

Einstein and conformally Einstein bi-invariant semi-Riemannian metrics

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Signed Statement

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Contents

Signed Statement	iii
Abstract	vii
Dedication	ix
Acknowledgements	xi
Introduction	xiii
1 Semi-Riemannian metrics and their curvature	1
1.1 Connections and curvature	2
1.2 Ricci and scalar curvature	7
1.3 Differential operators and more on curvature	9
1.4 The Schouten, Weyl, Cotton and Bach tensors	11
1.5 Conformally Einstein metrics	16
1.6 Obstructions to the metric being conformally Einstein	21
1.7 Conclusion	22
2 Bi-invariant metrics	23
2.1 Actions, invariance and bi-invariant metrics	23
2.2 Riemannian metric Lie algebras	26
2.3 Simple metric Lie algebras	27
2.4 Conclusion	32
3 Einstein and conformally Einstein bi-invariant metrics	35
3.1 The curvature tensors on Lie groups with bi-invariant metrics	35
3.2 The Schouten, Cotton, Bach and Weyl tensors and conformal to Einstein obstructions	37
3.3 Bi-invariant metrics with 2-step nilpotent Ricci tensor	41
3.4 Simple groups with bi-invariant metrics	42
3.5 Conclusion	47
4 Metric Lie algebras and the double extension procedure	49
4.1 Double extension of metric Lie algebras	50
4.2 Solvable metric Lie algebras	53
4.2.1 Nilpotent Lie algebras	54

4.3	Double extensions by 1-dimensional Lie algebras	56
4.3.1	Double extensions of abelian metric Lie algebras by 1-dimensional Lie algebras	60
4.4	Bach tensor for double extensions of signature $(2, n - 2)$	65
4.5	First obstruction for double extensions of signature $(2, n - 2)$	72
4.6	Conclusion	77
Conclusion and future research		79
	Future research	80
A APPENDICES		83
A.1	Vector fields and tensors	83
A.2	Contractions	84
A.3	Tensor derivations	86
A.4	Geodesics, the exponential map and normal coordinate systems	87
A.5	Proof of the Weyl tensor symmetries	89
A.6	Lie groups and Lie algebras	91
A.6.1	Conjugation and adjoint representations	92
A.7	Simple, semisimple and reductive Lie algebras	93
A.8	Elements of $\mathfrak{so}(\mathfrak{g}) \cap \text{GL}(\mathfrak{g})$	95
A.9	Nilpotent and solvable Lie Algebras	96
A.9.1	Properties of nilpotent Lie algebras	97
A.9.2	Properties of Solvable Lie algebras	98
A.9.3	Engel's Theorem	98
A.9.4	Lie's Theorem	99
A.10	Representations	99
A.10.1	Complexifications, realifications and real forms	101
A.11	The space of endomorphisms and the space of bilinear forms	103
A.12	Some proofs for Chapter 4	106
A.13	Notes on proof of Lemma 3.15	110
A.13.1	Abstract Jordan decomposition	110
A.13.2	Cartan subalgebras and toral subalgebras	112
Bibliography		113

Abstract

This thesis considers the geometric properties of bi-invariant metrics on Lie groups. On simple Lie groups, we show that there is always an Einstein bi-invariant metric; that when the Lie algebra is of complex type, there is another metric on a simple Lie group that is Bach-flat but not conformally Einstein and that when the metric is a linear combination of these aforementioned metrics, that the metric is not Bach-flat. This result can be used to describe all bi-invariant metrics on reductive Lie groups.

The thesis then considers bi-invariant metrics on Lie groups when the Lie algebra is created through a double extension procedure, as described initially by Medina [25]. We show two examples of bi-invariant metrics on non-reductive Lie groups that are Bach-flat but not conformally Einstein, however, we show that all Lorentzian bi-invariant metrics are conformally Einstein.

Dedication

To C.B.,
This will give us plenty to talk about.

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