_i}(\xi_2)}{2})^2}{(\frac{\overline{u_i}(\xi_2)-\underline{u_i}(\xi_2)}{2})^2}\leq 1$ and $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\eta_2)+\underline{u_i}(\eta_2)}{2})^2}{(\frac{\overline{u_i}(\eta_2)-\underline{u_i}(\eta_2)}{2})^2}\geq 1$. And then denote $\xi_3=\xi_2,\ \eta_3=r_2$ as $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2}>1$, while, denote $\xi_3=r_2,\ \eta_3=\eta_2$ as $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\underline{u_i}(r_2)+\overline{u_i}(r_2)}{2})^2}{(\frac{\underline{u_i}(r_2)-\overline{u_i}(r_2)}{2})^2}\leq 1$. Denote $r_3=(1-q)\xi_3+q\eta_3$, we have also $\xi_3<r_3<\eta_3$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\xi_3)+\underline{u_i}(\xi_3)}{2})^2}{(\frac{\overline{u_i}(\xi_3)-\underline{u_i}(\xi_3)}{2})^2}\leq 1$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\eta_3)+\underline{u_i}(\eta_3)}{2})^2}{(\frac{\overline{u_i}(\eta_3)-\underline{u_i}(\eta_3)}{2})^2}\geq 1$. ...... So always go on, we can obtain $\xi_m,\eta_m,r_m\in[0,1]$ with $\xi_m<r_m<\eta_m$ such that $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\xi_m)+\underline{u_i}(\xi_m)}{2})^2}{(\frac{\overline{u_i}(\xi_m)-\underline{u_i}(\xi_m)}{2})^2}\leq 1$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\eta_m)+\underline{u_i}(\eta_m)}{2})^2}{(\frac{\overline{u_i}(\eta_m)-\underline{u_i}(\eta_m)}{2})^2}\geq 1$ for any $m=1,2,\cdots$.

By the definitions of ξ_m,η_m,r_m, it is obvious that $[\xi_{m+1},\eta_{m+1}]\subset[\xi_m,\eta_m]$ $(m=1,2,\cdots)$ and $\lim_{m\to\infty}(\eta_m-\xi_m)=0$. By Closed Interval Theorem, we know that there exist r_0 with $\lim_{m\to\infty}\xi_m=\lim_{m\to\infty}\eta_m=r_0$ (By Squeeze Theorem, $\lim_{m\to\infty}r_m=r_0$) and $r_0\in[\xi_m,\eta_i]\subset[0,1]$ for all $m=1,2,\cdots$. By the continuity of $\underline{u_i}(r)$ and $\overline{u_i}(r)$ $(i=1,2,\cdots,n)$, we have $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}=\lim_{m\to\infty}\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\xi_m)+\underline{u_i}(\xi_m)}{2})^2}{(\frac{\overline{u_i}(\xi_m)-\underline{u_i}(\xi_m)}{2})^2}\leq 1$, $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}=\lim_{m\to\infty}\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\eta_m)+\underline{u_i}(\eta_m)}{2})^2}{(\frac{\overline{u_i}(\eta_m)-\underline{u_i}(\eta_m)}{2})^2}\geq 1$, so $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}=1$. The existence of r_0 have been shown. In the following, we show the uniqueness of r_0.

It is obvious that the uniqueness is valid as $r_0=1$. As $r_0\in[0,1)$, we assume the uniqueness is not valid, then there exist $\grave{r}_0\in[0,1]$ with $\grave{r}_0\neq r_0$ such that $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\grave{r}_0)+\underline{u_i}(\grave{r}_0)}{2})^2}{(\frac{\overline{u_i}(\grave{r}_0)-\underline{u_i}(\grave{r}_0)}{2})^2}=1$. Without loss of generality, let $\grave{r}_0<r_0$. Then by the strictly monotonicity of $\underline{u_i}(r)$ and $\overline{u_i}(r)$, we can see that $\underline{u_i}(\grave{r}_0)<\underline{u_i}(r_0)<\overline{u_i}(r_0)<\overline{u_i}(\grave{r}_0)$. Therefore, By Lemma 1 and from $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(\grave{r}_0)+\underline{u_i}(\grave{r}_0)}{2})^2}{(\frac{\overline{u_i}(\grave{r}_0)-\underline{u_i}(\grave{r}_0)}{2})^2}=1$, we know $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}>1$, this is in contradiction to $\sum_{i=1}^{n}\frac{(x_i^0-\frac{\overline{u_i}(r_0)+\underline{u_i}(r_0)}{2})^2}{(\frac{\overline{u_i}(r_0)-\underline{u_i}(r_0)}{2})^2}=1$. Thus we have completed the proof of the uniqueness of r_0 by reduction to absurdity.

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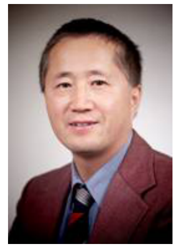

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