Einstein and conformally Einstein bi-invariant semi-Riemannian metrics

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Thesis submitted in partial fulfilment
of the requirements for the degree of
Master of Philosophy
in
Pure Mathematics
at
The University of Adelaide

THE UNIVERSITY
of ADELAIDE

School of Mathematical Sciences

September 2015

Signed Statement

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Contents

Abstract

This thesis considers the geometric properties of bi-invariant metrics on Lie groups. On simple Lie groups, we show that there is always an Einstein bi-invariant metric; that when the Lie algebra is of complex type, there is another metric on a simple Lie group that is Bach-flat but not conformally Einstein and that when the metric is a linear combination of these aforementioned metrics, that the metric is not Bach-flat. This result can be used to describe all bi-invariant metrics on reductive Lie groups.

The thesis then considers bi-invariant metrics on Lie groups when the Lie algebra is created through a double extension procedure, as described initially by Medina [25]. We show two examples of bi-invariant metrics on non-reductive Lie groups that are Bach-flat but not conformally Einstein, however, we show that all Lorentzian bi-invariant metrics are conformally Einstein.

Dedication

To C.B.,

This will give us plenty to talk about.

Acknowledgements

This thesis would never have got off the ground without the guidance, patience and expertise of my supervisor, Dr Thomas Leistner. He has lead me through the world of manifolds, semi-Riemannian geometry, Lie groups and Lie algebras, conformal geometry and bi-invariant metrics. His mentoring has been invaluable to me. Thank-you for all of your hard work Thomas.

Thanks also to my supervisor Professor Michael Murray for the corrections on multiple drafts and very sound advice on multiple topics.

Thanks to Dr Ray Vozzo for first introducing me to the world of pure maths through many discussions over coffee with Vincent Schlegel and Tyson Ritter. Without you three, I'm not sure I would have found my way quite so easily into pure mathematics. It was also Ray that suggested I ask Thomas to supervise me, and I thank him for that.

A few others have offered wise words of advice. Dr Jono Tuke, for the ever encouraging nudge towards statistics and valuable discussions on communication. Professor Nigel Bean, for mentoring me in the ways of the maths department and somehow always knowing the answers. Dr Nicholas Buchdahl, for your enthusiasm and pure joy when talking about mathematics; I was very lucky that Nick was my lecturer for two of my honours courses, ran a seminar series on geometric analysis which I learnt so much from, and could talk knowledgeably about any mathematical topic I came to him with.

Research can run rather slowly sometimes and it is with help from friends that can make the experience much more enjoyable. To my level 6 friends: Annie Conway, Steven Wade, Michael Lydeamore, Brett Chenoweth, Jo Varney, Andy Pfeiffer, Lyron Winderbaum, and many others, thank-you.

To my parents, thank-you for the roast dinners every Sunday evening, plus the leftovers that would last me through half of the following week. Thank-you also for always encouraging my diversity of interests.

At this point, I am also becoming extremely excited about my future studies, which I am to begin in October 2015. I would like to acknowledge the six referees who wrote references for me to apply for the Rhodes scholarship. They are: Mr Mike Mickan, Detective Sergeant Bernadette Martin PM, Professor Nigel Bean, Dr Jono Tuke, Dr Nick Buchdahl and Dr Thomas Leistner.

Finally, one of the most enjoyable aspects of my masters has been teaching and mentoring other students. Thank-you to all of the students who have contributed to this. And, to any students who may come across this thesis, I wish you all the best for you future endeavours and I do hope you continue to use mathematics in this ever changing and ever challenging world.