Einstein and conformally Einstein bi-invariant semi-Riemannian metrics

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Signed Statement

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Abstract

This thesis considers the geometric properties of bi-invariant metrics on Lie groups. On simple Lie groups, we show that there is always an Einstein bi-invariant metric; that when the Lie algebra is of complex type, there is another metric on a simple Lie group that is Bach-flat but not conformally Einstein and that when the metric is a linear combination of these aforementioned metrics, that the metric is not Bach-flat. This result can be used to describe all bi-invariant metrics on reductive Lie groups.

The thesis then considers bi-invariant metrics on Lie groups when the Lie algebra is created through a double extension procedure, as described initially by Medina [25]. We show two examples of bi-invariant metrics on non-reductive Lie groups that are Bach-flat but not conformally Einstein, however, we show that all Lorentzian bi-invariant metrics are conformally Einstein.

Dedication

To C.B.,

This will give us plenty to talk about.

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This thesis would never have got off the ground without the guidance, patience and expertise of my supervisor, Dr Thomas Leistner. He has lead me through the world of manifolds, semi-Riemannian geometry, Lie groups and Lie algebras, conformal geometry and bi-invariant metrics. His mentoring has been invaluable to me. Thank-you for all of your hard work Thomas.

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Introduction

It is well known that Lie groups with bi-invariant Riemannian metrics are always isometric to a product of Einstein manifolds. However, for indefinite metrics, this statement is far from true: there are many bi-invariant metrics, for example Lorentzian metrics, that are not Einstein. Instead, one may pose the question: is every bi-invariant metric, g, always conformal to an Einstein metric? That is, is there a positive function f such that the metric fg is an Einstein metric? We will see that this holds when the metric is Lorentzian, however we give two cases where the signature of the metric is $(\frac{n}{2}, \frac{n}{2})$ and $(2, n-2)$ in which this fails. We also reveal an interesting result about Bach-flat metrics; we show that there are Bach-flat bi-invariant metrics that are not conformally Einstein, and we give an example of a linear combination of Bach-flat metrics that is not Bach flat.

The example of a bi-invariant metric that is not conformally Einstein of signature $(\frac{n}{2}, \frac{n}{2})$ occurs in the literature in Medina [25, pg. 410]. This is an example of a bi-invariant metric on a simple Lie group that is not Einstein. In this instance, one considers a simple Lie group which is the realification of a complex simple Lie group. The metric is induced by the imaginary part of the Killing form from the complex Lie group. In addition to the observation that the metric is not Einstein, we show that this metric is Bach-flat and is not conformal to an Einstein metric.

The second class of examples is found in the realm of non-reductive Lie groups with bi-invariant metrics. It is derived from a solvable metric Lie algebra, created through a 'double extension procedure'. While presented in several sources, the classification of signature $(2, n-2)$ metric Lie algebras appears concisely in Baum and Kath [3, pg. 267] as well as a description of the double extension procedure. The geometry of the connected Lie groups induced by the metric Lie algebras is explored in this thesis, again showing that two cases can occur: one class that is conformal to an Einstein metric and two classes of metrics that are Bach-flat but *not* conformally Einstein.

Bach-flat metrics are of interest in differential geometry as they are a natural generalisation of Einstein metrics. In dimension 4, the Bach tensor is closely linked to conformal geometry as in this dimension it is conformally invariant, that is, it is invariant under conformal rescalings of the metric. In higher dimensions it also plays important roles; it appears in the ambient metric construction of conformal geometry as an ingredient of the conformally invariant obstruction tensor. In 2007, at the Midwest Geometry Conference, at the University of Iowa, in honour of Thomas P. Branson, a panel discussed open problems and the general direction of future research, see Peterson [32]. One particular topic was that of Bach-flat metrics; two panel members posed open questions about their geometry. In particular, the panel attendees were curious at to whether more Bach-flat but non-conformally Einstein metrics could be found.

The interest in Einstein metrics is easier to motivate; one need only read the introduction to Besse [4], to find three compelling reasons to study Einstein metrics. The first reason being that they are the natural way to extend the important properties of constant curvature in dimension 2 to higher dimensions without being too restrictive nor too general. The second, that they are the critical points of the total scalar curvature functional, whose simplicity and naturality point to Einstein metrics as optimising the geometry. The third, and from whence comes their name, is that they provide solutions to the Einstein field equations in a vacuum, with possibly non-zero cosmological constant. First proposed by Einstein in 1913, these field equations govern the interactions of space-time with gravity, mass and energy. In a vacuum, the equations reduce to finding a metric such that the Ricci tensor is a multiple of the metric.

When a metric is considered on a Lie group, one considers metrics that are compatible with the group structure. These are known as bi-invariant metrics. All reductive Lie groups, using the Killing form metric, are isometric to the direct sum of Einstein Lie groups. In fact, all Einstein Lie groups that are not Ricci-flat are semisimple. However, there is an abundance of non-semisimple Lie groups. The question is now, is there a way of generalising the Einstein condition to find metrics on other Lie groups that have similar properties? This points to conformally Einstein metrics, in which multiplying the bi-invariant metric by a non-zero smooth function results in another metric that is Einstein. Although no-longer bi-invariant, the conformally changed metric shares important properties with the original metric, such as angles. An example of a non-simple conformally Einstein Lorenztian Lie group is the Oscillator group. We describe several more examples which are Bach-flat but not conformally Einstein.

In summary, this thesis begins with a chapter introducing the background definitions and theorems from semi-Riemannian geometry. The aims of this section are to introduce notation, and to be a reference for preliminary material in semi-Riemannian geometry. This should allow readers with some background in differential geometry to be able to follow the thesis with reference to this chapter. The first half of this chapter covers the definition of a semi-Riemannian manifold, basic curvature definitions and results concerning the metric and curvature tensors, and follows the notation in O'Neill [29]. The second half defines Einstein metrics and conformally Einstein metrics, introduces the Schouten, Cotton, Weyl and Bach tensors, and proves important necessary conditions, which must be satisfied for conformally Einstein metrics. These are due to Gover and Nurowski [13].

The second chapter focuses on bi-invariant metrics on Lie groups. It contains the correspondence between connected Lie groups with bi-invariant metrics, called metric Lie groups, and Lie algebras with ad-invariant, symmetric, non-degenerate bilinear forms, known as metric Lie algebras. In light of this, the reader may conclude that one need only study metric Lie algebras to classify metric Lie groups. This chapter proceeds to show that all Riemannian metric Lie algebras are reductive, and then that all reductive metric Lie algebras can be orthogonally decomposed into their simple and abelian ideals. It then classifies simple metric Lie algebras into precisely two categories: real type and complex type. In the case of real type simple metric Lie algebras, the metric must be a multiple of the Killing form. In the case of complex type simple metric Lie algebras, the metric has signature $(\frac{n}{2}, \frac{n}{2})$ and comes from a 2-dimensional space of bilinear forms, where the real and imaginary parts of the Killing form of a complex Lie algebra form a basis for this space.

The third chapter combines the results of the second and third chapters, deducing results concerning Einstein metrics and conformally Einstein metrics on metric Lie algebras. It shows that the Killing form is an Einstein metric and that only semi-simple Lie groups can be equipped with Einstein bi-invariant metrics with non-zero Einstein constant. It then gives simplified formulas for the Schouten, Cotton, Weyl and Bach tensors in the case the metric is bi-invariant, and also simplifies the obstructions to conformally Einstein metrics in this case, noting that one of the obstructions reduces to the metric being Bach-flat. This chapter also simplifies further the obstructions for solvable metric Lie algebras and shows they have 2-step nilpotent Ricci tensor. It then shows that any bi-invariant metrics that have 2-step nilpotent Ricci tensor are Bach-flat. The last theorem in this chapter proves that if a simple metric Lie algebra of complex type is equipped with the metric induced from the imaginary part of the Killing form, then this metric is Bach-flat but not conformally Einstein. We then give examples of linear combinations of Bach-flat metrics which are not Bach-flat.

The final chapter introduces the double extension procedure and how to use it to create metric Lie algebras. It reviews the work of Medina and Revoy [26], who have proved that any indecomposable metric Lie algebra is either simple, one dimensional, or a double extension of a metric Lie algebra by simple and one-dimensional Lie algebras. In particular, it shows how to create several solvable metric Lie algebras, which are hence Bach-flat, and then shows whether they are conformally Einstein or not, using the obstructions.

Finally, there is an extensive appendix attached to this thesis. The appendix began as notes on definitions that the author required to clarify her understanding. They now remain as reference material and reminders for those who are less comfortable with the notations introduced from semi-Riemannan geometry and Lie theory, with some excerpts from representation theory. Some lengthy proofs, or proofs that do not provide significant insight, have been omitted from the main text and also included here.

Chapter 1

Semi-Riemannian metrics and their curvature

The aim of this section is to introduce the main concepts in semi-Riemannian geometry and fix the notation used in this thesis. This section also provides background information for those who have basic differential geometry knowledge, but may have limited knowledge of the results in the area of Einstein metrics and conformal geometry. The main reference for this section is O'Neill [29].

We denote by M a smooth manifold of dimension n. We denote by $\mathfrak{F}(M)$, the set of smooth real-valued functions on M, $\mathfrak{F}(M) = \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}$, and we denote by $\mathfrak{X}(M)$ the set of smooth vector fields on M, $\mathfrak{X}(M) = \{X : M \to TM \mid X(p) \in T_pM, X \text{ is smooth}\}$.

Recall from linear algebra that we can consider the *signature*, (p, q), of a non-degenerate symmetric bilinear form, b, on V, a finite dimensional vector space. Here, p and q are integers such that $p+q = n$ and p, q refer to the number of negative and positive eigenvalues of b respectively. Note that we can always consider $p \leq q$ as if $p > q$ we can consider $-b$, which has signature (q, p).

Definition 1.1. A *metric tensor*, g, is a non-degenerate $(0, 2)$ tensor field on M of constant signature.

That is, for any $p \in M$, the metric tensor at p, denoted g_p, is a non-degenerate, symmetric bilinear form on T_pM. It has the same signature for all points p and is non-degenerate. The common signature, (p, q), is called the *signature of* g. When the manifold is connected, non-degenerate $(0, 2)$ tensor fields always have constant signature, so this is often omitted in the definition. See also the definition of tensor and tensor field in Appendix A.1.

Locally, on a neighbourhood U, we can form a coordinate vector field basis of the tangent bundle[1], ∂_k and write $g_{ij} = g(\partial_i, \partial_j) \in \mathfrak{F}(U)$, so that if $V = V^i\partial_i$, and $W = W^i\partial_i$ then $g(V, W) = g_{ij}V^iW^j$. At each point $p \in M$, we can view the metric at this point, g_p as a matrix. As the metric is non-degenerate, g_p is invertible and we write g_p^{-1} as the inverse of this matrix. Then the entries of g_p^{-1} we denote g_p^{ij}. As this can be done at every

[1] The symbol ∂_k is a short hand for the coordinate vector field basis $\frac{\partial}{\partial x^k}$ defined by a coordinate functions $x^k : U \to \mathbb{R}$. See O'Neill [29, pg. 8] for further details on why this forms a basis.

point, this inverse then defines a $(2,0)$ tensor, g^{-1}, on the manifold. Locally, we write g^{ij} for the coefficients of the $(2,0)$ tensor locally.

Definition 1.2. A *semi-Riemannian manifold* is a smooth manifold M furnished with a metric tensor.

When the signature is $(0,n)$, M is a called a *Riemannian manifold* and each g_p is a positive definite inner product on T_pM. If the signature is (p,q) where $p \neq 0$, M is often referred to as a *pseudo-Riemannian manifold.* If the metric has signature $(1,n-1)$ where $n \geq 2$, the manifold is called a *Lorentzian manifold.*

1.1 Connections and curvature

Although locally diffeomorphic to $\mathbb{R}^n$, manifolds do not generally behave like vector spaces in the sense that they may be "curved". For instance, the sum of angles in a triangle will not be π when the triangle is drawn on a sphere.

Carl Friedrich Gauss was the first to introduce notions to describe how "curved" surfaces are using derivatives. Many books contain summaries of his work, for instance Spivak [36].

The connection describes a way to differentiate vector fields (and to differentiate tensors in general). In single variable calculus, the convexity of the graph of a function is determined by the second derivative. This leads to the definition of curvature as a "second derivative".

Definition 1.3. A *connection*, D, on a smooth manifold, M, is a function $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(D1) $D_V W$ is $\mathfrak{F}(M)$-linear in V,

(D2) $D_V W$ is $\mathbb{R}$-linear in W, and

(D3) $D_V(fW) = (Vf)W + fD_V W$ for $f \in \mathfrak{F}(M)$. This is known as the *Leibniz rule* and implies D is not a tensor in W.

We call $D_V W$ the *covariant derivative* of W with respect to V for the connection D.

Theorem 1.1. *On a semi-Riemannian manifold, M, there is a unique connection ∇ such that for all $X, V, W \in \mathfrak{X}(M)$*

(D4) $[V,W] = \nabla_V W - \nabla_W V$*, a property known as* torsion free*, and,*

(D5) $(\nabla_X g)(V,W) = Xg(V,W) - g(\nabla_X V, W) - g(V, \nabla_X W) = 0$*, that is, the connection is compatible with the metric. Here we use the covariant derivative of a* $(0,2)$ *tensor, which is defined in Appendix A.3.*

This unique connection ∇ is called the Levi-Civita connection *of M, and is characterised by the* Koszul formula *which holds for all $V, W, X \in \mathfrak{X}(M)$ as follows*

$$2g(\nabla_V W, X) = Vg(W,X) + Wg(X,V) - Xg(V,W) - g(V,[W,X]) + g(W,[X,V]) + g(X,[V,W]).$$

Proof. The sketch of the proof follows. For existence, fix W, V and check that the right-hand-side of the Koszul formula is $\mathfrak{F}(M)$-linear in X. By Proposition A.1 then there is a unique vector field, $\nabla_V W$ satisfying the Koszul formula.
Using the Koszul formula, we check that the axioms (D4)-(D5) hold for this vector field, thence ∇ is the Levi-Civita connection.
Now if $\nabla_V W$ is a vector field satisfying (D4) and (D5). Then on the right-hand-side of the Koszul formula, apply (D4) and (D5) both three times to get the left-hand-side, and hence the Koszul formula can be derived from the properties of the Levi-Civita connection. $\square$

For the full proof, see O'Neill [29, pg. 61].

Remark 1.1 (Covariant Derivative). We will use the tensor derivation known as the covariant derivative with the same symbol ∇. This is described in Definition A.5. In particular, it is defined such that $\nabla f(V) = \nabla_V f = df(V) = V(f)$ for $f \in \mathfrak{F}(M)$ and $V \in \mathfrak{X}(M)$, and that $\nabla W(V) = \nabla_V W$ is the Levi-Civitia connection on on $W \in \mathfrak{X}(M)$ with respect to V. However the covariant derivative is also defined on tensors of any type and is a generalisation of the Levi-Civita connection.

Definition 1.4. Let $x^1, \ldots, x^n$ be a coordinate system on a neighbourhood U in a semi-Riemannian manifold. The Christoffel symbols for the coordinate system are the real valued functions Γ^k_{ij} on U such that

$$\nabla_{\partial_i}(\partial_j) = \Gamma^k_{ij}\partial_k \qquad (1 \leq i,\, j \leq n)$$

where ∇ is the Levi-Civita connection.

Note we are assuming the summation convention that repeated indices are summed over, unless otherwise specified.

From (D4), we have that $\nabla_{\partial_j}(\partial_i) = \nabla_{\partial_i}(\partial_j)$ as the bracket is zero for coordinate bases, and hence $\Gamma^k_{ij} = \Gamma^k_{ji}$ for each k. We will often use the notation $\nabla_i = \nabla_{\partial_i}$ where $\partial_i, \ldots, \partial_n$ are a local coordinate system.

Proposition 1.2. *For a coordinate system $x^1, \ldots, x^n$ then*

$$\nabla_{\partial_i}(W^j\partial_j) = \left\{ \frac{\partial W^k}{\partial x^i} + \Gamma^k_{ij} W^j \right\} \partial_k$$

and the Christoffel Symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}(g^{mk}(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij})).$$

Proof. The proof of the first part uses the definition of the connection, and the definition of the Christoffel symbols, with an index swap. The second part starts with the Koszul formula for $\partial_i, \partial_j, \partial_m$, through which the inner product terms are zero, then attacking both sides with g^{mk} to give the result, noting that $g^{mk} g_{am} = \delta^k_a$.. $\square$

For the full proof, see O'Neill [29, pg. 62].

Example 1.1. The Christoffel symbols on a sphere can be calculated using the coordinates given by stereographic projection onto $\mathbb{R}^2$ with coordinates X_1, X_2. Let (x, y, z) be coordinates on S^2, as a subset of $\mathbb{R}^3$. Then using stereographic projection, we find that

$$\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and

$$\varphi^{-1}(X_1, X_2) = \left(\frac{2X_1}{1+X_1^2+X_2^2}, \frac{2X_2}{1+X_1^2+X_2^2}, \frac{-1+X_1^2+X_2^2}{1+X_1^2+X_2^2}\right)$$

for $x, y, z \in S^2 \setminus \{(0,0,1)\}$. We can write

$$\begin{aligned}\frac{\partial}{\partial X_1} =& \frac{2}{(1+X_1^2+X_2^2)^2}\left((1+X_2^2-X_1^2)\frac{\partial}{\partial x} - 2X_2X_1\frac{\partial}{\partial y} + 2X_1\frac{\partial}{\partial z}\right)\\ =&(1-z-x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + x(1-z)\frac{\partial}{\partial z}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial X_2} =& \frac{2}{(1+X_1^2+X_2^2)^2}\left((-2X_1X_2)\frac{\partial}{\partial x} + (1+X_1^2-X_2^2)\frac{\partial}{\partial y} + 2X_2\frac{\partial}{\partial z}\right)\\ =& -xy\frac{\partial}{\partial x} + (1-z-y^2)\frac{\partial}{\partial y} + y(1-z)\frac{\partial}{\partial z},\end{aligned}$$

which gives

$$g_{11} = \frac{4}{(1+X_1^2+X_2^2)^2} = (1-z)^2 = g_{22} \quad \text{and} \quad g_{12} = 0 = g_{21}.$$

Using the formula for the Christoffel symbols in Proposition 1.2, we find

$$\begin{aligned}\Gamma_{11}^1 =& \frac{-2X_1}{1+X_1^2+X_2^2} = -x = \Gamma_{21}^2 = \Gamma_{12}^2 = -\Gamma_{22}^1\\ \Gamma_{22}^2 =& \frac{-2X_2}{1+X_1^2+X_2^2} = -y = \Gamma_{12}^1 = \Gamma_{21}^1 = -\Gamma_{11}^2.\end{aligned}$$

The following tensor, first described by Bernhard Riemann in 1854, characterises the curvature of a semi-Riemannian manifold. An excellent description of Riemann's early work in this area can be found in Spivak [35].

Lemma 1.3. *Let M be a semi-Riemmannian manifold with Levi-Civita connection ∇. The function $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ given by*

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

is a (1,3) tensor field on M called the Riemannian curvature tensor *of M.*

See the definition of tensor and tensor field in Appendix A.1.

Proof. Additivity holds from all the components being additive, so checking the construction is $\mathfrak{F}(M)$-linear in each component is all that is required to show R is a tensor. This is a simple exercise in seeing that any introduced derivatives of the function from the first term are subtracted from the second. $\square$

See O'Neill [29, pg. 74] for the full proof.

We can consider a tensor field as an $\mathbb{R}$-multilinear function on individual tangent vectors, $X, Y, Z \in T_pM$. We can do this by finding vector fields that evaluate to these tangent vectors at p, finding the tensor field value (as an element of $\mathfrak{F}(M)$) on these vector fields and then evaluating at the point p. It does not matter about the behaviour of the tensor field away from p, so often we pick ones that locally make the brackets of the vector fields zero or the Christoffel symbols zero to simplify calculations.

The following are symmetries of the Riemannian curvature tensor at a point. These are important properties that will be used in calculations in later chapters.

Proposition 1.4. *For elements $X, Y, Z, V \in T_pM$ of the tangent space at point $p \in M$, the following hold for the Riemann curvature tensor, R:*

1. $R_{XY}Z = -R_{YX}Z$,
2. $g(R_{XY}Z, V) = -g(R_{XY}V, Z)$,
3. $R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$. *This is known as the first Bianchi identity, and*
4. $g(R_{XY}Z, V) = g(R_{ZV}X, Y)$.

Proof. Sketch: Extend the tangent vectors to vector fields with all brackets zero, that is extend them with constant coefficients relative to the coordinate system. The first result follows from the vector field choice, the second uses the Koszul formula to cancel out terms, the third follows from the definition of the Riemann curvature tensor, and the fourth uses repeated application of the other three results. $\square$

See O'Neill [29, pg. 75] for the full proof.

Definition 1.5. When $R_{XY}Z \equiv 0$, we say the manifold is *flat*.

Lemma 1.5. *When the manifold has dimension $n = 1$, the manifold is always flat.*

Proof. This follows from the skew-symmetry of the curvature tensor R from the first property in Proposition 1.4. $\square$

Remark 1.2. We will use the convention of O'Neill [29, pg. 76,83] and hence write locally $dx^l(R_{\partial_i \partial_j}\partial_k) = R^l{}_{kij}$. Note that we also write $R_{lkij} := g_{lb}R^b{}_{kij} = g(R_{\partial_i\partial_j}\partial_k, \partial_l)$.

If V is a vector field, which we can consider as it a $(1,0)$ tensor with $V(\theta) = \theta(V)$, for θ a 1-form on M. We then denote $V^i = dx^i(V)$. That is, $V = V^i\partial_i$. We also use the notation $\nabla_{\nabla_b a}V^c = \nabla_{\nabla_{\partial_b}\partial_a}dx^c(V)$

Note that the following now holds

$$\begin{aligned}(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c =& \nabla_a\nabla_b V^c - \nabla_b\nabla_a V^c\\ =&(\nabla_a\nabla_b)V^c - (\nabla_b\nabla_a)V^c\\ =&\nabla_a(\nabla_b V^c) - \nabla_{\nabla_a b}V^c - \nabla_b(\nabla_a V^c) + \nabla_{\nabla_b a}V^c\\ =&-dx_c(R_{\partial_a\partial_b}V)\\ =&-R^c{}_{dab}V^d,\end{aligned}$$

which highlights the difference in notation to Gover and Nurowski [13].

Rewriting Proposition 1.4 in this notation we have

$$R^l{}_{ijk} = -R^l{}_{ikj} \tag{1.1.1}$$

$$g_{al}R^l{}_{ijk} = R_{aijk} = -R_{iajk} \tag{1.1.2}$$

$$R^t{}_{ijk} + R^t{}_{jki} + R^t{}_{kij} = 0 \tag{1.1.3}$$

$$R_{lijk} = R_{kjil}. \tag{1.1.4}$$

We introduce the use of brackets to imply that the indices are skewed over these terms as follows

$$T_{[a_1a_2\ldots a_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sign}(\sigma)T_{a_{\sigma(1)}a_{\sigma(2)}\ldots a_{\sigma(n)}}$$

Is this notation, Equation (1.1.3) becomes $R^d{}_{[abc]} = 0$

Proposition 1.6. *For the Riemann curvature tensor R, we have*

$$R_{\partial_i\partial_j}\partial_k = 2\Gamma^l_{k[i}\Gamma^m_{j]l}\partial_m + 2(\partial_{[j}\Gamma^l_{i]k})\partial_l. \tag{1.1.5}$$

Proof. Using (D4), we can write the Riemann curvature tensor in terms of the Christoffel symbols as follows.

$$\begin{aligned}R_{\partial_i\partial_j}\partial_k &= \nabla_{\nabla_{\partial_i}\partial_j}\partial_k - \nabla_{\nabla_{\partial_j}\partial_i}\partial_k - \nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) + \nabla_{\partial_j}(\nabla_{\partial_i}\partial_k)\\ &= \nabla_{\Gamma^l_{ij}\partial_l}\partial_k - \nabla_{\Gamma^l_{ji}\partial_l}\partial_k - \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) + D_{\partial_j}(\Gamma^l_{ik}\partial_l)\\ &= \nabla_{\Gamma^l_{ij}\partial_l}\partial_k - \nabla_{\Gamma^l_{ji}\partial_l}\partial_k - \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) + \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l)\\ &= -\Gamma^l_{jk}\nabla_{\partial_i}(\partial_l) - \partial_i(\Gamma^l_{jk})\partial_l + \partial_j(\Gamma^l_{ik})\partial_l + \Gamma^l_{ik}\nabla_{\partial_j}(\partial_l)\\ &= 2\Gamma^l_{k[i}\Gamma^m_{j]l}\partial_m + 2(\partial_{[j}\Gamma^l_{i]k})\partial_l.\end{aligned}$$

□

Proposition 1.7 (Second Bianchi Identity)**.** *If $X, Y, X \in T_p(M)$ then*

$$(\nabla_X R)_{YZ} + (\nabla_Z R)_{XY} + (\nabla_Y R)_{ZX} = 0.$$

Proof. Choose a normal coordinate system in a neighbourhood of p, see Definition A.13 for details. Extend X, Y, Z, V to vector fields in this neighbourhood with the extensions having constant coefficients with respect to this coordinated system. Then the brackets vanish; the Christoffel symbols vanish at p and this along with Proposition 1.2 implies all the covariant derivatives are 0. By the product rule in Equation (A.3.1),

$$(\nabla_Z R)_{XY} Z = \nabla_Z (R_{XY} V) - R_{(\nabla_Z X) Y} V - R_{X(\nabla_Z Y)} V - R_{XY} \nabla_Z V,$$

which implies the middle two terms vanish at p. Dropping V for convenience

$$\begin{aligned}(\nabla_Z R)_{XY} =& \nabla_Z (R_{XY}) - R_{XY}(\nabla_Z) \\ =& [\nabla_Z, R_{XY}] \\ =& [\nabla_Z, \nabla_{[X,Y]} + [\nabla_X, \nabla_Y]] \\ =& [\nabla_Z, [\nabla_X, \nabla_Y]].\end{aligned}$$

Using the Jacobi identity, the result follows. □

1.2 Ricci and scalar curvature

From the symmetries of R, there is only one contraction of the curvature tensor that is non-zero.

Definition 1.6. If R is the Riemannian curvature tensor of M, we define the *Ricci curvature tensor* as the $(0, 2)$ tensor $\mathrm{Ric}(X, Y) = \mathrm{tr}\{V \mapsto R_{XV} Y\}$. This is known as a trace, or contraction, of the Riemann curvature tensor. Locally, picking a coordinate system $\partial_1, \ldots, \partial_n$, we define $R_{ij} := \mathrm{Ric}(\partial_i, \partial_j)$ and hence $R_{ab} = dx^c(R_{\partial_a, \partial_c} \partial_b) = R^c{}_{abc}$, using the convention of summing over repeated indices.

See also Remark 1.2 about which indices the contraction is over. Appendix A.2 contains more information on contractions in a broader sense.

Lemma 1.8. *The Ricci tensor is symmetric.*

Proof.

$$R_{ab} = R^c{}_{abc} = g^{ic} g_{ic} R^i{}_{abc} = g^{ic} R_{iabc} = g^{ic} R_{baic} = R^c{}_{bac} = R_{ba}$$

using Equation (1.1.4). □

Definition 1.7. The *scalar curvature* S of M is the metric contraction of the Ricci tensor. Note that this means one index must be raised using the metric and then the contraction taken. We write $S = R = R_i{}^i = g^{ij} R_{ij}$.

If a metric has $\mathrm{Ric} = 0$, it is called *Ricci-flat*, similarly if a tensor T is identically 0 on a manifold, the manifold is called *T flat.* It can immediately be seen that if the Riemann curvature tensor $R = 0$, that is if the metric is flat (from Definition 1.5), then it is Ricci-flat and the scalar curvature is also 0.

Both the Ricci and scalar curvature contain important information about the curvature of the manifold. For instance, solutions to the vacuum Einstein equations occur precisely when the Ricci tensor is a multiple of the metric. This case is particularly important and hence we define:

Definition 1.8. A metric, g, is called *Einstein* if $\mathrm{Ric} = \lambda g$ for some constant λ called the *Einstein constant* of g.

Note that immediately any Ricci-flat metric is Einstein with Einstein constant 0.

Lemma 1.9. *Whenever a metric is Einstein, the scalar curvature is $S = \lambda n$, where n is the dimension of the manifold and λ is the Einstein constant. Importantly, the scalar curvature is constant.*

Proof.
$$S = R^i{}_i = g^{ij} R_{ji} = c g^{ij} g_{ji} = \lambda \delta^i{}_i = \lambda n.$$
□

Example 1.2 (Sphere). We can equip S^2_r, the sphere of radius r, with the metric induced from the inner product in $\mathbb{R}^3$, $g = (dx)^2 + (dy)^2 + (dz)^2$. The Levi-Civita connection becomes
$$\nabla_X Y = X \circ Y + \frac{1}{r^2} g(X,Y) Id$$
That is, when evaluated at the point $p \in S^2_r$
$$\nabla_X Y|_p = X(Y(p)) + \frac{1}{r^2} g(X(p), Y(p)) p.$$
Using the definition of the Riemann curvature tensor, we have
$$R_{XY} Z = \frac{1}{r^2}(g(X,Z)Y - g(Y,Z)X).$$
We can write each tangent space of S^2_r as a subspace of the tangent space in $\mathbb{R}^3$. For example, at any point $p = (p_1, p_2, p_3) \in S^2_r$ except $(0,0,\pm r)$ and $(0, \pm r, 0)$ we can use the ordered basis
$$u^1|_p = p_2 \frac{\partial}{\partial x} - p_1 \frac{\partial}{\partial y} \ , \ u^2|_p = p_3 \frac{\partial}{\partial x} - p_1 \frac{\partial}{\partial z}.$$
That is, the vector fields
$$u^1 = y\frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \ , \ u^2 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}$$
are linear independent sections of TS^2_r except at $(0,0,\pm r)$ and $(0, \pm r, 0)$. Their duals are
$$U_1 = ydx - xdy \ , \ U_2 = zdx - xdz.$$
Note these are both the algebraic and metric duals as the metric is the dot product. In this basis, the metric becomes
$$[g_{ij}] = \begin{pmatrix} y^2 + x^2 & yz \\ yz & z^2 + x^2 \end{pmatrix}$$
with inverse
$$[g^{ij}] = \frac{1}{r^2 x^2} \begin{pmatrix} z^2 + x^2 & -yz \\ -yz & y^2 + x^2 \end{pmatrix}.$$
Then, we can write the Ricci tensor of X, Y as the trace of the map $V \to R_{XV} Y$. This map can be written as the matrix
$$\frac{1}{r^2} \begin{pmatrix} A_2 B_2 (z^2 + x^2) + A_1 B_2 (yz) & -(A_1 B_1 (yz) + A_2 B_1 (z^2 + x^2)) \\ -(A_1 B_2 (y^2 + x^2) + A_2 B_2 (yz)) & A_1 B_1 (y^2 + x^2) + A_2 B_1 (yz) \end{pmatrix}$$

where $X = A_1u^1 + A_2u^2$ and $Y = B_1u^1 + B_2u^2$, hence

$$\mathrm{Ric}(A_1u^1 + A_2u^2, B_1u^1 + B_2u^2) = \frac{1}{r^2}(A_2B_2(z^2 + x^2) + yz(A_1B_2 + A_2B_1) + A_1B_1(y^2 + x^2).$$

In matrix form, we have

$$[R_{ij}] = \frac{1}{r^2}\begin{pmatrix} y^2 + x^2 & y^2 \\ y^2 & z^2 + x^2 \end{pmatrix} = \frac{1}{r^2}[g_{ij}]$$

so (S^2_r, g) is Einstein, with Einstein constant of $\lambda = \frac{1}{r^2}$. Raising the first index, and letting $X^* = A_1U_1 + A_2U_2$ we have

$$\begin{aligned}\mathrm{Ric}(X^*, Y) =& \frac{(B_1(x^2 + y^2) - yzB_2)(A_1(z^2 + x^2) - A_2yz)}{x^2r^4} \\ &+ \frac{(B_2(x^2 + z^2) - yzB_1)(A_2(y^2 + x^2) - A_1yz)}{x^2r^4}.\end{aligned}$$

Hence the scalar curvature is

$$S = \mathrm{Ric}(U_1, u_1) + \mathrm{Ric}(U_2, u_2) = \frac{2}{r^2}$$

which is constant. This is of the form $n\lambda$ as required for an Einstein metric.

1.3 Differential operators and more on curvature

Definition 1.9. The *gradient*, $\mathrm{grad}\, f$ of a function $f \in \mathfrak{F}(M)$ is the vector field metrically equivalent to $df \in \mathfrak{X}(M)$. That is,

$$g(\mathrm{grad}\, f, X) = df(X) = Xf.$$

Definition 1.10. For a tensor A, the contraction of the new covariant slot in its covariant differential DA with one of its original slots is a called a *divergence*, $\mathrm{div}\, A$, of A. It is unique in the following two cases

- If V is a vector field, then $\mathrm{div}\, V = (\nabla_i V)^i = dx^i(\nabla_{\partial_i} V) \in \mathfrak{F}(M)$.
- If A is a symmetric $(0, 2)$ tensor, then $\mathrm{div}(A)_i = \nabla^j A_{ij} = g^{-1}(dx^k, dx^j)\nabla_{\partial_k} A(\partial_i, \partial_j)$

With this notation, we prove the twice contracted Bianchi Identity.

Corollary 1.10. *On a semi-Riemannian manifold, the Ricci tensor,* Ric *and scalar curvature S are connected by $2\nabla^i R_{ij} = \nabla_j S$. This is also written as $dS = 2\,\mathrm{div}\,\mathrm{Ric}$.*

This is a corollary to the 2nd Bianchi identity. Note that we use the convention that $\nabla f(V) = \nabla_V f = df(V) = V(f)$ for $f \in \mathfrak{F}(M)$ and $V \in \mathfrak{X}(M)$ as in Remark 1.1. This uses the definition of covariant derivative, which can be found in Definition A.5.

Proof. The 2nd Bianchi is

$$\nabla_i R^t{}_{ljk} + \nabla_k R^t{}_{lij} + \nabla_j R^t{}_{lki} = 0.$$

Contracting over t and k we have

$$\begin{aligned}\nabla_i R^k{}_{ljk} + \nabla_k R^k{}_{lij} + \nabla_j R^k{}_{lki} &= 0\\ \nabla_i R_{lj} + \nabla^k R_{klij} - \nabla_j R^k{}_{lik} &= 0\\ \nabla_i R_{lj} + \nabla^k R_{klij} - \nabla_j R_{li} &= 0,\end{aligned}$$

using definition of Ric and Equation (1.1.1). Note that we have raised the k on the 2nd R term and lowered it on the corresponding ∇.

Now raising i we have

$$\begin{aligned}\nabla_i R_{lj} + \nabla^k R_{klij} - \nabla_j R_{li} &= 0\\ \nabla_i R_{lj} - \nabla^k R_{ijlk} - \nabla_j R_{li} &= 0\\ \nabla^i R_{lj} - \nabla^k R^i{}_{jlk} - \nabla_j R^i_l &= 0.\end{aligned}$$

Finally, contracting over i and l we have

$$\begin{aligned}\nabla^i R_{ij} - \nabla^k R^i{}_{jik} - \nabla_j R^i_i &= 0\\ \nabla^i R_{ij} + \nabla^k R^i{}_{jki} - \nabla_j S &= 0\\ \nabla^i R_{ij} + \nabla^k R_{jk} - \nabla_j S &= 0\\ 2\nabla^i R_{ij} &= \nabla_j S\\ \text{or} \quad 2\,\mathrm{div}\,\mathrm{Ric} &= dS.\end{aligned}$$

□

Also note that in the above proof, the equation

$$2\nabla_{[i} R_{j]l} = -\nabla^k R_{klij}, \tag{1.3.1}$$

appears after contracting over t and k. This will prove useful in Lemma 1.18. Using the twice contacted Bianchi identity we have the following result about multiples of the metric.

Proposition 1.11. *On a semi-Riemannian manifold M, with metric g, if* $\mathrm{Ric} = fg$ *for some function $f \in \mathfrak{F}(M)$ then f is a constant when n, the dimension of the manifold, is greater than 2, and hence the metric is Einstein.*

Proof. The scalar curvature is

$$S = R^i{}_i = g^{ij} f g_{ji} = fn \Rightarrow \nabla_k S = n \nabla_k f.$$

However, using the twice contracted Bianchi identity,

$$\begin{aligned}\nabla_k S = 2\nabla^i (f g_{ik}) = 2(\nabla^i f) g_{ik} = 2(\nabla_i f) g^i_k &= 2(\nabla_k f)\\ &\Rightarrow (n-2)\nabla_k f = 0.\end{aligned}$$

Here $\nabla_k f = \nabla_{\partial_k} f = \partial_k f$ as in Definition A.5. So provided n is not 2, then f must be constant and the metric Einstein. Note that we do not consider $n = 1$ as this case is always flat by Lemma 1.5. □

Remark 1.3. Indeed one asks the question, what happens in dimension 2? It turns out that due to the symmetries and the low dimension of the space, these manifolds have been completely classified as outlined in Besse [4] and Millan and Parker [27]. The following results outline the details.

Proposition 1.12. *A semi-Riemannian manifold of dimension $n = 2$ has Riemann curvature and Ricci curvature as follows:*

$$R_{abcd} = h(g_{ac}g_{db} - g_{ad}g_{cd})$$

where $h \in \mathfrak{F}(M)$. From this it follows directly that $R_{ab} = hg_{ab}$.

A proof of this can be found in Millan and Parker [27, pg. 143]. We notice that in Example 1.2 we computer the Riemann curvature with constant $h = \frac{1}{r^2}$.

The function h is in fact equal to the sectional curvature, which is also equal to the Gaussian curvature in dimension 2 . The reader is directed to O'Neill [29, pg. 77] for the definition of sectional curvature and results concerning this construction. Importantly, we can see directly that not only does Proposition 1.11 not hold for surfaces, but in fact that every semi-Riemannian surface has Ricci curvature of the form fg, where g is the metric and $f \in \mathfrak{F}(M)$. This leads to the following lemma.

Lemma 1.13. *If $n = 2$, then a semi-Riemannian manifold is Einstein if and only if the Riemann curvature is constant.*

By constant Riemann curvature, we mean the function h is constant. One can see this follows directly from Proposition 1.12. See also Besse [4, pg. 49, pg. 342].

Importantly, this thesis concerns Einstein manifolds that are also Lie groups. The only connected Lie groups to be considered in dimension one and two are both abelian, which give flat metrics. See Example 2.1 for further details. Hence dimension two Einstein manifolds are not considered further in this thesis.

1.4 The Schouten, Weyl, Cotton and Bach tensors

The following sections describe properties of Einstein and conformally Einstein metrics using tensors such as the Weyl and Bach tensors. It introduces two obstructions for conformally Einstein metrics, which are explored in later chapters when considering bi-invariant metrics.

Definition 1.11. When the dimension of a semi-Riemannian manifold is $n \geq 3$, we define the *Schouten tensor*, P_{ab}. It is a (0,2) tensor such that

$$\mathsf{P}_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{S}{2(n-1)}g_{ab}\right)$$

where n is the dimension of the manifold. We denote its trace $\mathsf{J} = \mathsf{P}_a{}^a = g^{ai}\mathsf{P}_{ai}$.

Properties 1.1. *Properties of the Schouten tensor and its trace include:*

1. *The trace of the Schouten tensor is*

$$\mathsf{J} = \frac{S}{2(n-1)}. \tag{1.4.1}$$

2. *The Ricci tensor and the Schouten tensor are connected by*

$$R_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}g_{ab}. \tag{1.4.2}$$

3. *The Schouten tensor is symmetric.*

Proof. 1.

$$\begin{aligned}\mathsf{J} = \mathsf{P}_a{}^a =& \frac{1}{n-2}\left(R_a{}^a - \frac{S}{2(n-1)}g_a{}^a\right)\\ =& \frac{1}{n-2}\left(S - \frac{S}{2(n-1)}g_{ai}g^{ai}\right)\\ =& \frac{S}{n-2}\left(1 - \frac{n}{2(n-1)}\right)\\ =& \frac{S}{2(n-1)}.\end{aligned}$$

2.

$$(n-2)\mathsf{P}_{ab} + \mathsf{J}g_{ab} = \left(R_{ab} - \frac{S}{2(n-1)}g_{ab}\right) + \frac{Sg_{ab}}{2(n-1)} = R_{ab}.$$

3. Symmetry follows from the symmetry of the metric and the Ricci tensor.

□

The close relationship between the Schouten tensor and the Ricci tensor gives a corollary from the contracted Bianchi identity.

Corollary 1.14. *On a semi-Riemannian manifold of dimension $n \geq 3$, the covariant derivatives of Schouten tensor,* P *and its trace* J *are related by*

$$\nabla^a \mathsf{P}_{ab} = \nabla_b \mathsf{J}.$$

Note that we use the convention that $\nabla f(V) = \nabla_V f = df(V) = V(f)$ for $f \in \mathfrak{F}(M)$ and $V \in \mathfrak{X}(M)$ as in Remark 1.1. This uses the definition of covariant derivative, which can be found in Definition A.5.

Proof. Using Corollary 1.10 and Equation (1.4.1) we have

$$\begin{aligned}
\nabla^a \mathsf{P}_{ab} =& \frac{1}{n-2}\left(\nabla^a R_{ab} - \frac{\nabla^a (S g_{ab})}{2(n-1)}\right)\\
=& \frac{1}{n-2}\left(\nabla_b S - \frac{\nabla^a S g_{ab}}{2(n-1)}\right)\\
=& \frac{1}{n-2}\left(\frac{\nabla_b S}{2} - \frac{\nabla_b S}{2(n-1)}\right)\\
=& \frac{\nabla_b S}{(n-2)}\left(\frac{1}{2} - \frac{1}{2(n-1)}\right)\\
=& \frac{\nabla_b S}{2(n-1)}\\
=& \nabla_b \mathsf{J}.
\end{aligned}$$

This gives the result. □

Recall that a metric g on a manifold is Einstein if $\mathrm{Ric} = \lambda g$ for some constant λ. By Lemma 1.9, the scalar curvature is $S = \lambda n$ where n is the dimension of the manifold and λ is the Einstein constant. Using Equation (1.4.1) this implies the following lemma:

Lemma 1.15. *On a semi-Riemannian Einstein manifold, the trace of the Schouten tensor is $\mathsf{J} = \frac{\lambda n}{2(n-1)}$, where the dimension of the manifold is $n \geq 3$. Importantly, J is constant.*

Proposition 1.16. *When the dimension of the manifold is $n \geq 3$ then the metric, g, is Einstein if and only if*

$$\mathsf{P}_{ab} - \frac{1}{n}\mathsf{J} g_{ab} = 0. \tag{1.4.3}$$

Proof. The first direction follows from the definition of Einstein, Equation (1.4.2) and Equation (1.4.1). Assume $P_{ab} = \frac{1}{n}\mathsf{J} g_{ab}$ then

$$\begin{aligned}
R_{ab} &= (n-2)\mathsf{P}_{ab} + \mathsf{J} g_{ab}\\
&= (n-2)\frac{1}{n}\mathsf{J} g_{ab} + \mathsf{J} g_{ab}\\
&= \frac{2(n-1)}{n}\mathsf{J} g_{ab}\\
&= \frac{S}{n} g_{ab}
\end{aligned}$$

By Proposition 1.11, the Ricci tensor can only be of the form $\mathrm{Ric} = fg$ for some $f \in \mathfrak{F}(M)$ when the metric is Einstein. Assume now the metric is Einstein with Einstein constant λ, then the definition of the Schouten tensor and Lemma 1.15, we have

$$\begin{aligned}
\mathsf{P}_{ab} &= \frac{1}{n-2}\left(R_{ab} - \frac{S}{2(n-1)}g_{ab}\right) \\
&= \frac{1}{n-2}\left(\lambda g_{ab} - \frac{n\lambda}{2(n-1)}g_{ab}\right) \\
&= \frac{1}{n-2}\frac{\lambda(n-2)}{(n-1)}g_{ab} \\
&= \frac{n\lambda}{n(n-1)}g_{ab} \\
&= \frac{\mathsf{J}}{n}g_{ab}
\end{aligned}$$

and the result follows. □

Definition 1.12. When the dimension of a semi-Riemannian manifold is $n \geq 3$, the *Weyl curvature* C_{abcd} is the tensor defined by

$$C_{abcd} = -R_{abcd} - 2g_{c[a}\mathsf{P}_{b]d} - 2g_{d[b}\mathsf{P}_{a]c}.$$

We can rewrite the Weyl tensor as

$$C_{abcd} = -R_{abcd} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) + \frac{S}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}).$$

Properties 1.2. *The Weyl tensor has the following symmetries*

1.

$$C_{abcd} = C_{[ab][cd]} = C_{cdab}. \tag{1.4.4}$$

2.

$$C_{[abc]d} = 0.$$

For the proof, see Appendix A.5.

Properties 1.3. *The Weyl tensor is totally trace free.*

Totally trace free means all contractions of the tensor vanish. See Appendix A.2 for the definition of contractions and see Appendix A.5 for the proof of this proposition.

Definition 1.13. When the dimension of the manifold is $n \geq 3$, we define the Cotton tensor

$$A_{abc} = 2\nabla_{[b}\mathsf{P}_{c]a}.$$

Using the definition of the Schouten tensor, we can rewrite the Cotton tensor as

$$A_{abc} = \frac{1}{n-2}\left(\nabla_b R_{ca} - \nabla_c R_{ba} + \frac{1}{2(n-1)}(g_{ba}\nabla_c S - g_{ca}\nabla_b S)\right).$$

Properties 1.4. *The Cotton tensor is totally trace free.*

Proof. Using Corollary 1.14, we find that

$$\begin{aligned} A^a{}_{ac} &= 2\nabla_{[a}\mathsf{P}_{c]}{}^a \\ &= \nabla_a \mathsf{P}_c{}^a - \nabla_c \mathsf{P}^a{}_a \\ &= 0 = -A^a{}_{ca}, \end{aligned}$$

and

$$A^c{}_{aa} = 2\nabla_{[a}\mathsf{P}_{a]}{}^c = 0.$$

□

Lemma 1.17. *When the dimension of a semi-Riemannian manifold is $n \geq 3$, an Einstein metric has vanishing Cotton tensor.*

Proof. As the metric is Einstein, the trace of the Schouten tensor, J is constant by Lemma 1.15. Then by Proposition 1.16 we have that

$$A_{abc} = 2\nabla_{[b}\mathsf{P}_{c]a} = \frac{2}{n}(\nabla_{[b}\mathsf{J})g_{c]a} = 0.$$

□

Lemma 1.18. *When the dimension of a semi-Riemannian manifold is $n \geq 3$, the Cotton and Weyl tensors are connected by the following formula*

$$(n-3)A_{abc} = \nabla^d C_{dabc}.$$

Proof. Recall the twice contracted second Bianchi identity from Corollary 1.10, $2\nabla^i R_{ij} = \nabla_j S$, and from Equation (1.3.1),$\nabla_i R_{lj} + \nabla^k R_{klij} - \nabla_j R_{li} = 0$. Using these we have

$$\begin{aligned} \nabla^d C_{dabc} =& -\nabla^d R_{bcda} + \frac{\nabla^d}{n-2}(g_{ba}R_{dc} - g_{bd}R_{ac} + g_{cd}R_{ab} - g_{ca}R_{db}) \\ &+ \frac{\nabla^d S}{(n-2)(n-1)}(g_{db}g_{ac} - g_{ab}g_{dc}) \\ =& -\nabla^d R_{bcda} + \frac{1}{n-2}(g_{ba}\nabla^d R_{dc} - \nabla_b R_{ac} + \nabla_c R_{ab} - g_{ca}\nabla^d R_{db}) \\ &+ \frac{\nabla_b S g_{ac} - \nabla_c S g_{ab}}{(n-2)(n-1)} \\ =& \nabla^d R_{dacb} + \frac{1}{n-2}(\frac{1}{2}g_{ba}\nabla_c S - \nabla_b R_{ac} + \nabla_c R_{ab} - \frac{1}{2}g_{ca}\nabla_b S) \\ &+ \frac{\nabla_b S g_{ac} - \nabla_c S g_{ab}}{(n-2)(n-1)} \\ =& \nabla_b R_{ac} - \nabla_c R_{ab} + \frac{1}{n-2}(-\nabla_b R_{ac} + \nabla_c R_{ab}) + \frac{1}{2(n-2)}(g_{ba}\nabla_c S - g_{ca}\nabla_b S) \\ &+ \frac{\nabla_b S g_{ac} - \nabla_c S g_{ab}}{(n-2)(n-1)} \\ =& \frac{n-3}{n-2}(\nabla_b R_{ac} - \nabla_c R_{ab}) + \frac{n-3}{2(n-1)(n-2)}(g_{ab}\nabla_c S - g_{ac}\nabla_b S) \\ =& (n-3)A_{abc}. \end{aligned}$$

□

Definition 1.14. When the dimension of a semi-Riemannian manifold is $n \geq 3$, the Bach tensor is defined as
$$B_{ab} = \nabla^c A_{acb} + \mathsf{P}^{dc} C_{dacb}.$$

Properties 1.5. *The Bach tensor is symmetric and totally trace free.*

Proof. The Cotton and Weyl tensors are totally trace free, hence these properties carry to the Bach tensor. We have that

$$\begin{aligned}
\nabla^c A_{acb} =& 2\nabla^c \nabla_{[c} \mathsf{P}_{b]a} \\
=& \nabla^c \nabla_c \mathsf{P}_{ba} - \nabla^c \nabla_b \mathsf{P}_{ca} \\
=& \nabla^c \nabla_c \mathsf{P}_{ab} - \nabla_a \nabla_b \mathsf{J} && \text{using Corollary 1.14} \\
=& \nabla^c \nabla_c \mathsf{P}_{ab} - \nabla_a \nabla^c \mathsf{P}_{cb} && \text{using Corollary 1.14} \\
=& \nabla^c A_{bca}
\end{aligned}$$

and

$$\mathsf{P}^{dc} C_{dacb} = \mathsf{P}^{cd} C_{cbda} \qquad \text{by symmetry of P and Equation (1.4.4).}$$

□

Lemma 1.19. *When the dimension of a semi-Riemannian manifold is $n \geq 3$, an Einstein metric has vanishing Bach tensor.*

Proof. From Lemma 1.17, the Cotton tensor vanishes when the metric is Einstein. From Proposition 1.16, then the metric, g, is Einstein if and only if $\mathsf{P}_{ab} = \frac{1}{n}\mathsf{J} g_{ab}$. Then the Bach tensor reduces to the trace
$$B_{ab} = \frac{1}{n} \mathsf{J} g^{dc} C_{dacb},$$
which vanishes as the Weyl tensor is trace free. □

1.5 Conformally Einstein metrics

If a metric is not Einstein, what other properties may it have that make it behave nicely? From Proposition 1.11, we know that if $\mathrm{Ric} = fg$ for a function, $f \in \mathfrak{F}(M)$, then the function is constant when $n \geq 3$. However, we can multiply the metric by a non-zero smooth function, which will give another metric and consider whether this metric is Einstein as follows.

Definition 1.15. A semi-Riemannian metric, g, on a manifold M, is *conformal to Einstein* if there exists $\Upsilon \in \mathfrak{F}(M)$ such that $\widehat{g} = e^{2\Upsilon} g$ is an Einstein metric. If we let the Ricci tensor for $\widehat{g}$ be $\widehat{\mathrm{Ric}}$, then we must have $\widehat{\mathrm{Ric}} = \lambda \widehat{g} = \lambda e^{2\Upsilon} g$ for some constant λ.

Any two metrics $g, \widehat{g}$ such that there exists $\Upsilon \in \mathfrak{F}(M)$ with $\widehat{g} = e^{2\Upsilon} g$ are called *conformally related metrics.* Replacing g with $\widehat{g}$ is called a *conformal rescaling.*

Under conformal rescaling of the metric, if a tensor T transforms according to
$$T \to \widehat{T} = e^{w\Upsilon} T$$

then we say the tensor is *conformally covariant* (of weight $w \in \mathbb{R}$). If $w = 0$ it is called *conformally invariant.*

Proposition 1.20. *Let ∇ be the Levi-Civita connection for a manifold M equipped with a semi-Riemannian metric g. Let $\widehat{g} = e^{2\Upsilon} g$ be a conformal rescaling of g. Then the Levi-Civita connection for $\widehat{g}$, denoted $\widehat{\nabla}$ behaves as follows*

$$\nabla_V W - \widehat{\nabla}_V W = -(\nabla_V \Upsilon) W - (\nabla_W \Upsilon) V + \mathrm{grad}(\Upsilon) g(V, W) \tag{1.5.1}$$

for vector fields $V, W \in \mathfrak{X}(M)$.

Here we are using ∇ as the covariant derivative, defined in Definition A.5. This includes the convention that $\nabla f(V) = \nabla_V f = df(V) = V(f)$ for $f \in \mathfrak{F}(M)$ and $V \in \mathfrak{X}(M)$ as in Remark 1.1.

Proof. Using the Kozsul formula, we have

$$2g(\nabla_V W, X) = Vg(W, X) + Wg(X, V) - Xg(V, W) - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W])$$

and similarly for $\widehat{g}$. Multiplying the above by $e^{2\Upsilon}$ and then subtracting the Kozsul formula for $\widehat{g}$ we have

$$\begin{aligned}
&2e^{2\Upsilon} g(\nabla_V W, X) - 2\widehat{g}(\widehat{\nabla}_V W, X) \\
=&e^{2\Upsilon}\Big(Vg(W, X) + Wg(X, V) - Xg(V, W) - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W])\Big) \\
&- \Big(V\widehat{g}(W, X) + W\widehat{g}(X, V) - X\widehat{g}(V, W) - \widehat{g}(V, [W, X]) + \widehat{g}(W, [X, V]) + \widehat{g}(X, [V, W])\Big) \\
=&e^{2\Upsilon}\Big(Vg(W, X) + Wg(X, V) - Xg(V, W)\Big) - \Big(V\widehat{g}(W, X) + W\widehat{g}(X, V) - X\widehat{g}(V, W)\Big) \\
=&-\Big((Ve^{2\Upsilon})g(W, X) + (We^{2\Upsilon})g(X, V) - (Xe^{2\Upsilon})g(V, W)\Big) \\
=&-2e^{2\Upsilon}\Big((\nabla_V \Upsilon)g(W, X) + (\nabla_W \Upsilon)g(X, V) - (\nabla_X \Upsilon)g(V, W)\Big).
\end{aligned}$$

Hence

$$g(\nabla_V W - \widehat{\nabla}_V W, X) = -(\nabla_V \Upsilon)g(W, X) - (\nabla_W \Upsilon)g(X, V) + (\nabla_X \Upsilon)g(V, W)$$

and the result follows. ☐

We can write Equation (1.5.1) as a $(1, 1)$ tensor as follows

$$\nabla W - \widehat{\nabla} W = -(\nabla \Upsilon) \otimes W - (\nabla_W \Upsilon) Id + g(W, \cdot) \otimes \mathrm{grad}(\Upsilon).$$

Note that in index notation, we will let $\Upsilon_a = d\Upsilon(\partial_a) = \partial_a(\Upsilon)$. Then $\nabla_W \Upsilon Id$ corresponds to $W^c \nabla_c \Upsilon \delta_a{}^b = W^c \Upsilon_c \delta_a{}^b$, and $g(W, \cdot) \otimes \mathrm{grad}(\Upsilon)$ corresponds to $\Upsilon^a W^c g_{cb} = \Upsilon^a W_b$. Hence, in index notation, this equation becomes

$$\nabla_a V^b - \widehat{\nabla}_a V^b = -\Upsilon_a V^b + \Upsilon^b V_a - \Upsilon^c V_c \delta_a{}^b.$$

As commonly seen in the literature, for example in Eastwood [8], one may write

$$\nabla_a V^b - \widehat{\nabla}_a V^b = -\Gamma_{ac}{}^b V^c, \tag{1.5.2}$$

where $\Gamma_{ac}{}^b = \Upsilon_a \delta_c{}^b + \Upsilon_c \delta_a{}^b - \Upsilon^b g_{ac}$. Note that these Γ are not the Christoffel symbols.

Lemma 1.21. *Let ∇ be the Levi-Civita connection for a manifold M equipped with a semi-Riemannian metric g. Let $\widehat{g} = e^{2\Upsilon}g$ be a conformal rescaling of g. For a 1-form ω we have that the Levi-Civita conection for $\widehat{g}$, $\widehat{\nabla}$, behaves as follows*

$$(\widehat{\nabla}_V\omega)(W) = \nabla_V\omega(W) - \nabla_V\Upsilon\omega(W) + \nabla_X\Upsilon g(W,V) - \nabla_W\Upsilon\omega(V)$$

for vector fields $V, W \in \mathfrak{X}(M)$. Here $X \in \mathfrak{X}(M)$ is the unique vector field such that $\omega = g(X, \cdot)$, which exists by Proposition A.1. In index notation, we have

$$\begin{aligned}\nabla_a\omega_b - \widehat{\nabla}_a\omega_b =& {\Gamma_{ab}}^c\omega_c \\ =& \Upsilon_a\omega_b + \Upsilon_b\omega_a - \Upsilon^c\omega_c g_{ab},\end{aligned}$$

where Γ are as defined for Equation (1.5.2)*.*

Proof. If we wish to write $\widehat{\nabla}_V\omega$ for a 1-form ω, then we use Equation (A.3.2). Consider $\omega = g(X, \cdot)$, then

$$\begin{aligned}(\widehat{\nabla}_V\omega)(W) =& \widehat{\nabla}_V g(X,W) - g(X, \widehat{\nabla}_V W) \\ &= \widehat{\nabla}_V e^{-2\Upsilon}\widehat{g}(X,W) - g(X,\widehat{\nabla}_V W) \\ &= -2e^{-2\Upsilon}(\widehat{\nabla}_V\Upsilon)\widehat{g}(X,W) + e^{-2\Upsilon}\widehat{\nabla}_V\widehat{g}(X,W) - g(X,\widehat{\nabla}_V W) \\ &= -2(\widehat{\nabla}_V\Upsilon)g(X,W) + e^{-2\Upsilon}\widehat{g}(\widehat{\nabla}_V X, W) + e^{-2\Upsilon}\widehat{g}(X,\widehat{\nabla}_V W) - g(X,\widehat{\nabla}_V W) \\ &= -2(\widehat{\nabla}_V\Upsilon)g(X,W) + g(\widehat{\nabla}_V X, W) + g(X,\widehat{\nabla}_V W) - g(X,\widehat{\nabla}_V W) \\ &= -2(\widehat{\nabla}_V\Upsilon)g(X,W) + g(\widehat{\nabla}_V X, W).\end{aligned}$$

Similarly, we may also deduce that $(\nabla_V\omega)(W) = \nabla_V g(X,W) - g(X,\nabla_V W) = g(\nabla_V X, W)$. Combining these and the formula for $\widehat{\nabla}$ we have

$$\begin{aligned}(\widehat{\nabla}_V\omega)(W) &= -2(\widehat{\nabla}_V\Upsilon)g(X,W) + g(\widehat{\nabla}_V X, W) \\ &= g(\nabla_V X, W) + \nabla_V\Upsilon g(X,W) + \nabla_X\Upsilon g(W,V) - \nabla_W\Upsilon g(V,X) - 2\nabla_V\Upsilon g(X,W) \\ &= g(\nabla_V X, W) - \nabla_V\Upsilon g(X,W) + \nabla_X\Upsilon g(W,V) - \nabla_W\Upsilon g(V,X) \\ &= \nabla_V\omega(W) - \nabla_V\Upsilon\omega(W) + \nabla_X\Upsilon g(W,V) - \nabla_W\Upsilon\omega(V),\end{aligned}$$

which gives the result. □

Finally, we will also need the formulae for $\widehat{\nabla}$ applied to a $(2,0)$ tensor T_{ab}.

Lemma 1.22. *Let ∇ be the Levi-Civita connection for a manifold M equipped with a semi-Riemannian metric g. Let $\widehat{g} = e^{2\Upsilon}g$ be a conformal rescaling of g. For a $(2,0)$ tensor T, we have that the Levi-Civita conection for $\widehat{g}$, $\widehat{\nabla}$, behaves as follows*

$$\begin{aligned}(\widehat{\nabla}_V T)(X,Y) = (\nabla_V T)(X,Y) &- 2\nabla_V\Upsilon T(X,Y) - \nabla_X\Upsilon T(V,Y) - \nabla_Y\Upsilon T(X,V) \\ &+ g(V,X)T(\operatorname{grad}\Upsilon, Y) + g(V,Y)(\operatorname{grad}\Upsilon, X).\end{aligned}$$

In index notation, this is

$$\begin{aligned}\widehat{\nabla}_a T_{bc} &= \nabla_a T_{bc} - 2\Upsilon_a T_{bc} - \Upsilon_b T_{ac} - \Upsilon_c T_{ab} + g_{ab}T_{cd}\Upsilon^d + g_{ac}T_{bd}\Upsilon^d \\ &= \nabla_a T_{bc} - {\Gamma_{ab}}^d T_{dc} - {\Gamma_{ac}}^d T_{db},\end{aligned}$$

where Γ are as defined for Equation (1.5.2)*.*

Proof. Consider a $(2,0)$ tensor T, then

$$\begin{aligned}(\widehat{\nabla}_V T)(X,Y) &= \widehat{\nabla}_V T(X,Y) - T(\widehat{\nabla}_V X, Y) - T(X, \widehat{\nabla}_V Y)\\ &= \nabla_V T(X,Y) - T(\nabla_V X + (\nabla_V \Upsilon)X + (\nabla_X \Upsilon)V - \operatorname{grad}(\Upsilon) g(V,X), Y)\\ &\quad - T(X, \nabla_V Y + (\nabla_V \Upsilon)Y + (\nabla_Y \Upsilon)V - \operatorname{grad}(\Upsilon) g(V,Y))\\ &= (\nabla_V T)(X,Y) - 2\nabla_V \Upsilon T(X,Y) - \nabla_X \Upsilon T(V,Y) - \nabla_Y \Upsilon T(X,V)\\ &\quad + g(V,X) T(\operatorname{grad}\Upsilon, Y) + g(V,Y)(\operatorname{grad}\Upsilon, X),\end{aligned}$$

as required. □

The formulae in Lemma 1.21 and Lemma 1.22 also appear in the literature, for example in Curry and Gover [6, pg. 9] and Eastwood [8, pg. 61].

Proposition 1.23. *Let R be the Riemann curvature tensor,* Ric *the Ricci curvature and S the scalar curvature for a manifold M equipped with metric semi-Riemannian g. Let $\widehat{g} = e^{2\Upsilon} g$ be a conformal rescaling of g. Then the conformally transformed Riemann curvature tensor, Ricci curvature and scalar curvature, denoted $\widehat{R}$, $\widehat{\mathrm{Ric}}$ and $\widehat{S}$ respectively, are as follows*

$$\begin{aligned}\widehat{R}(X,Y)Z &= R(X,Y)Z - g(\nabla_X \operatorname{grad}(\Upsilon), Z)Y - g(\nabla_Y \operatorname{grad}(\Upsilon), X) + g(X,Z)\nabla_Y \operatorname{grad}(\Upsilon)\\ &\quad - g(Y,Z)\nabla_X \operatorname{grad}(\Upsilon) + (\nabla_Y \Upsilon)(\nabla_Z \Upsilon)X - (\nabla_X \Upsilon)(\nabla_Z \Upsilon)Y\\ &\quad - g(\operatorname{grad}\Upsilon, \operatorname{grad}\Upsilon)(g(Y,Z)X - g(X,Z)Y)\\ &\quad + \operatorname{grad}(\Upsilon)\Big((\nabla_X \Upsilon) g(Y,Z) - (\nabla_Y \Upsilon) g(X,Z)\Big),\\ \widehat{\mathrm{Ric}}(X,Y) &= \mathrm{Ric}(X,Y) - (\Delta\Upsilon + (n-2) g(\operatorname{grad}\Upsilon, \operatorname{grad}\Upsilon)) g(X,Y)\\ &\quad + (n-2) e^{\Upsilon} \nabla_X \nabla_Y (e^{-\Upsilon})\end{aligned}$$

and

$$\widehat{S} = e^{-2\Upsilon}(S - 2(n-1)\Delta\Upsilon - (n-2)(n-1) g(\operatorname{grad}(\Upsilon), \operatorname{grad}(\Upsilon)).$$

The proof is straightforward from the definitions and using Proposition 1.20. See Kühnel [21, pg. 349] or Besse [4, pg. 58] for further details. We can rewrite these formula for $\widehat{\mathrm{Ric}}$ and $\widehat{S}$ using index notation as follows:

$$\begin{aligned}\widehat{R}_{ab} &= R_{ab} - (\nabla_c \Upsilon^c + (n-2)\Upsilon^c \Upsilon_c) g_{ab} + (n-2)(\Upsilon_a \Upsilon_b - \nabla_a \Upsilon_b),\\ \widehat{S} &= e^{-2f}\big(S - 2(n-1)\nabla_a \Upsilon^a - (n-1)(n-2)\Upsilon^a \Upsilon_a\big).\end{aligned} \tag{1.5.3}$$

Lemma 1.24. *Einstein is not a conformally invariant property.*

Proof. Assume a semi-Riemannian metric g is conformally Einstein, then there is an $\Upsilon \in \mathfrak{F}(M)$ such that $\widehat{\mathrm{Ric}} = \lambda \widehat{g} = \lambda e^{2\Upsilon} g$. This is true if and only if

$$\mathrm{Ric}(X,Y) - (\Delta\Upsilon + (n-2) g(\operatorname{grad}\Upsilon, \operatorname{grad}\Upsilon)) g(X,Y) + (n-2) e^{\Upsilon} \nabla_X \nabla_Y (e^{-\Upsilon}) = \lambda e^{2\Upsilon} g(X,Y),$$

which is true if and only if

$$\mathrm{Ric}(X,Y) + (n-2) e^{\Upsilon} \nabla_X \nabla_Y (e^{-\Upsilon}) = (\lambda e^{2\Upsilon} + \Delta\Upsilon) + (n-2) g(\operatorname{grad}\Upsilon, \operatorname{grad}\Upsilon)) g(X,Y).$$

One can see directly that this means Einstein is not a conformally invariant property. □

Lemma 1.25. *When the dimension of a semi-Riemannian manifold is $n \geq 3$, the Schouten tensor changes conformally as*

$$\widehat{\mathsf{P}}(X,Y) = \mathsf{P}(X,Y) - \nabla_X(\nabla_Y(\Upsilon)) + \nabla_X\Upsilon\nabla_Y\Upsilon - \frac{1}{2}g(\operatorname{grad}\Upsilon, \operatorname{grad}\Upsilon)g(X,Y).$$

Proof. From direct computation using Proposition 1.23 and the definition of the Schouten tensor, the result follows. □

This result can also be found, for example, in Gover and Nurowski [13, pg. 455], Curry and Gover [6, pg. 12] and Eastwood [8, pg. 62]. In index notation, this reads

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon_c\Upsilon^c g_{ab}.$$

Lemma 1.26. *When the dimension of a semi-Riemannian manifold is $n \geq 3$, the Cotton tensor changes conformally as*

$$\widehat{A}_{abc} = A_{abc} + \Upsilon^d C_{dabc}.$$

Proof. Note the conformal change for the Levi-Civita connection applied to a $(2,0)$ tensor is

$$\begin{aligned}\widehat{\nabla}_a T_{bc} &= \nabla_a T_{bc} - 2\Upsilon_a T_{bc} - \Upsilon_b T_{ac} - \Upsilon_c T_{ab} + g_{ab}T_{cd}\Upsilon^d + g_{ac}T_{bd}\Upsilon^d \\ &= \nabla_a T_{bc} - \Gamma_{ab}{}^d T_{dc} - \Gamma_{ac}{}^d T_{db}.\end{aligned}$$

Using the Leibniz rule and the conformal change for $\widehat{\mathsf{P}}$ we have

$$\begin{aligned}\widehat{A}_{cab} =& 2\widehat{\nabla}_{[a}\widehat{\mathsf{P}}_{b]c} \\ =& \nabla_a\widehat{\mathsf{P}}_{bc} - 2\Upsilon_a\widehat{\mathsf{P}}_{bc} - \Upsilon_b\widehat{\mathsf{P}}_{ac} - \Upsilon_c\widehat{\mathsf{P}}_{ab} + g_{ab}\widehat{\mathsf{P}}_{cd}\Upsilon^d + g_{ac}\widehat{\mathsf{P}}_{bd}\Upsilon^d \\ &- \nabla_b\widehat{\mathsf{P}}_{ac} + 2\Upsilon_b\widehat{\mathsf{P}}_{ac} + \Upsilon_a\widehat{\mathsf{P}}_{bc} + \Upsilon_c\widehat{\mathsf{P}}_{ab} - g_{ba}\widehat{\mathsf{P}}_{cd}\Upsilon^d - g_{bc}\widehat{\mathsf{P}}_{ad}\Upsilon^d \\ =& \nabla_a\widehat{\mathsf{P}}_{bc} - \Upsilon_a\widehat{\mathsf{P}}_{bc} + g_{ac}\widehat{\mathsf{P}}_{bd}\Upsilon^d - \nabla_b\widehat{\mathsf{P}}_{ac} + \Upsilon_b\widehat{\mathsf{P}}_{ac} - g_{bc}\widehat{\mathsf{P}}_{ad}\Upsilon^d \\ =& \nabla_a(\mathsf{P}_{bc} - \nabla_b\Upsilon_c + \Upsilon_b\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{bc}) - \Upsilon_a(\mathsf{P}_{bc} - \nabla_b\Upsilon_c + \Upsilon_b\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{bc}) \\ &+ g_{ac}(\mathsf{P}_{bd} - \nabla_b\Upsilon_d + \Upsilon_b\Upsilon_d - \frac{1}{2}\Upsilon_e\Upsilon^e g_{bd})\Upsilon^d \\ &- \nabla_b(\mathsf{P}_{ac} - \nabla_a\Upsilon_c + \Upsilon_a\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{ac}) + \Upsilon_b(\mathsf{P}_{ac} - \nabla_a\Upsilon_c + \Upsilon_a\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{ac}) \\ &- g_{bc}(\mathsf{P}_{ad} - \nabla_a\Upsilon_d + \Upsilon_a\Upsilon_d - \frac{1}{2}\Upsilon_e\Upsilon^e g_{ad})\Upsilon^d \\ =& A_{cab} + R_{cdab}\Upsilon^d - \Upsilon_a\mathsf{P}_{bc} + g_{ac}\mathsf{P}_{bd}\Upsilon^d + \Upsilon_b\mathsf{P}_{ac} - g_{bc}\mathsf{P}_{ad}\Upsilon^d \\ &+ \nabla_a(\Upsilon_b\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{bc}) + \Upsilon_a\nabla_b\Upsilon_c - g_{ac}(\nabla_b\Upsilon_d)\Upsilon^d \\ &- \nabla_b(\Upsilon_a\Upsilon_c - \frac{1}{2}\Upsilon_d\Upsilon^d g_{ac}) - \Upsilon_b\nabla_a\Upsilon_c + g_{bc}(\nabla_a\Upsilon_d)\Upsilon^d \\ =& A_{cab} + \Upsilon^d C_{dcab}.\end{aligned}$$

□

Proposition 1.27. *A semi-Riemannian metric g on a manifold, M, is conformally Einstein if and only if there is an $\Upsilon \in \mathfrak{F}(M)$ such that*

$$\mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{n}(\mathsf{J} - \nabla^d \Upsilon_d + \Upsilon^d \Upsilon_d) g_{ab} = 0 \tag{1.5.4}$$

where $\Upsilon^a = \nabla^a \Upsilon = dx^a(\mathrm{grad}(\Upsilon))$ *and* $\Upsilon_a = \nabla_{\partial_a} \Upsilon = \partial_a \Upsilon = d\Upsilon(\partial_a)$.

Proof. Recall that g is conformally Einstein if and only if Equation (1.4.3) holds for $\widehat{g}$, that is

$$\widehat{\mathsf{P}}_{ab} - \frac{1}{n}\widehat{\mathsf{J}}\widehat{g}_{ab} = 0.$$

This is if and only if there is an $\Upsilon \in \mathfrak{F}(M)$ such that

$$\mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\nabla_c \Upsilon^c g_{ab} - \frac{1}{2n(n-1)}(S - 2(n-1)\nabla_c \Upsilon^c - (n-2)(n-1)\Upsilon^c \Upsilon_c) g_{ab} = 0.$$

Rearranging gives the result. □

This result can be found in Gover and Nurowski [13, pg. 456]

From the literature in conformal geometry, the following results are well known: The Cotton tensor in dimension $n = 3$ and the Weyl tensor in dimension $n \geq 3$ are conformally invariant, see [8, pg. 63] and [4, pg. 58]. When the dimension is $n \geq 4$ vanishing of Weyl tensor occurs if and only if the metric is conformally flat, that is the transformed metric is flat. See [4, pg. 60]. When $n = 3$ then manifold is conformally flat if and only if the Cotton tensor vanishes, see [21, pg. 352]. The Bach tensor is conformally covariant in dimension $n = 4$, see [4, pg. 135]. These are standard results are mentioned in Gover and Nurowski [13] and proved in Besse [4] and Kühnel [21].

Remark 1.4. The case of a semi-Riemannian manifold of dimension $n = 2$ is again simplified. From Besse [4, pg. 61], we have the following theorem:

Theorem 1.28. *Any 2-dimensional semi-Riemannian manifold is conformally flat.*

As this thesis concerns whether semi-Riemannian metrics are Einstein and conformally Einstein, semi-Riemannian surfaces are not considered further in this thesis.

1.6 Obstructions to the metric being conformally Einstein

Proposition 1.29 (Gover and Nurowski [13]). *If g is a semi-Riemannian conformally Einstein metric on a manifold M, then its Cotton and Bach and Weyl tensors satisfy*

$$A_{abc} + \Upsilon^d C_{dabc} = 0 \tag{1.6.1}$$

and

$$B_{ab} + (n-4)\Upsilon^d \Upsilon^c C_{dabc} = 0 \tag{1.6.2}$$

for some $\Upsilon \in \mathfrak{F}(M)$ where $\nabla^a \Upsilon = \Upsilon^a$. We call Equation (1.6.1) *and Equation* (1.6.2) *the first and second obstructions to Einstein conformality.*

Proof. For the first obstruction, we know that if $\widehat{g}$ is Einstein, then from Lemma 1.17, we know that $\widehat{g}$, must be Cotton flat. Applying Lemma 1.26, gives the result.

Starting with the first obstruction, we apply ∇^c, use the definition of the Bach tensor, $B_{ab} = \nabla^c A_{acb} + \mathsf{P}^{pc} C_{dacb}$ and use Lemma 1.18, $(n-3)A_{abc} = \nabla^d C_{dabc}$ as follows

$$\begin{aligned} 0 &= \nabla^c A_{abc} + \nabla^c(\Upsilon^d C_{dabc}) \\ &= -\nabla^c A_{acb} + (\nabla^c \Upsilon^d) C_{dabc} + \Upsilon^d \nabla^c C_{cbad} \\ &= -B_{ab} + \mathsf{P}^{dc} C_{dacb} + (\nabla^c \Upsilon^d) C_{dabc} + \Upsilon^d (n-3) A_{bad}. \end{aligned}$$

Again using the first obstruction, this reduces to

$$\begin{aligned} 0 =& - B_{ab} + \mathsf{P}^{dc} C_{dacb} + (\nabla^c \Upsilon^d) C_{dabc} - (n-3)\Upsilon^d \Upsilon^c C_{cbad} \\ =& - B_{ab} + \mathsf{P}^{dc} C_{dacb} + (\nabla^c \Upsilon^d - (n-3)\Upsilon^d \Upsilon^c) C_{dabc}. \end{aligned} \qquad (*)$$

Now note that as the metric is conformally Einstein, then Equation (1.5.4) holds, that is $\mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{n}(\mathsf{J} - \nabla^d \Upsilon_d + \Upsilon^d \Upsilon_d) g_{ab}$=0. Raising both indices and applying to C_{dabc} to Equation (1.5.4) we find that

$$(\mathsf{P}^{dc} - \nabla^c \Upsilon^d) C_{dabc} = (-\Upsilon^c \Upsilon^d + \frac{1}{n}(\mathsf{J} - \nabla^e \Upsilon_e + \Upsilon^e \Upsilon_e)) g^{cd} C_{dabc} = -\Upsilon^c \Upsilon^d C_{dabc}$$

as the Weyl tensor is trace free. Using this in the equation above, $(*)$, we have

$$0 = -B_{ab} + (\Upsilon^d \Upsilon^c - (n-3)\Upsilon^d \Upsilon^c) C_{dabc}$$

and the result follows. See also Gover and Nurowski [13, pg. 456]. $\square$

1.7 Conclusion

This chapter introduces important notation and results that the following chapters are based on. We have seen important definitions such as the Riemann curvature tensor, Einstein metrics and conformally Einstein metrics. We have summarised the literature in this area and compared our notation. The final result from Gover and Nurowski [13] will be of particular use in Chapter 3 and 4 as it involves only algebraic conditions on tensors as obstructions to conformally Einstein metrics.

Chapter 2

Bi-invariant metrics

A Lie group is a smooth manifold that has an algebraic group structure. This group structure is compatible with the smooth structure, that is, the multiplication and inversion maps are smooth maps on the manifold. Some preliminary results on Lie groups and Lie algebras can be found in the appendices, starting in Appendix A.6. Rossman [33] has excellent background material on Lie groups and Lie algebras that may be of use to a reader unfamiliar with this area.

To study the geometry of a Lie group, we would like a metric that is also compatible with the group structure. The Lie group structure creates canonical isomorphisms between each tangent space using the left action of each group element. A right action may also be considered. We would like our metric to be the same on each tangent space under this assumption. This gives the notion of a *left invariant* metric. If a metric is invariant under both left and right action, then it is called *bi-invariant*. We make this notion precise in the following section and study the metrics that arise.

Unless otherwise noted, we denote by G a Lie group with corresponding Lie algebra $\mathfrak{g}$.

2.1 Actions, invariance and bi-invariant metrics

Given a group, G, with binary operation $\cdot : G \times G \to G$ we can consider how the group acts on itself. Take an element h in G. The *left action* by h on G is a map $L_h : G \to G$ where $L_h(p) := h \cdot p$ for all $p \in G$. Often we just write $h = L_h$. Similarly, the *right action* by h on G is $R_h(p) = p \cdot h$.

When we consider actions on Lie group, multiplication by elements is a smooth map and can be differentiated. As the inverse map is also smooth, the resulting differential, $(dL_h)(p)$, is an isomorphism from T_pG to $T_{h\cdot p}G$ for any $h \in G$.

One defines the Lie algebra, $\mathfrak{g}$, as the vector space of all left-invariant vector fields equipped with the standard vector field bracket operation, see Equation (A.1.1). If we let e be the identity element, then the vector space of all left-invariant vector fields is isomorphic as a vector space to T_eG where the isomorphism is given by the evaluation map, $\mathfrak{X}(G) \ni X \mapsto X_e \in T_eG$. The inverse map given by $T_eG \ni Y \mapsto X \in \mathfrak{X}(G)$, where for $p \in G$, we have $X_p = (dL_p)(e)Y$. If one equips T_eG with the bracket given by the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (as defined in Appendix A.6.1), the vector space isomorphism is then a Lie algebra isomorphism.

As we have a canonical isomorphism between the Lie algebra and any tangent space, we would like that a semi-Riemannian metric, g, on G respects this isomorphism. That is,

$$g_{L_h(\widehat{h})}(dL_h|_{\widehat{h}}(X_{\widehat{h}}), dL_h|_{\widehat{h}}(Y_{\widehat{h}})) = g_{\widehat{h}}(X_{\widehat{h}}, Y_{\widehat{h}})$$

for any $h, \widehat{h} \in G$ and any $X, Y \in \mathfrak{X}(M)$. More succinctly, we must have

$$L_h^* g = g$$

and we say the metric is *left-invariant*. The metric g is a *right-invariant* metric if $R_h^* g = g$. With this property, for a connected Lie group, it is sufficient to consider only the metric evaluated at $\mathfrak{g} = T_e G$. We can evaluate the metric on all other tangent spaces using the left-invariance.

Remark 2.1 (Left-invariant metrics). If we consider how many left-invariant metrics exist on a Lie algebra, we are then just looking for non-degenerate symmetric bilinear forms on $\mathfrak{g}$ as a vector space. The space of symmetric bilinear forms has dimension $\frac{n(n+1)}{2}$. The non-degenerate ones are a non-empty open subset of this space. In fact, as they are symmetric, they can be diagonalised to a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_2)$ where λ_i are non-zero eigenvalues of the form.

Equivalently, every diagonal matrix of the form $\mathrm{diag}(\lambda_1, \ldots, \lambda_2)$ where λ_i are non-zero scalars, defines a symmetric, non-degenerate bilinear form on $\mathfrak{g}$. This can be extended by left-translation to a left-invariant metric on the Lie group. Hence, there is a left-invariant metric for every signature $(v, n - v)$, where $v = 0, 1, \ldots, n$ and $n = \dim(G)$.

This answers the question of existence and uniqueness for left-invariant metrics. Research concerning left-invariant metrics considers instead the geometry of the space and curvature results. An excellent description of Riemannian left-invariant metrics is available in the survey paper Milnor [28], and an equally excellent description of the pseudo-Riemannian ones in the survey paper Albuquerque [1].

If g is both a right- and left-invariant metric, it is called a *bi-invariant* metric. Both Milnor [28] and Albuquerque [1] end their papers with a foray into bi-invariant metrics, which are the main focus of this thesis.

Definition 2.1. A Lie group equipped with a bi-invariant metric is called a *metric Lie group* and its corresponding Lie algebra is called a *metric Lie algebra*.

From Appendix A.6.1, we define the Adjoint maps ad and Ad and we have $\mathrm{Ad}(h) = dL_h|_{h^{-1}} \circ dR_{h^{-1}}|_e$ for all $h \in G$. This gives the following property.

Lemma 2.1. *A left-invariant metric is bi-invariant if and only if it is* Ad*-invariant on* $\mathfrak{g}$.

This can be extended to the Lie algebra.

Lemma 2.2. *On a connected Lie group, a left-invariant metric is bi-invariant if and only if it is* ad*-invariant on* $\mathfrak{g}$.

Here, g is ad-invariant if $g(\mathrm{ad}(X)Y, Z) + g(Y, \mathrm{ad}(X)Z) = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Proof. Let g be a bi-invariant metric on a Lie algebra $\mathfrak{g}$. We will use Ad-invariance to show ad-invariance and vice-versa.

Assume g is ad-invariant. Take a sufficiently small neighbourhood, U, of the identity such that the exponential map, exp, is a diffeomorphism. The definition of exp and the diffeomorphism property can be found in Remark A.1. Take any element $h \in U$. Then as exp is a diffeomorphism in this neighbourhood, we must have $h = \exp(Z)$ for some $Z \in \mathfrak{g}$. Consider for any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} & g(\mathrm{Ad}(\exp(tZ))X, \mathrm{Ad}(\exp(tZ))Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} g(\exp(\mathrm{ad}(tZ))X, \exp(\mathrm{ad}(tZ))Y) \\
&= g\left(\left.\frac{d}{dt}\right|_{t=0} \exp(\mathrm{ad}(tZ))X, Y\right) + g\left(X, \left.\frac{d}{dt}\right|_{t=0} \exp(\mathrm{ad}(tZ))Y\right) \\
&= g(\mathrm{ad}(Z)X, Y) + g(X, \mathrm{ad}(Z)Y) \\
&= 0.
\end{aligned}$$

This implies $g(\mathrm{Ad}(\exp(tZ))X, \mathrm{Ad}(\exp(tZ))Y)$ is constant for all h and t. Equating when $t = 0$ and $t = 1$ we have $g(\mathrm{Ad}(h))X, \mathrm{Ad}(h)Y) = g(X, Y)$. By Lemma A.5, any neighbourhood of the identity generates a connected Lie group, hence this result holds for all $h \in G$.

Similarly, if g is Ad-invariant then for any $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} g(X, Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} g(\mathrm{Ad}(\exp(tZ))X, \mathrm{Ad}(\exp(tZ))Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} g(\exp(\mathrm{ad}(tZ))X, \exp(\mathrm{ad}(tZ))Y) \\
&= g\left(\left.\frac{d}{dt}\right|_{t=0} \exp(\mathrm{ad}(tZ))X, Y\right) + g\left(X, \left.\frac{d}{dt}\right|_{t=0} \exp(\mathrm{ad}(tZ))Y\right) \\
&= g(\mathrm{ad}(Z)X, Y) + g(X, \mathrm{ad}(Z)Y)
\end{aligned}$$

which gives ad-invariance. In both proofs, we used the identity $\exp(\mathrm{ad}) = \mathrm{Ad}(\exp)$, which can be found in Proposition A.6. □

Remark 2.2. As non-degeneracy and symmetry of the metric were not used, this proof holds for any left-invariant tensor on G. That is, a left-invariant tensor on a Lie group is Ad-invariant if and only if it is ad-invariant.

This result is more important than it seems at first. From left-invariance we know everything about how the metric behaves on each tangent space using left-translations. Bi-invariance initially seems that we have to consider how each group element h makes the metric $\mathrm{Ad}(h)$ invariant. However, this lemma tells us that on a connected group, we need only look at $\mathrm{ad}(X)$, for $X \in \mathfrak{g}$, and ignore the group elements. This cements the idea that classifying bi-invariant metrics on connected Lie groups is exactly the task of classifying metric Lie algebras, allowing algebraic techniques on $\mathfrak{g}$ to be used to classify the bi-invariant metrics.

Remark 2.3 (Abelian Lie groups). An important point for later is the notion of an abelian group. Recall a group G is abelian if $h_1h_2 = h_2h_1$ for all $h_1, h_2 \in G$. Then the map c_h is the identity map for all $h \in G$ and hence so is Ad and it follows that ad $= 0$. As $\exp : \mathfrak{g} \to G$ is a local diffeomorphism, and any neighbourhood of the identity generates a connected group, then ad $= 0$ if and only if a connected Lie group is abelian. See Appendix A.6.1 for the definitions of c_h, Ad, ad and exp.

As ad $= 0$, any scalar product on an abelian Lie algebra will have Ad- and ad-invariance trivially satisfied. Thus the class of left-invariant metrics, as described in Remark 2.1, is equal to the class of bi-invariant metrics on abelian Lie algebras.

The question of existence and uniqueness is much harder for semi-Riemannian bi-invariant metrics. There are Lie algebras that do not admit bi-invariant metrics, as in the following example.

Example 2.1. Let $\mathfrak{g}$ be a 2-dimensional Lie algebra with basis u, v and bracket $[u, v] = u$. This is the only non-abelian Lie algebra of dimension 2, and is solvable. See Definition A.20 for the definition of a solvable Lie algebra and Lemma A.9 for a proof of this fact. This Lie algebra admits no non-degenerate ad-invariant symmetric bilinear forms.

Proof. Assume A is a ad-invariant symmetric bilinear form. Then

$$\begin{aligned} 0 =& A([u, v], v) + A(v, [u, v]) = A(u, v) + A(v, u) = 2A(u, v), \\ 0 =& A([v, u], u) + A(u, [v, u]) = A(u, u) + A(u, u) = 2A(u, u). \end{aligned}$$

However, $\det(A) = A(u, u)A(v, v) - 2A(u, v) = 0$. Hence any ad-invariant, symmetric bilinear form on $\mathfrak{g}$ is degenerate. □

A general classification of bi-invariant metrics on Lie groups, in the sense of a list, is not known, however there are several cases that have been fully classified. They include the bi-invariant metrics on reductive Lie groups and the Riemannian bi-invariant metrics; both are explored further in the rest of this chapter. The progress towards a general list has been made in Kath and Olbrich [20] and in Baum and Kath [3], who have developed a classification scheme for metric Lie algebras. This is explored further in Chapter 4.

The reader may find it useful to now recall the definition of simple, semi-simple and reductive Lie algebra and some of their properties; see Appendix A.7 for details.

2.2 Riemannian metric Lie algebras

Proposition 2.3. *Let $\mathfrak{g}$ be a metric Lie algebra with a Riemannian bi-invariant metric. Then $\mathfrak{g}$ is reductive.*

The definition of reductive is that a Lie algebra is isomorphic to the direct sum of an abelian Lie algebra and a semisimple one. This can be found in Definition A.18.

Proof. For any ideal $\mathfrak{h} \subset \mathfrak{g}$ consider the orthogonal space under the metric $\mathfrak{h}^{\perp}$. As the metric is Riemannian, these spaces have zero intersection. Hence $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$.

Using the ad-invariance of the metric, first we show $\mathfrak{h}^{\perp}$ is a subalgebra and then that it is an ideal. Consider $u \in \mathfrak{h}$ and $v, w \in \mathfrak{h}^{\perp}$. Then, as $\mathfrak{h}$ is an ideal, $[u, v] \in \mathfrak{h}$ and $g([u, v], w) = 0$. However, by ad-invariance $g([u, v], w) = g(u, [v, w])$ hence $[v, w] \in \mathfrak{h}^{\perp}$ for all $v, w \in \mathfrak{h}^{\perp}$. Hence, $\mathfrak{h}^{\perp}$ is a subalgebra.

For $u \in \mathfrak{h}$ and $v \in \mathfrak{h}^{\perp}$ $g([u, v], [u, v]) = g(v, [[u, v], u) = 0$ as $\mathfrak{h}$ is an ideal and $[[u, v], u] \in \mathfrak{h}$. By non-degeneracy, $[u, v] = 0$. Hence $\mathfrak{h}^{\perp}$ is an ideal of $\mathfrak{g}$.

If we now consider any subideals of $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ we can continue this process, decomposing $\mathfrak{g}$ into ideals that are either abelian or simple. In particular, the centre $\mathfrak{z}$ is such that $[\mathfrak{z}, \mathfrak{g}] = 0$ is an abelian ideal and a part of the decomposition. This implies $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$ where $\mathfrak{h}_i$, $i = 1, \ldots, r$ are simple ideals of $\mathfrak{g}$, and therefore $\mathfrak{g}$ is reductive. □

We conclude here that classifying Riemannian metric Lie algebras is equivalent to classifying reductive Lie algebras. Fortunately, reductive Lie algebras have certain properties that makes classifying them easier.

Lemma 2.4. *If a metric Lie algebra* $\mathfrak{g}$ *is reductive, then* $\mathfrak{g}$ *decomposes into orthogonal ideals.*

Proof. Let $\mathfrak{g}$ be reductive and consider any semi-simple ideal $\mathfrak{h}_1 \subset \mathfrak{g}$. As $\mathfrak{g}$ is reductive, one can write $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $\mathfrak{h}_2$ a complementary ideal. By Proposition A.11, $[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_1$, and hence $g(\mathfrak{h}_1, \mathfrak{h}_2) = g([\mathfrak{h}_1, \mathfrak{h}_1], \mathfrak{h}_2) = g(\mathfrak{h}_1, [\mathfrak{h}_1, \mathfrak{h}_2]) = g(\mathfrak{h}_1, \{0\}) = \{0\}$. This implies that $\mathfrak{h}_2 \subset \mathfrak{h}_1^{\perp}$, where $\mathfrak{h}_1^{\perp} = \{X \in \mathfrak{g} \mid g(X, Y) = 0, \forall Y \in \mathfrak{h}_1\}$. If $X \in \mathfrak{h}_1 \cap \mathfrak{h}_1^{\perp}$, then $g(X, Y) = 0$ for all $Y \in \mathfrak{h}_1$ and for all $Y \in \mathfrak{h}_1^{\perp}$ and, by non-degeneracy of the metric, this implies $Y = 0$ and hence that $\mathfrak{h}_1^{\perp} = \mathfrak{h}_2$. □

This shows that if $\mathfrak{g}$ is reductive, that is $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}_i \oplus \ldots \oplus \mathfrak{h}_r$, then the metric g orthogonally decomposes to $g = g\big|_{\mathfrak{z}} \oplus g\big|_{\mathfrak{h}_1} \oplus \ldots \oplus g\big|_{\mathfrak{h}_r}$. Knowing the metric on each simple ideal and on the centre then describes the metric on $\mathfrak{g}$. Hence classifying reductive metric Lie algebras corresponds to classifying simple and abelian Lie algebras. Abelian Lie algebras have been considered in Remark 2.3, and the following section classifies simple metric Lie algebras.

2.3 Simple metric Lie algebras

To classify simple semi-Riemannian metric Lie algebras, we consider results about their bilinear forms which can be found in Di Scala et al. [7], of which further necessary theorems for this section can be found in Appendix A.11.

This first theorem shows how restrictive ad-invariance is, and shows that bilinear forms of simple Lie algebras fall into only two categories.

Lemma 2.5. *For a simple Lie algebra* $\mathfrak{h}$*, there are no skew-symmetric* $\mathrm{ad}(\mathfrak{h})$*-invariant bilinear forms.*

Proof. As $\mathfrak{h}$ is simple, by Proposition A.11 we have $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$.

Now if $B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{K}$ is a skew-symmetric $\mathrm{ad}(\mathfrak{h})$-invariant bilinear form, then using ad-invariance

$$\begin{aligned} B([x,y],z) + B(y,[x,z]) = 0 \text{ and } B([y,x],z) + B(x,[y,z]) = 0 \\ \Rightarrow B(y,[x,z]) + B(x,[y,z]) = 0 \\ B(y,[x,z]) - B(x,[z,y]) = 0 \\ B(y,[x,z]) + B([z,x],y) = 0 \end{aligned}$$

and using skew-symmetry

$$\begin{aligned} -B([x,z],y) + B([z,x],y) = 0 \\ \Rightarrow 2B([x,z],y) = 0 \qquad \forall x,y,z \in \mathfrak{h}. \end{aligned}$$

This implies that B is identically 0. □

Corollary 2.6. *A simple Lie algebra must be of real or complex type.*

Proof. This result follows from Lemma A.34 and Lemma 2.5. □

Remark 2.4. A simple Lie algebra of real type means the Lie algebra has a one dimensional space of non-degenerate ad-invariant bilinear forms and, equivalently, that the complexification of the Lie algebra is also simple. A simple Lie algebra of complex type can be complexified to a semi-simple Lie algebra, which is isomorphic to the direct sum of two simple ideals $W \oplus \overline{W}$, and, equivalently, the space of non-degenerate bi-invariant bilinear forms is two dimensional. In both cases, we can conclude that the ad-invariant forms are symmetric from Lemma 2.5. See Proposition A.32 and Remark A.7 for further details.

In Wilhelm Killing's early work on Lie theory he described a trace form which was invariant under the group action. However it was Élie Cartan who recognised importance of the form in the classification of a Lie algebra's bilinear forms. Sometimes known as the Cartan-Killing form or just the Killing form, it is an ad-invariant scalar product on the Lie algebra. More historical details can be found in Varadarajan [38]. Recall the definition of the Killing form and some of its properties.

Definition 2.2. The *Killing form* of a Lie algebra $\mathfrak{g}$ (or Lie group) is the bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by $K(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$.

Properties 2.1. *The Killing form is symmetric, and bilinear by the properties of the bracket and trace. Using the Jacobi identity, we have $K([X,Y],Z) = K(X,[Y,Z])$ for all $X,Y,Z \in \mathfrak{g}$ and hence the Killing form is* ad*-invariant. By Remark 2.2 it is therefore also Ad-invariant, that is $K(Ad(g)X, Ad(g)Y) = K(X,Y)$ for all $X,Y \in \mathfrak{g}$ and $g \in G$.*

It seems the Killing form is a candidate to induce a bi-invariant metric. However, the Killing form may be degenerate.

Proposition 2.7. *The Killing form defines a metric Lie algebra if and only if the Lie algebra is semi-simple.*

Proof. From the definition of the Killing form, it is always a symmetric, ad-invariant bilinear form on a Lie algebra. The non-degeneracy is a result of the Cartan Criterion, which states that the Killing form is non-degenerate if and only if the Lie algebra is semi-simple. See Theorem A.10 for details. $\square$

The reader may wish to refer to Appendix A.10.1 on complexifications, realifications and real forms as a reference for the following theorem and proof.

Theorem 2.8. *Let G is a Lie group with a simple Lie algebra, $\mathfrak{g}$. If G admits a bi-invariant metric g, then either*

1. *the Lie algebra $\mathfrak{g}$ is of real type and the metric g is a multiple of the Killing form, K, or*

2. *the Lie algebra $\mathfrak{g}$ is of complex type, that is, it is the realification of a complex Lie algebra $\mathfrak{h}$, and the metric g is a real linear combination of $K_{\mathbb{R}}$ and $K_{\mathbb{I}}$, the real and complex parts of the Killing form on $\mathfrak{h}$ respectively. The metric g has signature (n, n), where n is the dimension of $\mathfrak{h}$.*

This result is mentioned Medina [25, pg. 410] however is not well known. Note immediately that any simple Riemannian metric Lie algebra would have to be of real type.

Proof. First consider if $\mathfrak{g}$ is odd dimensional. Then the proof is similar to that of Albuquerque [1, Theorem 3], which is as follows. Consider the linear bijection S of $\mathfrak{g}$ such that

$$g(x, y) = K(S(x), y) \qquad \forall x, y \in \mathfrak{g}.$$

Then

$$\begin{aligned} K(S(\mathrm{ad}(z)x), y) = g(\mathrm{ad}(z)x, y) = g(x, \mathrm{ad}(z)y) =& K(S(x), \mathrm{ad}(z)y) = K(\mathrm{ad}(z)S(x), y) \\ K(S(\mathrm{ad}(z)x) - \mathrm{ad}(z)S(x), y) =& 0 \qquad \forall x, y, z \in \mathfrak{g} \end{aligned}$$

by non-degeneracy of K. Hence $S(\mathrm{ad}(z)x) = \mathrm{ad}(z)S(x)$, so S commutes with ad.

As $\mathfrak{g}$ has odd dimension, then there is at least one real eigenvalue of S with associated eigenvalue λ and eigenvector y. Then for all $x \in \mathfrak{g}$,

$$S[x, y] = S(\mathrm{ad}(x)y) = \mathrm{ad}(x)S(y) = \mathrm{ad}(x)(\lambda y) = \lambda[x, y].$$

This means the eigenspace is an ideal of $\mathfrak{g}$ and $\mathfrak{g}$ is simple so this is all of $\mathfrak{g}$. Then

$$g(x, y) = K(S(x), y) = K(\lambda x, y) = \lambda K(x, y)$$

and the result holds.

Now consider $\mathfrak{g}$ even dimensional. If we do the same construction for S and S has a real eigenvalue, then the proof is the same as above. Assume now that S does not have a real eigenvalue, then it has at least two complex ones λ and $\overline{\lambda}$. Consider $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$.

The same proof above shows that the eigenspaces, $\mathfrak{h}$, $\overline{\mathfrak{h}}$ corresponding to λ, $\overline{\lambda}$ are ideals, and, as they are distinct, we must have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \overline{\mathfrak{h}}$. Following Remark A.6, we have that $\mathfrak{g} \cong \mathfrak{h}_{\mathbb{R}} \cong \overline{\mathfrak{h}_{\mathbb{R}}}$.

Applying Corollary 2.6 and Remark 2.4, the space of ad-invariant bilinear forms on $\mathfrak{g}$ must be two dimensional and entirely symmetric.

Finally, take any real form of $\mathfrak{h}$, which always exists by Remark A.5. The definition of a real form can be found in Definition A.27. Let $Y_1, \ldots, Y_n$ be a basis for this real form and consider these Y_i now as elements of $\mathfrak{h}$ and note that $Y_1, \ldots, Y_n$ form a basis for $\mathfrak{h}$. We may now write $K_\mathfrak{h}$, the Killing form of $\mathfrak{h}$, as a matrix using this basis.

Define $K_\mathbb{R}(X, Z) = \mathrm{Re}K_\mathfrak{h}(X, Z)$ and $K_\mathbb{I}(X, Z) = \mathrm{Im}K_\mathfrak{h}(X, Z)$ as bilinear functions $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by taking $X, Z \in \mathfrak{g}$ then considering them as elements of $\mathfrak{h}$, then applying $K_\mathfrak{h}$, then taking the real and imaginary parts respectively. Then writing $K_\mathbb{R}$ and $K_\mathbb{I}$ using $Y_1, \ldots, Y_n, iY_1, \ldots, iY_n$ as a basis for $\mathfrak{g}$, we have

$$K_\mathbb{R} = \begin{pmatrix} K_\mathfrak{h} & 0 \\ 0 & -K_\mathfrak{h} \end{pmatrix} \qquad \text{and} \qquad K_\mathbb{I} = \begin{pmatrix} 0 & K_\mathfrak{h} \\ K_\mathfrak{h} & 0 \end{pmatrix}$$

which describe linearly independent and non-degenerate forms on $\mathfrak{g}$. By construction, they are also ad-invariant. Let $X_1, \ldots, X_n$ be the eigenvectors of $K_\mathfrak{h}$ with eigenvalues λ_j. Then $X_1, \ldots, X_n$ and $iX_1, \ldots, iX_n$ are the eigenvectors of $K_\mathbb{R}$ with eigenvalues λ_j and $-\lambda_j$ respectively, which gives a signature of (n, n). Similarly, the eigenvectors of $K_\mathbb{I}$ are $X_j + iX_j$ and $X_j - iX_j$ with eigenvalues λ_j and $-\lambda_j$ respectively, which also gives a signature of (n, n). As $K_\mathbb{R}$ and $K_\mathbb{I}$ are symmetric ad-invariant bilinear forms on $\mathfrak{g}$, the metric g must be a real linear combination of the two and the result follows. $\square$

Note that from Proposition 2.7, we know the Killing form of $\mathfrak{g}$ must define a metric on $\mathfrak{g}$, which appears not to be considered in the second case of Theorem 2.8. We note the following lemma which shows that the Killing form of $\mathfrak{g}$ is included in the second case; we show the Killing form of $\mathfrak{g}$ is a multiple of the real part of the Killing form of $\mathfrak{h}$.

Lemma 2.9. *The Killing form of $\mathfrak{g}$ in the second case of the above theorem, Theorem 2.8, is $K_\mathfrak{g} = 2K_\mathbb{R}$.*

Proof. Let $Y_1, \ldots, Y_n$ be a basis for $\mathfrak{h}$ then $Y_1, \ldots, Y_n, iY_1, \ldots, iY_n$ is (isomorphic to) a basis for $\mathfrak{g}$. For $a, b = 1, \ldots, n$, we find that

$$\mathrm{ad}^\mathfrak{g}_{Y_a} = \begin{pmatrix} \mathrm{ad}^\mathfrak{h}_{Y_a} & 0 \\ 0 & \mathrm{ad}^\mathfrak{h}_{Y_a} \end{pmatrix} \qquad \text{and} \qquad \mathrm{ad}^\mathfrak{g}_{iY_a} = \begin{pmatrix} 0 & -\mathrm{ad}^\mathfrak{h}_{Y_a} \\ \mathrm{ad}^\mathfrak{h}_{Y_a} & 0 \end{pmatrix}.$$

Computing the Killing form we find that

$$\begin{pmatrix} \mathrm{ad}^\mathfrak{h}_{Y_a} & 0 \\ 0 & \mathrm{ad}^\mathfrak{h}_{Y_a} \end{pmatrix} \begin{pmatrix} \mathrm{ad}^\mathfrak{h}_{Y_b} & 0 \\ 0 & \mathrm{ad}^\mathfrak{h}_{Y_b} \end{pmatrix} = \begin{pmatrix} \mathrm{ad}^\mathfrak{h}_{Y_a} \mathrm{ad}^\mathfrak{h}_{Y_b} & 0 \\ 0 & \mathrm{ad}^\mathfrak{h}_{Y_a} \mathrm{ad}^\mathfrak{h}_{Y_b} \end{pmatrix},$$

which implies $K_\mathfrak{g}(Y_a, Y_b) = \mathrm{tr}\{\mathrm{ad}^\mathfrak{g}_{Y_a} \circ \mathrm{ad}^\mathfrak{g}_{Y_b}\} = 2\,\mathrm{tr}\{\mathrm{ad}^\mathfrak{h}_{Y_a} \circ \mathrm{ad}^\mathfrak{h}_{Y_b}\} = 2K_\mathfrak{h}(Y_a, Y_b)$. Also

$$\begin{pmatrix} 0 & -\mathrm{ad}^\mathfrak{h}_{Y_a} \\ \mathrm{ad}^\mathfrak{h}_{Y_a} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{ad}^\mathfrak{h}_{Y_b} \\ \mathrm{ad}^\mathfrak{h}_{Y_b} & 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{ad}^\mathfrak{h}_{Y_a} \mathrm{ad}^\mathfrak{h}_{Y_b} & 0 \\ 0 & -\mathrm{ad}^\mathfrak{h}_{Y_a} \mathrm{ad}^\mathfrak{h}_{Y_b} \end{pmatrix},$$

which implies $K_\mathfrak{g}(iY_a, iY_b) = \mathrm{tr}\{\mathrm{ad}^\mathfrak{g}_{iY_a} \circ \mathrm{ad}^\mathfrak{g}_{iY_b}\} = -2\,\mathrm{tr}\{\mathrm{ad}^\mathfrak{h}_{Y_a} \circ \mathrm{ad}^\mathfrak{h}_{Y_b}\} = -2K_\mathfrak{h}(Y_a, Y_b)$.

Finally

$$\begin{pmatrix} \mathrm{ad}^{\mathfrak{h}}_{Y_a} & 0 \\ 0 & \mathrm{ad}^{\mathfrak{h}}_{Y_a} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{ad}^{\mathfrak{h}}_{Y_b} \\ \mathrm{ad}^{\mathfrak{h}}_{Y_b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{ad}^{\mathfrak{h}}_{Y_a}\mathrm{ad}^{\mathfrak{h}}_{Y_b} \\ \mathrm{ad}^{\mathfrak{h}}_{Y_a}\mathrm{ad}^{\mathfrak{h}}_{Y_b} & 0 \end{pmatrix},$$

which implies $K_{\mathfrak{g}}(Y_a, iY_b) = \mathrm{tr}\{\mathrm{ad}^{\mathfrak{g}}_{Y_a} \circ \mathrm{ad}^{\mathfrak{g}}_{iY_b}\} = 0$.

Hence $K_{\mathfrak{g}} = 2K_{\mathbb{R}}$. □

See also Helgason [16, pg. 180] for a similar proof.

Example 2.2. Consider the realification of the Lie group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \quad \text{where} \quad a, b, c, d \in \mathbb{C} \right\}.$$

Its corresponding Lie algebra is

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}.$$

Over the complex numbers, $\mathfrak{sl}_2(\mathbb{C})$ has basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Over the real numbers, it has basis

$$e_1,\ e_2,\ e_3,\ ie_1,\ ie_2,\ ie_3.$$

Its bracket operations are

$$[e_1, e_2] = -2e_1,\ [e_2, e_3] = -2e_3,\ [e_1, e_3] = e_2.$$

It is a simple Lie algebra over the real and complex numbers. The proof is as follows: Over the complex numbers, assume there is an ideal I with non-zero element, $x = ae_1 + be_2 + ce_3$. Note that $[[x, e_1], e_1] = -2ce_1$, $[[x, e_2], e_3] = -2ae_2$ and $[[x, e_3], e_2] = be_3$. As x is non-zero, at least one of a, b, c are non-zero, hence at least one e_i is in the ideal. Using the bracket relations, then the other e_i's must be in the set, so the ideal is equal to $\mathfrak{sl}_2(\mathbb{C})$. The same argument follows over the real numbers.

The Killing form on $\mathfrak{sl}_2(\mathbb{C})$, using the basis given, has matrix

$$K = 4\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Diagonalising, we have

$$K \cong 4\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence it has signature $(1, 2)$.

From the above formulae, we have

$$K_{\mathbb{R}} = 4\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$K_{\mathbb{I}} = 4\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These both have signature $(3,3)$.

Any ad-invariant scalar product on $\mathfrak{sl}_2(\mathbb{C})$ is a real linear combination of these two. Finally, consider the reductive Lie algebra $\mathfrak{gl}_2(\mathbb{C}) = M_2(\mathbb{C})$. Over the complex numbers, this splits into its centre $\mathfrak{z} = \text{span}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathfrak{sl}_2(\mathbb{C})$. Over the real numbers we have

$$\mathfrak{gl}_2(\mathbb{C}) = M_2(\mathbb{C}) = \mathbb{C} \cdot I \oplus \mathfrak{sl}_2(\mathbb{C}).$$

As $\mathbb{C} \cdot I$ is abelian, ad-invariance is trivial, hence we can describe ad-invariant scalar products of signature $(0,2), (1,1), (2,0)$. For example, picking a basis $e_4 = I, e_5 = iI$ for $\mathbb{C} \cdot I$, we can describe metrics on $\mathbb{C} \cdot I$ by linear combinations of

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } g_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There are lots of other examples! In fact, any non-degenerate real symmetric 2×2 matrix describes a metric, and these are distinguished by their eigenvalues.

Using $K_{\mathbb{I}}$ and $K_{\mathbb{R}}$ and a non-degenerate real symmetric 2×2 matrix, we can describe ad-invariant scalar products on $\mathfrak{gl}_2(\mathbb{C})$ of signature $(3,5), (4,4)$ and $(5,3)$.

2.4 Conclusion

From Lemma 2.4, any reductive metric Lie algebra orthogonally decomposes to a direct sum of simple and abelian ideals. Hence describing ad-invariant, non-degenerate symmetric, bilinear forms on abelian and simple Lie algebras describes them on reductive Lie algebras. The abelian case is classified in Remark 2.3, which explains that any non-degenerate, symmetric bilinear form is ad-invariant on an abelian Lie algebra. Theorem 2.8 describes all the possible bi-invariant metrics on simple Lie algebras, showing that every simple Lie algebra is a metric Lie algebra. Here the metric is either described by the Killing form, or the metric is a linear combination of the real and imaginary parts of the Killing form of a complex Lie algebra and has signature $(\frac{n}{2}, \frac{n}{2})$.

There is in fact a complete list of simple groups and simple Lie algebras, and their Killing forms are well known. Wilhelm Killing and Élie Cartan developed this list; Killing made significant progress in 1888 to 1890 and Cartan providing further details in his thesis in 1894. Along with Theorem 2.8, the list completely classifies semi-Riemannian reductive metric Lie algebras. As metric Lie algebras correspond to connected metric Lie groups from Lemma 2.2, this classifies connected, reductive metric Lie groups.

Finally, Proposition 2.3 implies that the Riemannian metric Lie algebras are reductive. We note from Helgason [16, pg. 132] and Milnor [28, pg. 324] that a simple Lie group has a Riemannian bi-invariant metric if and only if it has negative definite Killing form which is if and only if it is compact. The negative of the Killing form then induces a Riemannian metric on the Lie group. Hence Riemannian metric Lie groups are precisely the direct sum of compact simple Lie algebras and abelian metric Lie algebras.

Chapter 3

Einstein and conformally Einstein bi-invariant metrics

In this section we consider bi-invariant metrics that are Einstein or conformally Einstein, combining the results of the first two chapters. We show how the obstructions to conformally Einstein metrics simplify for bi-invariant metrics. We show that on simple Lie groups, the Killing form is always an Einstein metric however there are metrics on simple Lie groups are not Einstein nor conformally Einstein, although they are Bach-flat. We also show how the obstructions and curvature tensors from Chapter 1 simplify in the case that the Lie algebra is solvable.

3.1 The curvature tensors on Lie groups with bi-invariant metrics

The curvature tensors on a metric Lie algebra $\mathfrak{g}$ have a close relationship with the bracket and the Killing form.

Proposition 3.1. *Let G be a Lie group equipped with a bi-invariant metric, g. For any $X, Y, Z \in \mathfrak{g}$ we have that*

$$\begin{aligned}
\nabla_X Y =& \frac{1}{2}[X, Y] \\
R_{XY} Z =& \frac{1}{4}[[X, Y], Z] \\
R_{XY} Z(U) =& \frac{1}{4} g([X, Y], [Z, U]) \\
\mathrm{Ric}(X, Y) =& -\frac{1}{4} K(X, Y).
\end{aligned}$$

Note that as the Killing form is ad*-invariant, hence so is the Ricci tensor.*

Proof. Pick any three elements in the Lie algebra, V, W, X. Now note that on the Lie algebra, (D5) is

$$Xg(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W) = \frac{1}{2}\left(g(\mathrm{ad}(X)V, W) + g(V, \mathrm{ad}(X)W)\right) = 0$$

due to the ad-invariance of the metric which will be useful in the following calculation.

Using this fact, bi-invariance and symmetry of metric and skew symmetry of the bracket, the Kozsul formula gives

$$\begin{aligned}
2g(\nabla_V W, X) =& g(W, [X, V]) + g(X, [V, W]) - g(V, [W, X]) \\
2g(\nabla_V W, X) =& -g(W, [V, X]) + g([V, W], X) + g([W, V], X) \\
2g(\nabla_V W, X) =& g([V, W], X]) \\
\Rightarrow \nabla_V W =& \frac{1}{2}[V, W].
\end{aligned}$$

Using the Jacobi identity, we have

$$\begin{aligned}
R_{XY}Z &= \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \\
&= \frac{1}{2}[[X, Y], Z] - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z \\
&= \frac{1}{2}[[X, Y], Z] - \frac{1}{2}\nabla_X [Y, Z] + \frac{1}{2}\nabla_Y [X, Z] \\
&= \frac{1}{2}[[X, Y], Z] + \frac{1}{4}[[Y, Z], X] - \frac{1}{4}[[X, Z], Y] \\
&= \frac{1}{2}[[X, Y], Z] + \frac{1}{4}[[Y, Z], X] - \frac{1}{4}[[X, Y], Z] - \frac{1}{4}[[Y, Z], X] \\
&= \frac{1}{4}[[X, Y], Z].
\end{aligned}$$

We then have

$$\begin{aligned}
R_{XY}Z(U) &= g(R_{XY}Z, U) \\
&= g(\frac{1}{4}[[X, Y], Z], U) \\
&= \frac{1}{4}g([X, Y], [Z, U]).
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{Ric}(X, Y) &= \mathrm{tr}\{V \mapsto R_{XV}Y\} \\
&= \mathrm{tr}\{V \mapsto \frac{1}{4}[[X, V], Y]\} \\
&= -\mathrm{tr}\{V \mapsto \frac{1}{4}[Y, [X, V]]\} \\
&= -\frac{1}{4}\mathrm{tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X)) \\
&= -\frac{1}{4}K(X, Y).
\end{aligned}$$

□

Lemma 3.2. *The Killing form defines an Einstein bi-invariant metric on a connected Lie group with semisimple Lie algebra.*

Proof. By Proposition 3.1, the Killing form is a multiple of the Ricci tensor on the Lie algebra, and hence, by its non-degeneracy in Proposition 2.7, it defines an Einstein metric on the Lie group. □

Note that if each simple ideal is of real type, the Killing form defines the only bi-invariant metric. We consider the case when the simple ideal is of complex type in Theorem 3.13. See also Remark A.7 for the definitions of complex and real type.

Lemma 3.3. *If a metric Lie algebra has non-zero Ricci tensor, then the metric is Einstein only if the Lie algebra is semi-simple.*

Proof. If the metric, g, is Einstein with Einstein constant λ, then using Proposition 3.1,

$$\lambda g = \mathrm{Ric} = -\frac{1}{4}K.$$

If λ is non-zero, then as g is non-degenerate, K must be also, and by the Cartan Criterion Theorem A.10, it must be semi-simple. Otherwise, λ must be zero, which implies the Ricci tensor is zero and so too is the Killing form. In this case the Killing form is degenerate and hence the Lie algebra cannot be semi-simple by Theorem A.10. □

Another way to consider this is that a non-semi-simple metric Lie algebra induces an Einstein metric on the Lie group only when the Einstein constant is zero, and the Ricci tensor is also zero. Of course, there are several metric Lie algebras that are not semi-simple and have non-zero Ricci tensor, which can therefore not be Einstein. One such class of examples appears in Theorem 4.13 and there are various others in Chapter 4. To consider the property that they are instead conformally Einstein, we will use the conformally Einstein definition from Definition 1.15 and the obstructions to conformally Einstein metrics in Proposition 1.29. These obstructions use the Schouten, Weyl, Cotton and Bach tensors, so we now simplify these tensors when the metric is bi-invariant on a Lie group.

3.2 The Schouten, Cotton, Bach and Weyl tensors and conformal to Einstein obstructions

Consider the Schouten tensor

$$\mathsf{P}_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{S}{2(n-1)}g_{ab}\right).$$

Then if X, Y are vector fields on a Lie group with bi-invariant metric, we have

$$\mathsf{P}(X,Y) = \frac{1}{n-2}\left(\mathrm{Ric}(X,Y) - \frac{S}{2(n-1)}g(X,Y)\right).$$

Hence

Lemma 3.4. *On a metric Lie group, if X, Y are elements of a metric Lie algebra then the Schouten tensor on the Lie algebra is*

$$\mathsf{P}(X,Y) = \frac{1}{n-2}\left(-\frac{1}{4}K(X,Y) - \frac{S}{2(n-1)}g(X,Y)\right)$$

and it is also ad-invariant.

To consider the Cotton tensor, we make use of the following lemma.

Lemma 3.5. *On a metric Lie group with Lie algebra, $\mathfrak{g}$, let R be the Riemannian curvature tensor, then*

$$\nabla R = 0, \quad \nabla \operatorname{Ric} = 0 \text{ and } \nabla S = 0.$$

Proof. Consider $(\nabla_X R)_{YZ}W$ for $X, Y, Z, W \in \mathfrak{g}$. Using the Jacobi identity and bi-invariance of the metric, we have

$$\begin{aligned}
(\nabla_X R)_{YZ}W &= \frac{1}{8}\left([X,[[Y,Z],W]] - [[[X,Y],Z],W] - [[Y,[X,Z]],W] - [[Y,Z],[X,W]]\right) \\
&= \frac{1}{8}\left([X,[[Y,Z],W]] - [[X,[Y,Z]],W] - [[Y,Z],[X,W]]\right) \\
g((\nabla_X R)_{YZ}W, V) &= \frac{1}{8}\left(g([X,[[Y,Z],W]],V) - g([[X,[Y,Z]],W],V) - g([[Y,Z],[X,W]],V)\right) \\
&= \frac{-1}{8}\left(g([Y,Z],[W,[X,V]]) + g([Y,Z],[X,[V,W]]) + g([Y,Z],[V,[W,X]])\right) \\
&= \frac{1}{8}\left(-g([Y,Z],[W,[X,V]] + [X,[V,W]] + [V,[W,X]])\right) \\
&= 0.
\end{aligned}$$

This is true for all $V \in \mathfrak{g}$. Thence, as g is non-degenerate, $\nabla_X R \equiv 0$. Now, on $\mathfrak{g}$, using the ad-invariance of the metric we have

$$\begin{aligned}
(\nabla_V \operatorname{Ric})(X,Y) &= -\frac{1}{4}\big(\nabla_V(K(X,Y)) - K(\nabla_V X, Y) - K(X, \nabla_V Y)\big) \\
&= -\frac{1}{4}\nabla_V(K(X,Y)) \\
&= 0.
\end{aligned}$$

By the contracted Bianchi identity, $2\nabla^i R_{ij} = \nabla_j S$ then we also have $\nabla_j S = 0$.

As the Lie algebra is the set of all left invariant vector fields on G then any vector field, $V \in \mathfrak{X}(M)$ can be written $V = V^i X_i$ where $X_1, \ldots, X_n$ form a basis for the Lie algebra and $V^i \in \mathfrak{F}(G)$. As tensors are $\mathfrak{F}(G)$-linear, then if a tensor is 0 on the Lie algebra, it must be zero on the entire tangent bundle. Hence each of these tensors vanish on the entire Lie group. □

We now consider the Cotton tensor.

Proposition 3.6. *On a metric Lie group, the Cotton tensor, A, vanishes.*

Proof. On the metric Lie algebra, from Lemma 3.5, we have $\nabla_j S = 0$. This implies,

$$\nabla_a P_{bc} = \frac{1}{n-2}\Big(\nabla_a R_{bc} - \frac{(\nabla_a S)}{2(n-1)} g_{bc}\Big) = 0.$$

Hence we conclude the result on the Lie algebra

$$A_{abc} = 2\nabla_{[b}\mathsf{P}_{c]a} = 0.$$

As in the proof of Lemma 3.5, we deduce that this is true on the entire Lie group. □

To simplify the Bach tensor for a bi-invariant metric, we use the following identity.

Lemma 3.7. *Let G be a metric Lie group with Lie algebra $\mathfrak{g}$ and bi-invariant metric g. We can show that $-R^j{}_a R_{bj} = R^{ij} R_{jbia}$. In fact if $E_1, \ldots, E_n$ is an orthonormal basis for $\mathfrak{g}$, then on the Lie algebra*

$$-R^j{}_a R_{bj} = \frac{\epsilon_j}{16} K([E_a, E_j], [E_b, E_j]) = R^{ij} R_{jbia}. \tag{3.2.1}$$

Proof. For an orthonormal basis, $g(E_i, E_i) = \epsilon_i$ and we write $g_{ij} = \epsilon_i \delta_{ij}$. For the left hand side, we have

$$\begin{aligned}
R^i{}_a R_{bi} &= g^{ik} R_{ak} R_{bi} = \epsilon_i R_{ai} R_{bi} \\
&= \epsilon_i \frac{-1}{4} K(E_a, E_i) R^j{}_{bij} \\
&= -\epsilon_i \epsilon_j \frac{1}{4} K(E_a, E_i) R_{jbij} \\
&= -\epsilon_i \epsilon_j \frac{1}{16} K(E_a, E_i) g([E_i, E_j], [E_b, E_j]) \\
&= \frac{-\epsilon_i \epsilon_j}{16} K(E_a, E_i) g(E_i, [E_j, [E_b, E_j]]) \\
&= \frac{-\epsilon_j}{16} K(E_a, \epsilon_i g(E_i, [E_j, [E_b, E_j]]) E_i) \\
&= \frac{-\epsilon_j}{16} K(E_a, [E_j, [E_b, E_j]]) \\
&= \frac{-\epsilon_j}{16} K([E_a, E_j], [E_b, E_j]).
\end{aligned}$$

For the right hand side, we have

$$\begin{aligned}
R^{ij} R_{jbia} &= \epsilon_i \epsilon_j R_{ij} R_{jbia} \\
&= \frac{-\epsilon_i \epsilon_j}{16} K(E_i, E_j) g([E_i, E_a], [E_b, E_j]) \\
&= \frac{-\epsilon_i \epsilon_j}{16} K(E_i, E_j) g(E_i, [E_a, [E_b, E_j]]) \\
&= \frac{-\epsilon_j}{16} K(\epsilon_i g(E_i, [E_a, [E_b, E_j]]) E_i, E_j) \\
&= \frac{-\epsilon_j}{16} K([E_b, E_j], [E_j, E_a]) \\
&= \frac{\epsilon_j}{16} K([E_a, E_j], [E_b, E_j]).
\end{aligned}$$

As the Lie algebra is the set of left invariant vector fields on G then any vector field, $V \in \mathfrak{X}(M)$ can be written $V = V^i X_i$ where $X_1, \ldots, X_n$ form a basis for the Lie algebra and $V^i \in \mathfrak{F}(G)$. As tensors are $\mathfrak{F}(G)$-linear, then the V^i can be factored out of $-R^j{}_a R_{bj}$ and $R^{ij} R_{jbia}$. Then applying this result on the Lie algebra means that these are equal for all elements of $\mathfrak{X}(M)$. Hence the result holds on the Lie group.

□

Proposition 3.8. *For a metric Lie group with metric g, the Bach tensor is*

$$B_{bd} = \frac{1}{(n-2)^2}\left(nR_b{}^aR_{da} - \frac{Sn}{(n-1)}R_{bd} + g_{db}\left(\frac{S^2}{n-1} - R^{ac}R_{ac}\right)\right).$$

Proof. From Proposition 3.6, the Cotton tensor is 0 and hence the Bach tensor is

$$B_{bd} = \nabla^c A_{bcd} + \mathsf{P}^{ac}C_{abcd} = \mathsf{P}^{ac}C_{abcd}.$$

Now, consider $g^{ac}C_{abcd} = C^a{}_{bad} = 0$ as the Weyl is totally trace free. Expanding the remaining terms using Lemma 3.7, we have

$$\begin{aligned}
B_{bd} =&\mathsf{P}^{ac}C_{abcd}\\
=&\frac{1}{n-2}\left(R^{ac} - \frac{S}{2(n-1)}g^{ac}\right)C_{abcd}\\
=&\frac{1}{n-2}R^{ac}C_{abcd}\\
=&\frac{1}{n-2}\Big(-R^{ac}R_{cdab} + \frac{1}{n-2}(g_{cb}R^{ac}R_{ad} - g_{ca}R^{ac}R_{bd} + g_{da}R^{ac}R_{bc} - g_{db}R^{ac}R_{ac})\\
&+\frac{S}{(n-2)(n-1)}(R^{ac}g_{ac}g_{bd} - R^{ac}g_{bc}g_{ad})\Big)\\
=&\frac{1}{n-2}\Big(R^{ac}R_{bacd} + \frac{1}{n-2}(R^a{}_bR_{ad} - (S)R_{bd} + R^a{}_bR_{da} - g_{db}R^{ac}R_{ac})\\
&+\frac{S}{(n-2)(n-1)}(Sg_{bd} - R_{bd})\Big)\\
=&\frac{1}{n-2}\Big(R^a{}_bR_{ad} + \frac{1}{n-2}(2R^a{}_bR_{ad} - (S)R_{bd} - g_{db}R^{ac}R_{ac}) + \frac{S}{(n-2)(n-1)}(Sg_{bd} - R_{bd})\Big)\\
=&\frac{1}{n-2}\left(\left(\frac{2}{n-2}+1\right)R^a{}_bR_{ad} - \frac{(S)R_{bd}}{n-2}\left(1+\frac{1}{n-1}\right) + \frac{g_{bd}}{n-2}\left(-R^{ac}R_{ac} + \frac{S^2}{n-1}\right)\right)\\
=&\frac{1}{n-2}\left(\frac{n}{n-2}R_b{}^aR_{da} - \frac{Sn}{(n-1)(n-2)}R_{bd} + \frac{g_{db}}{n-2}\left(\frac{S^2}{n-1} - R^{ac}R_{ac}\right)\right).
\end{aligned}$$

Hence the result holds. □

We now recall Proposition 1.29 which is as follows: If g is a conformally Einstein metric on a manifold, then its Cotton and Bach and Weyl tensors satisfy

$$A_{abc} + \Upsilon^d C_{dabc} = 0$$

and

$$B_{ab} + (n-4)\Upsilon^d\Upsilon^c C_{dabc} = 0$$

where $\nabla^a\Upsilon = \Upsilon^a$ for some $\Upsilon \in \mathfrak{F}(X)$.

When the metric is bi-invariant on a Lie group G, we use Proposition 3.8 and Proposition 3.6 and conclude that Proposition 1.29 reduces to

Proposition 3.9. *On a metric Lie group G, if the metric is conformally Einstein then the following obstructions must be satisfied:*

$$\Upsilon^a C_{abcd} = 0 \tag{3.2.2}$$

and

$$B_{bd} = 0 \tag{3.2.3}$$

where $\nabla^a \Upsilon = \Upsilon^a$ *for some* $\Upsilon \in \mathfrak{F}(G)$. *Hence we require that*

$$\Upsilon^a C_{abcd} = \Upsilon^a \Big(- R_{cdab} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) + \frac{S}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}) \Big) = 0$$

and

$$B_{bd} = \frac{1}{n-2}\left(\frac{n}{n-2}R_b{}^a R_{da} - \frac{Sn}{(n-1)(n-2)}R_{bd} + \frac{g_{db}}{n-2}\left(\frac{S^2}{n-1} - R^{ac}R_{ac}\right)\right) = 0.$$

3.3 Bi-invariant metrics with 2-step nilpotent Ricci tensor

We consider now several results that concern bi-invariant metrics with 2-step nilpotent Ricci tensor. This property simplifies the obstructions further. We further show that solvable Lie algebras have this property so the results can be applied to them.

Definition 3.1. We call a tensor, A_{ab}, *2-step nilpotent* if its square $A^2_{ab} = A_{ac}A^c{}_b$ is 0.

Lemma 3.10. *If the Ricci tensor of a semi-Riemannian manifold is 2-step nilpotent, then the scalar curvature* $S = 0$.

Proof. Let λ be any eigenvalue of $\mathrm{Ric}^a{}_b$ with corresponding eigenvector $X \in \mathfrak{g}$. Then $0 = \mathrm{Ric}^2(X, X) = \lambda^2 g(X, X)$. As the metric is non-degenerate, this is true if and only if $\lambda^2 = 0$, hence $\lambda = 0$. Finally, $S = R^a{}_a = \sum \lambda_i = 0$. □

Proposition 3.11. *If the Ricci tensor of a metric Lie group is 2-step nilpotent, then the metric is Bach-flat and hence satisfies the second obstruction. The Weyl tensor reduces to*

$$C_{abcd} = -R_{abcd} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}).$$

Proof. By Lemma 3.10, the scalar curvature $S = 0$. As $\mathrm{Ric}^2_{ab} = R^j_a R_{bj} = 0$ then the formula for the Bach tensor from Proposition 3.8 is

$$\begin{aligned} B_{bd} &= \frac{1}{n-2}\left(\frac{n}{n-2}R_b{}^a R_{da} - \frac{Sn}{(n-1)(n-2)}R_{bd} + \frac{g_{db}}{n-2}\left(\frac{S^2}{n-1} - R^{ac}R_{ac}\right)\right) \\ &= \frac{1}{n-2}\left(\frac{n}{n-2}\,\mathrm{Ric}^2_{bd} + \frac{\mathrm{Ric}^2{}_a{}^a}{n-2} g_{db}\right) \\ &= 0. \end{aligned}$$

The Weyl tensor is

$$\begin{aligned} C_{abcd} &= -R_{abcd} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) \\ &\quad + \frac{S}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}) \\ &= -R_{abcd} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}). \end{aligned}$$

□

We can show that solvable Lie algebras have 2-step nilpotent Ricci tensor and conclude the following proposition.

Proposition 3.12. *If G is a metric Lie group with solvable Lie algebra $\mathfrak{g}$ and metric g, then*

- *the Ricci tensor is 2-step nilpotent,*
- *the scalar curvature $S = 0$,*
- *the Bach tensor $B = 0$,*
- *the second conformal to Einstein obstruction from Equation* (3.2.3) *is identically 0, and*
- *the first conformal to Einstein obstruction from Equation* (3.2.2) *reduces to*
$$\Upsilon^a C_{abcd} = \Upsilon^a \Big(- R_{cdab} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) \Big) = 0$$
for some $\Upsilon \in \mathfrak{F}(G)$ such that $\Upsilon^a = \nabla^a \Upsilon$.

Proof. Recall that on a metric Lie algebra, from Lemma 3.7 we have
$$-\operatorname{Ric}^2_{ab} = -R^j_a R_{bj} = \frac{\epsilon_j}{16} K([X_a, E_j], [X_b, E_j]).$$

Now, Corollary A.27 tells us that whenever $\mathfrak{g}$ is solvable, $K_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = \{0\}$. We can conclude that any solvable Lie algebra has 2-step nilpotent Ricci tensor. Lemma 3.10 concludes that the scalar curvature is 0 and the final results follow from Proposition 3.11. $\square$

3.4 Simple groups with bi-invariant metrics

This section considers whether the bi-invariant metrics on a simple Lie group, as described in Theorem 2.8, are conformally Einstein. From Lemma 3.2 we know the Killing form on a semisimple Lie group is always an Einstein bi-invariant metric. However, on simple Lie groups with Lie algebra of complex type, the metric can be a linear combination of the Killing form and another metric $K_{\mathbb{I}}$. We show on these metric Lie algebras that when the metric is a multiple of $K_{\mathbb{I}}$ only, it is Bach-flat but not conformally Einstein, and when the metric is not a multiple of the Killing form or $K_{\mathbb{I}}$ is a linear combination of the Killing form and $K_{\mathbb{I}}$. In this case, it is not Bach-flat and hence not conformally Einstein.

Theorem 3.13. *Let G be a Lie group with simple Lie algebra $\mathfrak{g}$. When $\mathfrak{g}$, is of real type, then the only bi-invariant metric it can be equipped with is a non-zero multiple of the Killing form. This is an Einstein metric by Lemma 3.2.*

When $\mathfrak{g}$, is of complex type, G has a 2-parameter family of bi-invariant metrics signature (n, n). This family is spanned by $K_{\mathbb{I}}$ and $K_{\mathbb{R}} = 2K_{\mathfrak{g}}$, where $K_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$ and $K_{\mathbb{I}}$ and $K_{\mathbb{R}}$ are defined in Theorem 2.8. Let such a metric be $g = aK_{\mathbb{R}} + bK_{\mathbb{I}}$. Then we have the Ricci tensor, scalar curvature and Bach tensor of g are
$$\operatorname{Ric} = -\frac{1}{2}K_{\mathbb{R}}, \quad S = \frac{-an}{a^2+b^2}, \text{ and } B = \frac{2abn(n-1)}{2(a^2+b^2)^2(2n-1)(2n-2)^2}(bK_{\mathbb{R}} - aK_{\mathbb{I}}).$$

Three distinct cases can then occur:

- *We have $b = 0$ and the metric is a non-zero multiple of $K_{\mathbb{R}} = 2K_{\mathfrak{g}}$ only, and is hence an Einstein metric by Lemma 3.2.*

- *We have $a = 0$ and the metric is a non-zero multiple of $K_{\mathbb{I}}$ only, has vanishing scalar curvature, is Bach-flat but is not conformally Einstein.*

- *Both a and b are non-zero, and the metric is not Bach flat and hence not conformally Einstein.*

From Lemma 1.19 we know that all Einstein metrics are Bach-flat, hence this theorem gives an example of two Bach-flat metrics whose non-trivial linear combinations are not Bach-flat.

Proof. Theorem 2.8 shows that if $\mathfrak{g}$ is of real type, the only metric is a multiple of the Killing form, and if $\mathfrak{g}$ is of complex type and the metric has signature (n, n) and is a non-trivial linear combination of $K_{\mathbb{I}}$ and $K_{\mathbb{R}} = 2K_{\mathfrak{g}}$. When $\mathfrak{g}$ is of complex type, we will show that the metric $K_{\mathbb{I}}$ is scalar-flat, Bach-flat, but the metric is not conformally Einstein by showing there is no $\Upsilon \in \mathfrak{F}(G)$ such that $\Upsilon^a C_{abcd} = 0$. We will show that when the metric is a linear combination of $K_{\mathbb{I}}$ and $K_{\mathbb{R}}$ such that it is not a multiple of $K_{\mathbb{I}}$ or $K_{\mathbb{R}}$, then the metric is not Bach-flat, and is hence not conformally Einstein.

Assume now that $\mathfrak{g}$ is of complex type. Hence, it must be even dimensional. Assume it has dimension $2n$ and is the realification of $\mathfrak{h}$, a complex Lie algebra of dimension n. If $K_{\mathfrak{h}}$ is the Killing form of $\mathfrak{h}$, $K_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$, then using the basis as in Theorem 2.8, we have

$$K_{\mathfrak{g}} = 2\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}} \end{pmatrix} = 2K_{\mathbb{R}} \qquad \text{and} \qquad K_{\mathbb{I}} = \begin{pmatrix} 0 & K_{\mathfrak{h}} \\ K_{\mathfrak{h}} & 0 \end{pmatrix}.$$

We let $g = aK_{\mathbb{R}} + bK_{\mathbb{I}}$. The Ricci tensor is $\mathrm{Ric} = -\frac{1}{4}K_{\mathfrak{g}} = -\frac{1}{2}K_{\mathbb{R}}$. Denote the matrix of coeffecients of Ric as [Ric] and similarly for g. We note that

$$[g]^{-1} = \frac{-1}{a^2+b^2}\begin{pmatrix} -aK_{\mathfrak{h}}^{-1} & -bK_{\mathfrak{h}}^{-1} \\ -bK_{\mathfrak{h}}^{-1} & aK_{\mathfrak{h}}^{-1} \end{pmatrix}.$$

The scalar curvature is then

$$S = \mathrm{tr}([\mathrm{Ric}][g]^{-1}) = -\frac{1}{2}\frac{-1}{a^2+b^2}\,\mathrm{tr}\left(\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}} \end{pmatrix}\begin{pmatrix} -aK_{\mathfrak{h}}^{-1} & -bK_{\mathfrak{h}}^{-1} \\ -bK_{\mathfrak{h}}^{-1} & aK_{\mathfrak{h}}^{-1} \end{pmatrix}\right) = \frac{-an}{a^2+b^2}.$$

Note this means the scalar curvature is 0 when the metric is a multiple of $K_{\mathbb{I}}$ only. Consider that $R^a_b R_{da} = \big([\mathrm{Ric}][g]^{-1}[\mathrm{Ric}]\big)_{bd}$ where

$$\begin{aligned}
[\mathrm{Ric}][g]^{-1}[\mathrm{Ric}] =& \frac{-1}{a^2+b^2}\frac{1}{4}\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}} \end{pmatrix}\begin{pmatrix} -aK_{\mathfrak{h}}^{-1} & -bK_{\mathfrak{h}}^{-1} \\ -bK_{\mathfrak{h}}^{-1} & aK_{\mathfrak{h}}^{-1} \end{pmatrix}\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}} \end{pmatrix} \\
=& \frac{1}{4(a^2+b^2)}(aK_{\mathbb{R}} - bK_{\mathbb{I}}).
\end{aligned}$$

We also have $R_{ac}R^{ac} = \big([\mathrm{Ric}][g]^{-1}[\mathrm{Ric}][g]^{-1}\big)^a_{\ a}$ and hence

$$\begin{aligned}\big([\mathrm{Ric}][g^{-1}][\mathrm{Ric}][g]^{-1}\big)^a_{\ a} =& \frac{-1}{(a^2+b^2)^2}\frac{1}{4}\operatorname{tr}\left\{\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}}\end{pmatrix}\begin{pmatrix} -aK_{\mathfrak{h}}^{-1} & -bK_{\mathfrak{h}}^{-1} \\ -bK_{\mathfrak{h}}^{-1} & aK_{\mathfrak{h}}^{-1}\end{pmatrix}\right. \\ & \left.\begin{pmatrix} K_{\mathfrak{h}} & 0 \\ 0 & -K_{\mathfrak{h}}\end{pmatrix}\begin{pmatrix} -aK_{\mathfrak{h}}^{-1} & -bK_{\mathfrak{h}}^{-1} \\ -bK_{\mathfrak{h}}^{-1} & aK_{\mathfrak{h}}^{-1}\end{pmatrix}\right\} \\ =& \frac{n(a^2-b^2)}{2(a^2+b^2)^2}.\end{aligned}$$

Hence we have the Bach tensor

$$B = \frac{2abn(n-1)}{2(a^2+b^2)^2(2n-1)(2n-2)^2}(bK_{\mathbb{R}} - aK_{\mathbb{I}}).$$

As $K_{\mathbb{R}}$ and $K_{\mathbb{I}}$ are linearly independent, then provided both a and b are non-zero, the Bach tensor is not flat and hence the metric is not conformally Einstein. In the case $b = 0$, the metric is a multiple of the Killing form, which is Einstein. When $a = 0$, the metric is a multiple of $K_{\mathbb{I}}$. We show this is not conformally Einstein by considering the first obstruction: $\Upsilon^a C_{abcd} = 0$.

As the Lie algebra is the set of all left-invariant vector fields, we can take a basis for the Lie algebra, $X_1, \ldots, X_{2n}$. Then any vector field, $V \in \mathfrak{X}(G)$ can be written as $V = f^a X_a$ for some functions $f^a \in \mathfrak{F}(G)$. Hence we can write $\operatorname{grad}\Upsilon = \Upsilon^a X_a$. If we consider evaluating $\operatorname{grad}\Upsilon = \Upsilon^a X_a$ at a particular point of $p \in G$, for instance the identity element, then $\Upsilon^a|_p X_a|_p$ is actually an $\mathbb{R}$-linear combination of $X_a|_p$. However due to left invariance, $X_a|_p$ is canonically isomorphic to X_a, hence $\operatorname{grad}\Upsilon$ is an element of the Lie algebra, $\mathfrak{g}$. Hence, if $\Upsilon^a C_{abcd} = 0$, this must be true at every point $p \in G$ on the Lie group, so this can only occur if there is at least one $W \in \mathfrak{g}$ such that $C(W, \cdot, \cdot, \cdot) = 0$ on the Lie algebra. Note, due to the symmetries of the Weyl tensor, the W can be inserted into any of the coordinates.

Finally, note that $\Upsilon = 0$ satisfies $\Upsilon^a C_{abcd} = 0$. In fact this is true for constant Υ too. However, if we take $\Upsilon = c \in \mathbb{R}$, this implies that $\widehat{g} = e^{2c}K_{\mathbb{I}}$ is an Einstein metric. However, we know $K_{\mathbb{I}}$ is not an Einstein metric, so no multiples of it are Einstein either. Hence we need to consider finding non-constant Υ that satisfy this. This means there must be at least one point in G such that $\operatorname{grad}\Upsilon \neq 0$. Hence we will consider finding non-zero $W \in \mathfrak{g}$ that satisfy $C(W, X, Y, Z) = 0$ for all Lie algebra elements $X, Y, Z \in \mathfrak{g}$.

For $K_{\mathbb{I}}$, the scalar curvature is 0. Hence the Weyl tensor on the Lie algebra reduces to

$$C_{abcd} = -R_{abcd} + \frac{1}{2n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}).$$

For $X, Y, Z, W \in \mathfrak{g}$ we see that

$$\begin{aligned} C(X,Y,Z,W) =& \\ & - K_{\mathbb{I}}([Z,W],[X,Y]) - \frac{1}{4n-4}\Big(K_{\mathbb{I}}(Z,Y)K_{\mathbb{R}}(X,W) - K_{\mathbb{I}}(Z,X)K_{\mathbb{R}}(Y,W) \\ & + K_{\mathbb{I}}(W,X)K_{\mathbb{R}}(Y,Z) - K_{\mathbb{I}}(W,Y)K_{\mathbb{R}}(X,Z)\Big) \\ =& -\mathrm{Im}\big(K_{\mathfrak{h}}([Z,W],[X,Y])\big) - \frac{1}{4n-4}\Big(\mathrm{Im}\big(K_{\mathfrak{h}}(Z,Y)K_{\mathfrak{h}}(X,W)\big) - \mathrm{Im}\big(K_{\mathfrak{h}}(Z,X)K_{\mathfrak{h}}(Y,W)\big)\Big) \\ =& \,\mathrm{Im}\bigg(- K_{\mathfrak{h}}([Z,W],[X,Y]) - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(Z,Y)K_{\mathfrak{h}}(X,W) - K_{\mathfrak{h}}(Z,X)K_{\mathfrak{h}}(Y,W)\Big)\bigg)\end{aligned}$$

where the second and third line considers X, Y, Z, W as elements of $\mathfrak{h}$.

From Lemma A.39 we see that there is a $W \in \mathfrak{h}$ such that $C(X, Y, Z, W) = 0$ for all $X, Y, Z \in \mathfrak{h}$ if and only if for all $X, Y, Z \in \mathfrak{h}$. This W satisfies

$$-K_{\mathfrak{h}}([Z, W], [X, Y]) - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(Z, Y)K_{\mathfrak{h}}(X, W) - K_{\mathfrak{h}}(Z, X)K_{\mathfrak{h}}(Y, W)\Big) = 0.$$

Using the ad-invariance of $K_{\mathfrak{h}}$ we can rewrite the first term so we have

$$K_{\mathfrak{h}}(Z, [[X, Y], W]) - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(Z, Y)K_{\mathfrak{h}}(X, W) - K_{\mathfrak{h}}(Z, X)K_{\mathfrak{h}}(Y, W)\Big) = 0.$$

Dualising using the metric $K_{\mathfrak{h}}$ we are now looking for an element $W \in \mathfrak{h} \setminus \{0\}$ such that for all $X, Y \in \mathfrak{h}$ and for all $Z \in \mathfrak{h}^*$

$$Z^*\left([[X, Y], W] - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(X, W)Y - K_{\mathfrak{h}}(Y, W)X\Big)\right) = 0.$$

This simplifies to finding $W \in \mathfrak{h} \setminus \{0\}$ such that

$$[[X, Y], W] - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(X, W)Y - K_{\mathfrak{h}}(Y, W)X\Big) = 0 \tag{3.4.1}$$

for all $X, Y \in \mathfrak{h}$.

Recall the Lie algebra, $\mathfrak{sl}_2(\mathbb{C})$ from Example 2.2

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C}\right\}.$$

Lemma 3.14. *When $\mathfrak{h} = \mathfrak{sl}_2\mathbb{C}$ there is no such $W \in \mathfrak{h} \setminus \{0\}$ that satisfies Equation* (3.4.1).

Proof. Recall the basis of $\mathfrak{sl}_2\mathbb{C}$, e_1, e_2, e_3 such that

$$[e_1, e_2] = -2e_1,\ [e_2, e_3] = -2e_3,\ [e_1, e_3] = e_2.$$

In this basis, the Killing form is

$$K_{\mathfrak{h}} = 4\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $W = ae_1 + be_2 + ce_3$ and $X = e_1, Y = e_2$. Then Equation (3.4.1) becomes

$$\begin{aligned} 0 =& [[e_1, e_2], W] - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(e_1, W)e_2 - K_{\mathfrak{h}}(e_2, W)e_1\Big) \\ =& -2[e_1, ae_1 + be_2 + ce_3] - \frac{1}{8}\Big(K_{\mathfrak{h}}(e_1, ae_1 + be_2 + ce_3)e_2 - K_{\mathfrak{h}}(e_2, ae_1 + be_2 + ce_3)e_1\Big) \\ =& 4be_1 - 2ce_2 - \frac{1}{8}\Big((0a + 0b + 4c)e_2 - (0a + 8b + 0c)e_1\Big) \\ =& 5be_1 - \frac{5}{2}ce_2 \end{aligned}$$

which implies $b = 0, c = 0$, and $W = ae_1$. Now let $W = ae_1$ and $X = e_3, Y = e_2$, then Equation (3.4.1) becomes

$$\begin{aligned} 0 =& [[e_3, e_2], ae_1] - \frac{1}{8}\Big(K_{\mathfrak{h}}(e_3, ae_1)e_2 - K_{\mathfrak{h}}(e_2, ae_1)e_3\Big) \\ =& 2a[e_3, e_1] - \frac{1}{8}\Big(4ae_2\Big) \\ =& -2ae_2 - \frac{1}{2}ae_2 = -\frac{5}{2}ae_2. \end{aligned}$$

This implies $a = 0$ and $W = 0$. Hence there is no non-zero $W \in \mathfrak{h}$ which satisifies Equation (3.4.1). $\square$

Lemma 3.15. *If $\mathfrak{h}$ is a complex simple Lie algebra then there is no such $W \in \mathfrak{h} \setminus \{0\}$ that satisfies Equation* (3.4.1).

Proof. Assuming that $\mathfrak{h} \neq \mathfrak{sl}_2\mathbb{C}$ then $\mathfrak{h}$ has a Cartan subalgebra, $\mathfrak{h}_0$, of dimension $\dim(\mathfrak{h}_0) \geq 2$. See Remark A.10 for more details. Using the Jordan decomposition, Theorem A.42, write $W = W_N + W_S$ where ad_{W_N} is nilpotent and ad_{W_S} is diagonalisable. Pick a Cartan subalgebra $\mathfrak{h}_0$ where $W_S \in \mathfrak{h}_0$, which can be done by Lemma A.45. Note that $W_N \notin \mathfrak{h}_0$ as ad_{W_N} is not diagonalisable. Then, as in Theorem A.46, write $\mathfrak{h}$ in its root space decomposition

$$\mathfrak{h} = \mathfrak{h}_0 \oplus^{\perp} \bigoplus_{\alpha \in \Delta}^{\perp} \{\mathfrak{h}_\alpha \oplus \mathfrak{h}_{-\alpha}\}$$

where the subalgebras are orthogonal with respect to the metric $K_{\mathfrak{h}}$. Note that $K_{\mathfrak{h}}$ is also non-degenerate on $\mathfrak{h}_0$.

Let X, Y be any two elements of $\mathfrak{h}_0$, then we have $[X, Y] = 0$ and hence we are searching for a W such that

$$K_{\mathfrak{h}}(X, W)Y - K_{\mathfrak{h}}(Y, W)X = 0.$$

As $\dim(\mathfrak{h}_0) \geq 2$, then it follows that this only holds if $K_{\mathfrak{h}}(Y, W) = 0$ for all $Y \in \mathfrak{h}_0$. However, as $\mathfrak{h}_0$ is orthogonal to each $\mathfrak{h}_\alpha$ then $K(Y, W_N) = 0$ and hence $0 = K_{\mathfrak{h}}(Y, W) = K_{\mathfrak{h}}(Y, W_S + W_N) = K_{\mathfrak{h}}(Y, W_S)$ for all $Y \in \mathfrak{h}_0$. As $K_{\mathfrak{h}}$ is non-degenerate on $\mathfrak{h}_0$ then this implies that $W_S = 0$ and hence $W = W_N \in \mathfrak{h}_\beta$ for some $\beta \in \Delta$ (or perhaps $-\beta \in \Delta$).

Finally, take $X \in \mathfrak{h}_0$ and $Y \in \mathfrak{h}_{\pm\beta}$ such that $K_{\mathfrak{h}}(Y, W) \neq 0$ which is possible by the non-degeneracy of $K_{\mathfrak{h}}$. Then W must satisfy

$$\begin{aligned} 0 =& [[X, Y], W] - \frac{1}{4n-4}\Big(K_{\mathfrak{h}}(X, W)Y - K_{\mathfrak{h}}(Y, W)X\Big) \\ =& \beta(X)[Y, W] + \frac{1}{4n-4}K_{\mathfrak{h}}(Y, W)X. \end{aligned}$$

If $W \neq 0$ then we require $\beta(X)[Y, W] = -\frac{1}{4n-4}K_{\mathfrak{h}}(Y, W)X \neq 0$ for all $X \in \mathfrak{h}_0$. Then $\beta(X) \neq 0$ for all $X \in \mathfrak{h}_0$. However, as $\beta : \mathfrak{h}_0^* \to \mathbb{C}$ and $\mathfrak{h}_0$ has dimension ≥ 2, then $\dim\ker(\beta) = \dim(\mathfrak{h}_0) - 1 \geq 1$, which means the kernel is non-trivial. This is a contradiction, hence W must be 0. $\square$

As we have shown the only way the first obstruction can be satisfied is if the gradient is 0, however we have shown that this means that the metric is not conformally Einstein. Hence the metric $K_{\mathbb{I}}$ on $\mathfrak{g}$ is Bach-flat, but not Einstein nor conformal to Einstein. $\square$

3.5 Conclusion

In this chapter we have combined the first two chapters and shown that there is always an Einstein metric on a simple Lie algebra, which is hence Bach-flat. We show that when the Lie algebra is of complex type, the metric may induced from linear combinations of the imaginary and real part of the Killing form from a complex Lie algebra. When it is purely a multiple of the real part, it is Einstein. When it is purely a mulitple of the complex part, it is Bach-flat but not conformally Einstein. When it is a linear combination of both the real and complex part, it is not Bach flat and hence not conformally Einstein. As Lie algebras of complex type only have metrics of signature (n, n), we can also conclude that all Riemannian bi-invariant metrics occur with Lie algebras of real type and hence are Einstein metrics.

This chapter has also considered the conformal to Einstein obstructions when the metric has 2-step nilpotent Ricci tensor. This will be of use in the following chapter when solvable metric Lie algebras are considered.

Chapter 4

Metric Lie algebras and the double extension procedure

This next section begs the question, can one classify metrics on non-reductive Lie algebras? The Oscillator algebra, a solvable Lie algebra, is one such example explored here. However a complete list of non-reductive metric Lie algebras currently does not exist. Instead, Medina [25] first developed the concept of constructing non-reductive metric Lie algebras using a process called the double extension procedure. Medina proved the following theorem with Revoy:

Theorem 4.1 (Medina and Revoy [26]). *Every indecomposable metric Lie algebra is simple, 1-dimensional or a double extension by a simple or 1-dimensional Lie algebra.*

Here, indecomposable means that the Lie algebra cannot be written as the orthogonal sum of two non-trivial metric Lie algebras. This is the analogous to the way simple and abelian ideals break down reductive metric Lie algebras.

This theorem is widely known in the literature of metric Lie algebras, including in the following papers: Figueroa-O'Farrill and Stanciu [11], Kath and Olbrich [20] and Baum and Kath [3]. Kac [19, pg. 28] has a similar result for solvable metric Lie algebras in his Exercises 2.10 and 2.11, of which a complete proof can be found in Favre and Santharoubane [10, pg. 456]. Medina [25] gave the first classification of Lorentzian metric Lie algebras, which is also given in Baum and Kath [3], and explored here in Theorem 4.14.

Figueroa-O'Farrill and Stanciu [11] presented some further preliminary results on double extensions and gave several examples of metric Lie algebras constructed in this way, with the aim of applying it to conformal field theory.

Kath and Olbrich [20] extended these preliminary results into a full classification scheme, where they have used this as a stepping stone for the classification of symmetric spaces. Baum and Kath [3] use this scheme to present explicit classification results for several cases, including the Lorenzian metric Lie algebras, the signature $(2, n-2)$ metric Lie algebras, and low dimensional signature $(\frac{n}{2}, \frac{n}{2})$ metric Lie algebras, as well focusing on the holonomy groups of the resulting metrics.

As we are interested in the curvature properties of these metrics, we note that Lemma 3.3 states that any non-semi-simple Lie algebra can only be Einstein if it is Ricci-flat. Hence computing the Ricci curvature immediately shows whether the metric is Einstein or not.

It then becomes important to consider whether the metric is conformal to Einstein. Baum and Kath [3] give related curvature results, which are recreated here in a different format and used to consider whether the metrics are conformally Einstein.

4.1 Double extension of metric Lie algebras

In the following chapter, we will use the word *isometry* to mean a metric-preserving diffeomorphism in the case of a Lie group, and a metric-preserving Lie algebra isomorphism in the case of a Lie algebra. Here, a *metric* on a Lie algebra is the bilinear form on the Lie algebra which is the restriction of the metric on the Lie group to the Lie algebra.

When a Lie algebra, $\mathfrak{g}$, is isomorphic to the sum of two ideals, $\mathfrak{i}$ and $\mathfrak{j}$, we write $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{j}$. When it only contains one ideal, say $\mathfrak{i} \subset \mathfrak{g}$ but has vector space complement $\mathfrak{j}$, then we write $\mathfrak{g} = \mathfrak{i} \rtimes \mathfrak{j}$ or $\mathfrak{g} = \mathfrak{j} \ltimes \mathfrak{i}$. If $\mathfrak{g}$ is isomorphic as a vector space to the direct sum of two subspaces, $\mathfrak{i}, \mathfrak{j}$, we write $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{j}$ say that this is a vector space decomposition. If a metric Lie algebra is isometric to the orthogonal sum of two ideals, $\mathfrak{i}$ and $\mathfrak{j}$, we write $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{j}$ and state that this is an orthogonal isometric decomposition. If a double extension of a metric Lie algebra is orthogonally isometric to the sum of two or more non-trivial ideals, we say the double extension is *decomposable* or *decomposes*.

Definition 4.1. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be a metric Lie algebra and $(\mathfrak{h}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ a Lie algebra with ad-invariant, symmetric (possibly degenerate) bilinear form. Let $\pi : \mathfrak{h} \to \mathrm{Der}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be a Lie algebra homomorphism from $\mathfrak{h}$ to the Lie algebra of anti-symmetric derivations on $\mathfrak{g}$. That is, $\pi([H_1, H_2]) = [\pi(H_1), \pi(H_2)]$ for all $H_1, H_2 \in \mathfrak{h}$, $\langle \pi(H)x, y \rangle_{\mathfrak{g}} + \langle x, \pi(H)y \rangle_{\mathfrak{g}} = 0$ for all $H \in \mathfrak{h}, x, y \in \mathfrak{g}$, and $\pi(H)[x, y] = [\pi(H)x, y] + [x, \pi(H)y]$ for all $H \in \mathfrak{h}$ and $x, y \in \mathfrak{g}$.

Let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$ and let $\mathrm{ad}^*_{\mathfrak{h}}(H) : \mathfrak{h}^* \to \mathfrak{h}^*$ be the coadjoint representation of $\mathfrak{h}$, that is $\mathrm{ad}^*_{\mathfrak{h}}(H)\alpha = \alpha \circ \mathrm{ad}_{\mathfrak{h}}(H)$. Define $\beta : \mathfrak{g}^2 \to \mathfrak{h}^*$ by

$$\beta(X, Y)(H) = \langle \pi(H)X, Y \rangle_{\mathfrak{g}}.$$

As a vector space, define $\mathfrak{d} := \mathfrak{h}^* \oplus \mathfrak{g} \oplus \mathfrak{h}$. Using this notation, we define the *double extension of* $\mathfrak{g}$ *by* $\mathfrak{h}$ as the metric Lie algebra $\mathfrak{d}_\pi$, which is $\mathfrak{d}$ equipped the following bracket

$$\left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix} \right]_{\mathfrak{d}_\pi} = \begin{pmatrix} \beta(X, \widehat{X}) + \mathrm{ad}^*_{\mathfrak{h}}(H)\widehat{\alpha} - \mathrm{ad}^*_{\mathfrak{h}}(\widehat{H})\alpha \\ [X, \widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X} - \pi(\widehat{H})X \\ [H, \widehat{H}]_{\mathfrak{h}} \end{pmatrix}$$

and with bilinear form

$$\left\langle \begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix} \right\rangle_{\mathfrak{d}_\pi} = \langle H, \widehat{H} \rangle_{\mathfrak{h}} + \langle X, \widehat{X} \rangle_{\mathfrak{g}} + \alpha(\widehat{H}) + \widehat{\alpha}(H). \tag{4.1.1}$$

Note that this bilinear form is ad-invariant, symmetric, non-degenerate and with

$$\mathrm{signature}(\langle \cdot, \cdot \rangle_{\mathfrak{d}}) = \mathrm{signature}(\langle \cdot, \cdot \rangle_{\mathfrak{g}}) + (\dim(\mathfrak{h}), \dim(\mathfrak{h})). \tag{4.1.2}$$

Also that

$$\mathrm{ad}_{\mathfrak{d}_\pi} \begin{pmatrix} \alpha \\ X \\ H \end{pmatrix} = \begin{pmatrix} \mathrm{ad}^*_{\mathfrak{h}}(H) & \beta(X, \cdot) & -\mathrm{ad}^*_{\mathfrak{h}}(\cdot)\alpha \\ 0 & \mathrm{ad}_{\mathfrak{g}}(X) + \pi(H) & -\pi(\cdot)X \\ 0 & 0 & \mathrm{ad}_{\mathfrak{h}}(H) \end{pmatrix}.$$

Remark 4.1. From this point, we will often write $\mathfrak{d}$ for the double extension $\mathfrak{d}_\pi$ when the specific derivation is not referred to. In the case where $\mathfrak{h}$ is one dimensional, then there is an $A \in \mathfrak{so}(\mathfrak{g})$ such that $\pi(x) = xA$ for all $x \in \mathfrak{h}$, and we will refer to this double extension as $\mathfrak{d}_A$, again dropping the subscript when it is not explicitly necessary to refer to it. See Definition A.19 for the definition of $\mathfrak{so}(\mathfrak{g})$.

Proposition 4.2. *The Ricci curvature of a double extension is*

$$\mathrm{Ric}_{\mathfrak{d}_\pi}(\cdot,\cdot) = -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} \\ 0 & \mathrm{tr}\{\pi(\cdot)\,\mathrm{ad}_{\mathfrak{g}}(\cdot)\} & 2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)\pi(\cdot)\} \end{pmatrix}$$

and the scalar curvature is $S = S_{\mathfrak{g}}$ where $S_{\mathfrak{g}}$ is the scalar curvature of $\mathfrak{g}$.

For the proof see Appendix A.12.

Proposition 4.3. *When a Lie algebra $\mathfrak{g}$ is double extended by a simple Lie algebra, $\mathfrak{h}$, the double extension is not Einstein.*

Proof: Sketch. By Lemma 3.3, we know that a degenerate Ricci curvature must be 0 for the metric to be Einstein. As the Ricci curvature is always degenerate in the $\mathfrak{h}^*$ direction, then we need $K_{\mathfrak{g}} = 0$, $2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)\pi(\cdot)\} = 0$ and $\mathrm{tr}\{\pi(\cdot)\,\mathrm{ad}_{\mathfrak{g}}(\cdot)\} = 0$ for the metric to be Einstein.

In the case where $\mathfrak{h}$ is simple, then the Killing form is a positive multiple of the trace form $\mathrm{tr}\{XY\}$ for $X, Y \in \mathfrak{h}$, where we view $\mathfrak{h}$ as a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. As π is a homomorphism, then $\pi(\mathfrak{h})$ is isomorphic to an ideal of $\mathfrak{h}$. However, as $\mathfrak{h}$ is simple, this is either $\mathfrak{h}$ or 0.

If $\pi(\mathfrak{h}) = 0$, then we conclude that $K_{\mathfrak{h}} = 0$ for the double extension to be Ricci-flat. However, as $\mathfrak{h}$ is simple, we know this is non-zero, so this is impossible.

In the case where $\pi(\mathfrak{h})$ is non-zero, then π is an isomorphism onto its image. One can show that $\mathrm{tr}\{\pi(H_1)\pi(H_2)\} = cK_{\mathfrak{h}}(H_1, H_2)$ where c is positive. This can be shown using properties of Cartan decompositions of $\pi(\mathfrak{h})$ and $\mathfrak{so}(\mathfrak{g})$, and by applying Theorem 3.6 of Onishchik and Vinbergs [30, pg. 207], which states if a simple Lie algebra is subalgebra of another Lie algebra, then this implies the embedding is canonical with respect to the Cartan decomposition. Further details on Cartan decompositions can be found in Onishchik and Vinbergs [30].

Hence $2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)\pi(\cdot)\} = (2 + c)K_{\mathfrak{h}}(\cdot,\cdot)$. This non-zero as $\mathfrak{h}$ is simple. Hence the double extension cannot be Ricci-flat and is not Einstein. □

This result is also mentioned in Baum and Kath [3, pg. 261] in Theorem 4.1.

The following propositions show that if we consider only decomposable double extensions, then is it necessary to use outer derivations and to have no simple ideals in the metric Lie algebra we are extending.

Proposition 4.4. *If $\pi(x)$ is an inner derivation for all $x \in \mathfrak{h}$, then $\mathfrak{d}_\pi$ decomposes isometricly into $\mathfrak{g} \oplus (\mathfrak{h} \ltimes \mathfrak{h}^*)$.*

Proof (Sketch). Define $\phi : \mathfrak{h} \to \mathfrak{g}$ such that $\pi(h) = [\phi(h), \cdot]$. As π is a homomorphism, we can show ϕ is a homomorphism using the Jacobi identity as follows:

$$\begin{aligned}[\phi([h_1, h_2]), \cdot] &= \pi([h_1, h_2]) = [\pi(h_1), \pi(h_2)] = \pi(h_1)\pi(h_2) - \pi(h_2)\pi(h_1) \\ &= [\phi(h_1), [\phi(h_2), \cdot]] - [\phi(h_2), [\phi(h_1), \cdot]] \\ &= [[\phi(h_1), \phi(h_2)], \cdot].\end{aligned}$$

Define $\phi^\flat : \mathfrak{h} \to \mathfrak{g}^*$ where $\phi^\flat(h) = \langle \phi(h), \cdot \rangle_{\mathfrak{g}}$ for $h \in \mathfrak{h}$, and $\phi^\sharp : \mathfrak{g} \to \mathfrak{h}^*$ as its transpose, that is $\phi^\sharp(g) = \langle \phi(\cdot), g \rangle$ for $g \in \mathfrak{g}$. One can show these are intertwiners (see Definition A.26) of the action of $\mathfrak{h}$. That is, for any $h_1, h_2 \in \mathfrak{h}$

$$\phi^\flat([h_1, h_2])(\cdot) = \langle \phi(h_1 h_2), \cdot \rangle = \langle \phi(h_1)\phi(h_2), \cdot \rangle = \langle \phi(h_1), [\phi(h_2), \cdot] \rangle = \phi^\flat(h_1)([\phi(h_2), \cdot])$$

and for any $h \in \mathfrak{h}, g \in \mathfrak{g}$

$$\phi^\sharp([\phi(h), g])(\cdot) = \langle \phi(\cdot), [\phi(h), g] \rangle = \langle [\phi(\cdot), \phi(h)], g \rangle = \phi^\sharp(g)([h, \cdot]) = \mathrm{ad}^*_{\mathfrak{h}}(h)\phi^\sharp(g).$$

Finally, by direct computation, we can show that

$$\Psi(x, h, \alpha) = (x + \phi(h), h, \alpha - \phi^\sharp(x)) \qquad x \in \mathfrak{g}, h \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$$

is an isometry from $\mathfrak{d}_\pi$ to $\mathfrak{g} \oplus (\mathfrak{h} \ltimes \mathfrak{h}^*)$. Here, the bracket on $\mathfrak{h} \ltimes \mathfrak{h}^*$ is

$$\left[\begin{pmatrix} h \\ \alpha \end{pmatrix}, \begin{pmatrix} \widehat{h} \\ \widehat{\alpha} \end{pmatrix} \right] = \begin{pmatrix} [h, \widehat{h}] \\ \mathrm{ad}^*_{\mathfrak{h}}(h)(\widehat{\alpha}) - \mathrm{ad}^*_{\mathfrak{h}}(\widehat{h})(\alpha) \end{pmatrix}$$

and the metric is

$$\left\langle \begin{pmatrix} h \\ \alpha \end{pmatrix}, \begin{pmatrix} \widehat{h} \\ \widehat{\alpha} \end{pmatrix} \right\rangle = \langle h, \widehat{h} \rangle_{\mathfrak{h}} + \langle \phi(h), \phi(\widehat{h}) \rangle_{\mathfrak{g}} + \alpha(\widehat{h}) + \widehat{\alpha}(h).$$

See Figueroa-O'Farrill and Stanciu [11, pg. 4128] Proposition 6.1 for further details. Note the corrected form of their isomorphism that appears here. $\square$

Theorem 4.5. *If $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra such that it is a direct sum of ideals, $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{s}$ with $\mathfrak{g}'$ semisimple, then any double extension is decomposable.*

Proof (Sketch). As $\mathfrak{g}'$ is semisimple, then we have the isomorphism $\mathrm{der}(\mathfrak{g}) \cong \mathrm{der}(\mathfrak{g}') \oplus \mathrm{der}(\mathfrak{s})$ by Proposition A.13. Here, $\mathrm{der}(\mathfrak{g}')$ are all inner derivations by Proposition A.12. Then we use an appropriate form of the isomorphism from Proposition 4.4

$$\Psi(x, a, h, \alpha) = (x + \phi(h), a, h, \alpha - \phi^\sharp(x)) \qquad x \in \mathfrak{g}', a \in \mathfrak{s}, h \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$$

to show the isometry $\mathfrak{d}_\pi(\mathfrak{g}' \oplus \mathfrak{s}, \mathfrak{h}) \cong \mathfrak{g}' \oplus \mathfrak{d}_\pi(\mathfrak{s}, \mathfrak{h})$.

Finally, note that $\mathfrak{g}'$ and $\mathfrak{s}$ are orthogonal as $\langle \mathfrak{g}', \mathfrak{s} \rangle = \langle [\mathfrak{g}', \mathfrak{g}'], \mathfrak{s} \rangle = \langle \mathfrak{g}', [\mathfrak{g}', \mathfrak{s}] \rangle = 0$. However, the metric on $\mathfrak{d}_\pi(\mathfrak{s}, \mathfrak{h})$ is not the standard metric induced by the double extension (as in Equation (4.1.1)) but is in fact

$$\left\langle \begin{pmatrix} a \\ h \\ \alpha \end{pmatrix}, \begin{pmatrix} \widehat{a} \\ \widehat{h} \\ \widehat{\alpha} \end{pmatrix} \right\rangle_{\mathfrak{d}_\pi} = \langle h, \widehat{h} \rangle_{\mathfrak{h}} + \langle a, \widehat{a} \rangle_{\mathfrak{s}} + \langle \phi(h), \phi(\widehat{h}) \rangle_{\mathfrak{g}} + \alpha(\widehat{h}) + \widehat{\alpha}(h).$$

See the full proof in Figueroa-O'Farrill and Stanciu [11, pg. 4129] Theorem 6.5. $\square$

4.2 Solvable metric Lie algebras

Consider any indecomposable solvable metric Lie algebra, $\mathfrak{d}$, with dimension $n \geq 2$. As the Lie algebra is non-simple and not 1-dimensional, then by Theorem 4.1, it must be a double extension of a metric Lie algebra $\mathfrak{g}$ by a 1-dimensional or simple Lie algebra, $\mathfrak{h}$. Consider if $\mathfrak{h}$ is simple, then the bracket is

$$\left[\begin{pmatrix}\alpha\\ X\\ H\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}\\ \widehat{X}\\ \widehat{H}\end{pmatrix}\right]_{\mathfrak{d}_\pi} = \begin{pmatrix}\beta(X,\widehat{X}) + \mathrm{ad}^*_{\mathfrak{h}}(H)\widehat{\alpha} - \mathrm{ad}^*_{\mathfrak{h}}(\widehat{H})\alpha\\ [X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X} - \pi(\widehat{H})X\\ [H,\widehat{H}]_{\mathfrak{h}}\end{pmatrix}$$

and subsequently that $[\mathfrak{h},\mathfrak{h}] \subset \mathrm{proj}_{\mathfrak{h}}[\mathfrak{d},\mathfrak{d}]$. Induction shows that $\mathfrak{h}^p \subset \mathrm{proj}_{\mathfrak{h}}(\mathfrak{d}^p)$ and $\mathfrak{h}^{(p)} \subset \mathrm{proj}_{\mathfrak{h}}(\mathfrak{d}^{(p)})$. Hence $\mathfrak{d}$ is solvable requires $\mathfrak{h}$ to be solvable, and we conclude that $\mathfrak{h}$ must not be simple. Hence, for $\mathfrak{d}$ to be a indecomposable solvable Lie algebra, it must be a double extension by a 1-dimensional Lie algebra.

In this case, the bracket is

$$\left[\begin{pmatrix}\alpha\\ X\\ H\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}\\ \widehat{X}\\ \widehat{H}\end{pmatrix}\right]_{\mathfrak{d}_A} = \begin{pmatrix}\langle AX,\widehat{X}\rangle\\ [X,\widehat{X}]_{\mathfrak{g}} + HA\widehat{X} - \widehat{H}AX\\ 0\end{pmatrix}$$

where $\pi : \mathfrak{h} \to \mathrm{der}(\mathfrak{g})$ has $\pi(x) = xA$ for all $x \in \mathfrak{h}$.

A standard fact from Lie theory is that $\mathfrak{d}$ is solvable if and only if $[\mathfrak{d},\mathfrak{d}]$ is nilpotent, see Corollary A.25 for details. Consider that an element of $[[\mathfrak{d},\mathfrak{d}],[\mathfrak{d},\mathfrak{d}]]$ has the form

$$\begin{pmatrix}\langle A([X_1,\widehat{X}_1]_{\mathfrak{g}} + H_1A\widehat{X}_1 - \widehat{H}_1AX_1), [X_2,\widehat{X}_2]_{\mathfrak{g}} + H_2A\widehat{X}_2 - \widehat{H}_2AX_2\rangle\\ [[X_1,\widehat{X}_1]_{\mathfrak{g}} + H_1A\widehat{X}_1 - \widehat{H}_1AX_1, [X_2,\widehat{X}_2]_{\mathfrak{g}} + H_2A\widehat{X}_2 - \widehat{H}_2AX_2]\\ 0\end{pmatrix}$$

where $X_1, \widehat{X}_1, X_2, \widehat{X}_2 \in \mathfrak{g}$ and $H_1, \widehat{H}_1, H_2, \widehat{H}_2 \in \mathfrak{h}$.

Note that $\mathrm{proj}_{\mathfrak{g}}([[\mathfrak{d},\mathfrak{d}],[\mathfrak{d},\mathfrak{d}]]) \subset [\mathfrak{g},\mathfrak{g}]$ and subsequently that $\mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]^p) \subset \mathfrak{g}^p$. Also, $\mathrm{proj}_{\mathfrak{h}^*}([\mathfrak{d},\mathfrak{d}]^p) = \langle A\,\mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]), \mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]^{p-1})\rangle \subset \langle A\mathfrak{g}, \mathfrak{g}^{p-1}\rangle$. If $\mathfrak{g}$ is nilpotent, then there is $p \in \mathbb{N}$ such that $\mathfrak{g}^p = \mathfrak{g}^{p-1} = 0$ and hence we have $[\mathfrak{d},\mathfrak{d}]^{p-1} = 0 = [\mathfrak{d},\mathfrak{d}]^p$. That is, $\mathfrak{g}$ nilpotent implies that $\mathfrak{d}$ is solvable.

Consider also that $\mathfrak{g}^{(p)} \subset \mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]^p)$. Hence, at the very least, it is necessary for $\mathfrak{g}$ to be solvable for the double extension $\mathfrak{d}$ to be solvable. Note that if A was invertible, then $\mathfrak{g}^p = \mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]^p)$ and it would be required that $\mathfrak{g}$ would have to be nilpotent for $\mathfrak{d}$ to be solvable. If A is not invertible, another condition on A may be possible to relax the nilpotent condition on $\mathfrak{g}$ to just solvable. For instance, if $\mathrm{Im}(A) \subset [\mathfrak{g},\mathfrak{g}]$, then it is easy to see that $[\mathfrak{g},\mathfrak{g}]^p = \mathrm{proj}_{\mathfrak{g}}([\mathfrak{d},\mathfrak{d}]^p)$ and if $\mathfrak{g}$ is solvable, this would conclude that $\mathfrak{d}$ is solvable also.

To summarise these observations, we have the following theorem.

Theorem 4.6 (Sufficient Conditions for Solvability). *Let $\mathfrak{g}$ be a metric Lie algebra and consider any double extension $\mathfrak{d}$ of $\mathfrak{g}$. Then for $\mathfrak{d}$ to be solvable, it must be a 1-dimensional double extension of $\mathfrak{g}$ with some further conditions. Let A be the anti-symmetric derivation of $\mathfrak{g}$ used to double extend it to $\mathfrak{d}$, then*

(1) if $\mathfrak{g}$ is solvable and $\mathrm{Im}(A) \subset [\mathfrak{g}, \mathfrak{g}]$*, or*

(2) if $\mathfrak{g}$ is a nilpotent Lie algebra

then $\mathfrak{d}$ is solvable.

In fact, from Theorem 4.6 and from Proposition 3.12 we have a stronger version of Lemma 4.12, that is:

Corollary 4.7. *If G is a connected Lie group with Lie algebra that is the 1-dimensional extension $\mathfrak{d}$ of a metric Lie algebra, $\mathfrak{g}$, such that either Theorem 4.6 (1) or (2) hold, then*

- *The Ricci tensor is 2-step nilpotent,*
- *the scalar curvature $S_{\mathfrak{d}} = 0$,*
- *the Bach tensor $B_{\mathfrak{d}} = 0$,*
- *the second conformal to Einstein obstruction is identically 0, and*
- *the first conformal to Einstein obstruction reduces to finding a $\Upsilon \in \mathfrak{F}(G)$ such that on the Lie algebra*

$$\Upsilon^a C_{abcd} = \Upsilon^a \Big(- R_{cdab} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac})\Big) = 0,$$

where $\nabla^a \Upsilon = \Upsilon^a$.

4.2.1 Nilpotent Lie algebras

A Lie algebra, $\mathfrak{n}$, is nilpotent whenever its derived series terminates. We know that nilpotent Lie algebras are also solvable, hence the argument in Section 4.2 tells us they must be extension of a metric Lie algebra $\mathfrak{g}$ by 1-dimensional Lie algebras. If we consider the bracket with respect to the decomposition $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$, we have

$$\left[\begin{pmatrix} \tilde{\alpha} \\ \tilde{X} \\ \tilde{H} \end{pmatrix}, \left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix}\right]\right] =$$

$$\begin{pmatrix} \beta(\tilde{X}, [X, \widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X} - \pi(\widehat{H}X)) \\ [\tilde{X}, [X, \widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X} - \pi(\widehat{H})X]_{\mathfrak{g}} + \pi(\tilde{H})([X, \widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X} - \pi(\widehat{H})X) \\ 0 \end{pmatrix}.$$

Let A be the anti-symmetric derivation such that $xA = \pi(x)$ for all $x \in \mathbb{R} \cong \mathfrak{h}$. From this, we can see firstly that $\mathrm{proj}_{\mathfrak{h}^*}(\mathfrak{n}^p) \subset \langle A\mathfrak{g}, \mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^{p-1})\rangle$. Then note that $\mathfrak{g}^2 \subset \mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^2)$ and $\mathrm{Im}(A^2) \subset \mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^2)$, and by induction, $\mathfrak{g}^n + \mathrm{Im}(A^n) \subset \mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^n)$. Hence $\mathfrak{n}$ is nilpotent requires at least $\mathfrak{g}$ to be nilpotent and A to be a nilpotent derivation.

In fact we can show these are the only conditions necessary.

Theorem 4.8. *If $\mathfrak{g}$ is a metric Lie algebra, and $\mathfrak{d}$ is a double extension of $\mathfrak{g}$, then $\mathfrak{d}$ is nilpotent if and only if it is a one-dimensional double extension of $\mathfrak{g}$, where $\mathfrak{g}$ is nilpotent, and the anti-symmetric derivation used is nilpotent.*

Proof. The "only if" part of the proof is above. For the "if" part, we will first show that

$$[\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] \subset \sum_{i=0}^{p} A^i [\mathfrak{g}, \mathfrak{g}^k]. \tag{4.2.1}$$

Consider initial case, $p = 1$, we use the derivation property of A to find that

$$[\mathfrak{g}, A\mathfrak{g}^k] = A[\mathfrak{g}, \mathfrak{g}^k] - [A\mathfrak{g}, \mathfrak{g}^k] \subset A[\mathfrak{g}, \mathfrak{g}^k] + [\mathfrak{g}, \mathfrak{g}^k] = \sum_{i=0}^{p} A^i [\mathfrak{g}, \mathfrak{g}^k].$$

Assume true for the pth case. Consider the $p+1$th case and again using the derivation property of A we have

$$\begin{aligned}
[\mathfrak{g}, \sum_{i=0}^{p+1} A^i \mathfrak{g}^k] =& A[\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] - [A\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] \\
\subset& A[\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] + [\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] \\
\subset& A \sum_{i=0}^{p} A^i [\mathfrak{g}, \mathfrak{g}^k] + \sum_{i=0}^{p} A^i [\mathfrak{g}, \mathfrak{g}^k] \qquad \text{by induction hypothesis} \\
\subset& \sum_{i=0}^{p+1} A^i [\mathfrak{g}, \mathfrak{g}^k].
\end{aligned}$$

Hence Equation (4.2.1) holds.

Using Equation (4.2.1), we proceed to show that

$$\mathrm{proj}_{\mathfrak{g}}(\mathfrak{d}^n) \subset \sum_{i=0}^{n} A^i \mathfrak{g}^{n-i}.$$

For $n = 1$ we see that

$$\mathrm{proj}_{\mathfrak{g}}(\mathfrak{d}^1) = [\mathfrak{g}, A\mathfrak{g}] + A\mathfrak{g} - A\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}] + A\mathfrak{g} = \sum_{i=0}^{1} A^i \mathfrak{g}^{n-i},$$

hence the initial case holds. Assume true for the nth case and consider

$$\begin{aligned}
\mathrm{proj}_{\mathfrak{g}}(\mathfrak{d}^n) = [\mathfrak{g}, \mathrm{proj}_{\mathfrak{g}}(\mathfrak{d}^{n-1})] + A\,\mathrm{proj}_{\mathfrak{g}}(\mathfrak{d}^{n-1}) \subset& [\mathfrak{g}, \sum_{i=0}^{n} A^i \mathfrak{g}^{n-i}] + A \sum_{i=0}^{n} A^i \mathfrak{g}^{n-i} \\
\subset& \sum_{i=0}^{n} A^i \mathfrak{g}^{n+1-i} + \sum_{i=0}^{n} A^{i+1} \mathfrak{g}^{n-i} \\
=& \sum_{i=0}^{n+1} A^i \mathfrak{g}^{n-i}.
\end{aligned}$$

Now if A is a nilpotent endomorphism and $\mathfrak{g}$ is a nilpotent Lie algebra then there are $n_1, n_2 \in \mathbb{N}$ such that $A^{n_1} = 0$ and $\mathfrak{g}^{n_2} = 0$. Let $N = 2\max\{n_1, n_2\}$. Then one of i or $n - i$ is always greater than $\frac{N}{2}$ which is greater than or equal to n_1 and n_2. Hence $\sum_{i=0}^{N} A^i \mathfrak{g}^{N-i} = 0$, which implies that $\mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^N) = 0$. As $\mathrm{proj}_{\mathfrak{h}^*}(\mathfrak{n}^p) \subset \langle A\mathfrak{g}, \mathrm{proj}_{\mathfrak{g}}(\mathfrak{n}^{p-1})\rangle$, then $\mathfrak{d}^{N+1} = 0$. Hence $\mathfrak{d}$ is nilpotent. $\square$

4.3 Double extensions by 1-dimensional Lie algebras

We have shown that nilpotent and solvable metric Lie algebras are 1-dimensional double extensions of metric Lie algebras. We now consider double extensions by 1-dimensional Lie algebras in general. That is, we consider $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$ where $\mathfrak{h} = \mathbb{R}H_0 \cong \mathfrak{h}^* = \mathbb{R}\alpha_0$.

A 1-dimensional Lie algebra is trivially abelian, hence the adjoint and co-adjoint maps $\mathrm{ad}_{\mathfrak{h}}$ and $\mathrm{ad}^*_{\mathfrak{h}}$ are both equal to zero. As in Remark 4.1, we use $A \in \mathfrak{so}(\mathfrak{g})$ in the place of the anti-symmetric map π, as for all $x \in \mathfrak{h}$ we must have $\pi(x) = xA$ for some $A \in \mathfrak{so}(\mathfrak{g})$. We also drop α_0 and H_0 whenever necessary. With respect to the vector space decomposition $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$ the bracket, metric and Ricci tensor of $\mathfrak{d}$ simplify to

$$\left[\begin{pmatrix}\alpha\\ X\\ H\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}\\ \widehat{X}\\ \widehat{H}\end{pmatrix}\right]_{\mathfrak{d}_\pi} = \begin{pmatrix}\langle AX, \widehat{X}\rangle_{\mathfrak{g}}\\ [X,\widehat{X}]_{\mathfrak{g}} + HA\widehat{X} - \widehat{H}AX\\ 0\end{pmatrix},$$

$$\mathrm{ad}\begin{pmatrix}\alpha\\ X\\ H\end{pmatrix} = \begin{pmatrix}0 & \langle AX,\cdot\rangle & 0\\ 0 & \mathrm{ad}_{\mathfrak{g}}(X) + HA & -AX\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathrm{Ric} = -\frac{1}{4}\begin{pmatrix}0 & 0 & 0\\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\cdot)A\}\\ 0 & \mathrm{tr}\{A\,\mathrm{ad}_{\mathfrak{g}}(\cdot)\} & \mathrm{tr}\{A^2\}\end{pmatrix},$$

and

$$\langle,\rangle_{\mathfrak{d}_\pi} = \begin{pmatrix}0 & 0 & 1\\ 0 & \langle,\rangle_{\mathfrak{g}} & 0\\ 1 & 0 & 0\end{pmatrix}.$$

Remark 4.2. If we take the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ on $\mathbb{R}H_0$ to be non-zero, one can show that the resulting double extension is always isometric to $\langle\cdot,\cdot\rangle_{\mathfrak{h}} = 0$ using the isometry

$$\phi(\begin{pmatrix}\alpha\\ X\\ H\end{pmatrix}) = \begin{pmatrix}\alpha - \frac{cH}{2}Id\\ X\\ H\end{pmatrix} \quad \text{where } \langle H,\widehat{H}\rangle_{\mathfrak{h}} = cH\widehat{H}.$$

Here we again write vectors with respect to the decomposition $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$.

The following proposition shows when two 1-dimensional double extensions are isomorphic.

Proposition 4.9 (Favre and Santharoubane [10]). *Let $\mathfrak{g}$ be a metric Lie algebra and A, $\widehat{A}$ two antisymmetric derivations of $\mathfrak{g}$. Let $\mathfrak{d}_A$ and $\mathfrak{d}_{\widehat{A}}$ between two 1-dimensional double extensions of $\mathfrak{g}$, defined by anti-symmetric derivations $A, \widehat{A}$ respectively. Then there exists an isometry*

$$\phi : \mathfrak{d}_A \to \mathfrak{d}_{\widehat{A}}$$

if and only if there is a $\lambda_0 \in \mathbb{R}\setminus\{0\}$, $X_0 \in \mathfrak{g}$ and ψ_0, an isometry of $\mathfrak{g}$, such that

$$\psi_0\widehat{A}\psi_0^{-1} = \lambda_0 A + \mathrm{ad}(X_0).$$

The isometry has the form

$$\phi = \begin{pmatrix} \frac{1}{\lambda_0} & \frac{-1}{\lambda_0}\langle X_0, \psi_0(\cdot)\rangle_{\mathfrak{g}} & -\frac{1}{2}\langle X_0, X_0\rangle_{\mathfrak{g}} \\ 0 & \psi_0 & X_0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Proof. Assume $\phi : \mathfrak{d}_{\widehat{A}} \to \mathfrak{d}_A$ is an isomorphism and, with respect to the decomposition $\mathfrak{d} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$, we can represent ϕ as a matrix

$$\phi = \begin{pmatrix} a & b^t & c \\ d & e & f \\ g & h^t & i \end{pmatrix}.$$

By Lemma A.37, any isomorphism must send the centre to the centre, so d, g, h must all be 0. This can also be shown using the homomorphism property; we have that for any α, $\widehat{\alpha} \in \mathbb{R}$, X, $\widehat{X} \in \mathfrak{g}$, H, $\widehat{H} \in \mathbb{R}$ then

$$\begin{aligned} &\begin{pmatrix} \langle A(\alpha d + eX + Hf), \widehat{\alpha} d + e\widehat{X} + \widehat{H} f\rangle \\ [\alpha d + eX + Hf, \widehat{\alpha} d + e\widehat{X} + \widehat{H} f]_{\mathfrak{g}} \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ (\alpha g + h^t X + iH)A(\widehat{\alpha} d + e\widehat{X} + \widehat{H} f) - (\widehat{\alpha} g + h^t \widehat{X} + i\widehat{H})A(\alpha d + eX + Hf) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a\langle \widehat{A}X, \widehat{X}\rangle + b^t([X, \widehat{X}]_{\mathfrak{g}} + H\widehat{A}\widehat{X} - \widehat{H}\widehat{A}X) \\ d\langle \widehat{A}X, \widehat{X}\rangle + e([X, \widehat{X}]_{\mathfrak{g}} + H\widehat{A}\widehat{X} - \widehat{H}\widehat{A}X) \\ g\langle \widehat{A}X, \widehat{X}\rangle + h^t([X, \widehat{X}]_{\mathfrak{g}} + H\widehat{A}\widehat{X} - \widehat{H}\widehat{A}X) \end{pmatrix} \end{aligned}$$

We notice immediately that $g = 0$ and $h^t = 0$. If we set $\widehat{X} = 0$ and $\widehat{H} = 0 = H$, then we require that $[eX, d] = 0$ for all $X \in \mathfrak{g}$. If we let only $H = \widehat{H} = 0$, the second entry reduces to $[eX, e\widehat{X}] = e[X, \widehat{X}] + d\langle \widehat{A}X, \widehat{X}\rangle$, from which applying the Jacobi identity and that $[eX, d] = 0$, we conclude that $d = 0$, and hence that e is a Lie algebra endomorphism of $\mathfrak{g}$.

Note that as ϕ is an isometry, $\det(\phi)$ is non-zero. We also have $\det(\phi) = ai\det(e)$, so we require $\det(e)$ to be non-zero. This means means e is a Lie algebra automorphism.

From the second entry, we now deduce that

$$H(iAe\widehat{X} + [f, e\widehat{X}]_{\mathfrak{g}}) - \widehat{H}(iAeX + [f, eX]_{\mathfrak{g}}) = e(H\widehat{A}\widehat{X} - \widehat{H}\widehat{A}X),$$

which occurs if and only if $e\widehat{A}e^{-1} = iA + \mathrm{ad}_{\mathfrak{g}}(f)$.

We now have

$$\phi = \begin{pmatrix} a & b^t & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}.$$

Using the isometry property, we have that

$$\begin{aligned} &\alpha(\widehat{H}) + \widehat{\alpha}(H) + \langle X, \widehat{X}\rangle_{\mathfrak{g}} = \\ &\langle eX + Hf, e\widehat{X} + \widehat{H} f\rangle_{\mathfrak{g}} + i(a\widehat{\alpha}(H) + b^t\widehat{X}H + cH\widehat{H}) + i(a\alpha(\widehat{H}) + b^t X\widehat{H} + c\widehat{H}H). \end{aligned}$$

If we let $H = \widehat{H} = 0$ then $\langle X, \widehat{X}\rangle_{\mathfrak{g}} = \langle eX, e\widehat{X}\rangle_{\mathfrak{g}}$ for all X and hence e must be an isometry of $\mathfrak{g}$.

If we let $X = \widehat{X} = 0$ and $\alpha = \widehat{\alpha} = 0$ then we deduce that $ic = -\frac{1}{2}\langle f, f\rangle_{\mathfrak{g}}$.

If we let $\widehat{H} = 0$ and $\widehat{X} = 0$, this implies $ai = 1$. If we let $\widehat{H} = 0$, $\widehat{\alpha} = 0$ and $X = 0$, we deduce that $b^t = \frac{-1}{i}\langle f, e(\cdot)\rangle_{\mathfrak{g}}$.

We may now write

$$\phi = \begin{pmatrix} a & -a\langle f, e(\cdot)\rangle_{\mathfrak{g}} & -\frac{1}{2}\langle f, f\rangle_{\mathfrak{g}} \\ 0 & e & f \\ 0 & 0 & \frac{1}{a} \end{pmatrix}.$$

If one now checks the isometry and homomorphism property, these are both satisfied. As $\det(\phi)$ is non-zero, ϕ is an isometry of the double extensions. Note that A and $\widehat{A}$ are connected by the formula $e\widehat{A}e^{-1} = \frac{1}{a}A + \mathrm{ad}_{\mathfrak{g}}(f)$. If we let $\psi_0 = e$, $\lambda_0 = \frac{1}{a}$ and $X_0 = f$ the results follows. $\square$

This theorem was first stated and proven in Favre and Santharoubane [10] and appears also in Baum and Kath [3].

To check whether the metrics are conformally Einstein, we will need to consider the formula for the square of the Ricci tensor, Ric^2. Let $Y_1, \ldots, Y_n$ be an orthonormal basis for $\mathfrak{g}$, and let $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $X_i = \begin{pmatrix} 0 \\ Y_i \\ 0 \end{pmatrix}$, $X_{n+1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be a basis for $\mathfrak{d} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$ with respect to this decomposition. Then

$$\begin{aligned} \mathrm{Ric}^2(X_0, Z) &= 0 \quad \text{for all } Z \in \mathfrak{d}, \\ \mathrm{Ric}^2(X_i, X_k) &= \frac{1}{16}K_{\mathfrak{g}}(Y_i, Y_j)\epsilon_j K_{\mathfrak{g}}(Y_j, Y_k) = \mathrm{Ric}^2_{\mathfrak{g}}(Y_i, Y_k), \\ \mathrm{Ric}^2(X_i, X_{n+1}) &= \frac{1}{16}K_{\mathfrak{g}}(Y_i, Y_j)\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(Y_j)\pi(1)\}, \text{ and} \\ \mathrm{Ric}^2(X_{n+1}, X_{n+1}) &= \frac{1}{16}\operatorname{tr}\{\pi(1)\,\mathrm{ad}_{\mathfrak{g}}(Y_j)\}\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(Y_j)\pi(1)\}. \end{aligned}$$

This formula can be found by direct calculation, see Lemma A.35. If we let $Y_1, \ldots, Y_n$ be an orthonormal basis for $\mathfrak{g}$ such that $\langle Y_i, Y_j\rangle = \epsilon_i\delta_{ij}$, then $\mathrm{Ric}^2 = 0$ if and only if

$$\begin{aligned} 0 =& \frac{1}{16}\Big(K_{\mathfrak{g}}(Y_i, Y_j)\epsilon_j K_{\mathfrak{g}}(Y_j, Y_k) + K_{\mathfrak{g}}(Y_i, Y_j)\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(Y_j)\pi(H)\} \\ &+ \operatorname{tr}\{\pi(\widehat{H})\,\mathrm{ad}_{\mathfrak{g}}(Y_j)\}\epsilon_j K_{\mathfrak{g}}(Y_j, Y_k) + \operatorname{tr}\{\pi(\widehat{H})\,\mathrm{ad}_{\mathfrak{g}}(Y_j)\}\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(Y_j)\pi(H)\}\Big) \end{aligned}$$

for all $H, \widehat{H} \in \mathbb{R}$. Hence, we have the following, which can be found in Baum and Kath [3, pg. 257].

Proposition 4.10 (Baum and Kath [3])**.** *The Ricci tensor of $\mathfrak{d}$ is 2-step nilpotent if and only if the Ricci tensor of $\mathfrak{g}$ is 2-step nilpotent and for any othornormal basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ then*

$$K_{\mathfrak{g}}(X_i, X_j)\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(X_j)\pi(H)\} = 0 \tag{4.3.1}$$

$$\operatorname{tr}\{\pi(\widehat{H})\,\mathrm{ad}_{\mathfrak{g}}(X_j)\}\epsilon_j \operatorname{tr}\{\mathrm{ad}_{\mathfrak{g}}(X_j)\pi(H)\} = 0 \tag{4.3.2}$$

for all $H, \widehat{H} \in \mathbb{R}$.

Remark 4.3. This proposition gives us another way to see that the metric on any 1-dimensional double extension of a nilpotent Lie algebra is Bach-flat, as in Corollary 4.7. This is due to the following lemma:

Lemma 4.11. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let A be any derivation of $\mathfrak{g}$. Then $A \circ \mathrm{ad}_X$ is nilpotent endomorphism of $\mathfrak{g}$ for any $X \in \mathfrak{g}$.*

When Lemma 4.11 applies, then we can consider that Remark A.3 shows that for any nilpotent endomorphism there is a basis of $\mathfrak{g}$ such that the endomorphism is strictly upper triangular. The trace of a strictly upper triangluar matrix is zero. As the trace is independent of change of basis, then the trace of $A \circ \mathrm{ad}_X$ must be zero. This means that Equation (4.3.1) and Equation (4.3.2) hold. Applying Proposition 4.10, then any a 1-dimensional double extension of a nilpotent Lie algebra has 2-step nilpotent Ricci tensor. Applying Proposition 3.11 shows that the metric on any 1-dimensional double extension of a nilpotent Lie algebra is Bach-flat. We prove Lemma 4.11 below.

Proof. Define $\mathfrak{g}_A^k$ such that $\mathfrak{g}_A^k = [\mathfrak{g}, A\mathfrak{g}_A^{k-1}]$ for $k \in \mathbb{N}$ and $\mathfrak{g}_A^0 = \mathfrak{g}$. Note that $(A \circ \mathrm{ad}_X)^k(Y) \in A\mathfrak{g}_A^k$ for all $X, Y \in \mathfrak{g}$. From Equation (4.2.1), we have

$$[\mathfrak{g}, \sum_{i=0}^{p} A^i \mathfrak{g}^k] \subset \sum_{i=0}^{p} A^i [\mathfrak{g}, \mathfrak{g}^k].$$

We now show

$$\mathfrak{g}_A^k \subset \sum_{i=0}^{k} A^i \mathfrak{g}^k. \tag{4.3.3}$$

Consider the initial case, $k = 1$. Then, by Equation (4.2.1), we have

$$\mathfrak{g}_A^1 = [\mathfrak{g}, A\mathfrak{g}] \subset \sum_{i=0}^{1} A^i [\mathfrak{g}, \mathfrak{g}] = \sum_{i=0}^{1} A^i \mathfrak{g}^1.$$

Assume true for the kth case. Consider the $k + 1$th case.

$$\begin{aligned}
\mathfrak{g}_A^{k+1} &= [\mathfrak{g}, A\mathfrak{g}_A^k] \\
&= A[\mathfrak{g}, \mathfrak{g}_A^k] - [A\mathfrak{g}, \mathfrak{g}_A^k] \\
&\subset A[\mathfrak{g}, \mathfrak{g}_A^k] + [\mathfrak{g}, \mathfrak{g}_A^k] \\
&\subset A[\mathfrak{g}, \sum_{i=0}^{k} A^i \mathfrak{g}^k] + [\mathfrak{g}, \sum_{i=0}^{k} A^i \mathfrak{g}^k] && \text{by induction hypothesis} \\
&\subset A \sum_{i=0}^{k} A^i [\mathfrak{g}, \mathfrak{g}^k] + \sum_{i=0}^{k} A^i [\mathfrak{g}, \mathfrak{g}^k] && \text{by claim 1} \\
&= \sum_{i=0}^{k+1} A^i \mathfrak{g}^{k+1}.
\end{aligned}$$

Hence the second claim holds.

We have shown

$$(A \circ \mathrm{ad}_X)^k(Y) \in A\mathfrak{g}_A^k \subset \sum_{i=0}^{k} A^i \mathfrak{g}^k \quad \forall X, Y \in \mathfrak{g}.$$

Now if $\mathfrak{g}$ is a nilpotent Lie algebra, then there is a $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$. Then for this k, the right hand side of the above equation is zero. Hence $(A \circ \mathrm{ad}_X)^k = 0$ for all $X \in \mathfrak{g}$. That is, $A \circ \mathrm{ad}_X$ is a nilpotent endomorphism. □

4.3.1 Double extensions of abelian metric Lie algebras by 1-dimensional Lie algebras

We will now look closely at the case for when $\mathfrak{g}$ is an abelian Lie algebra. The first result about such a Lie algebra is as follows.

Lemma 4.12. *If $\mathfrak{g}$ is an abelian metric Lie algebra, then $\mathfrak{d}$, a double extension of $\mathfrak{g}$ by a 1-dimensional Lie algebra, has Ricci tensor*

$$\mathrm{Ric}\left(\begin{pmatrix}\alpha, \\ X \\ H\end{pmatrix}, \begin{pmatrix}\widehat{\alpha} \\ \widehat{X} \\ \widehat{H}\end{pmatrix}\right) = -\frac{1}{4} H\widehat{H} \operatorname{tr} A^2,$$

which is 2-step nilpotent. Here we write the elements of $\mathfrak{d}$, $\begin{pmatrix}\alpha, \\ X \\ H\end{pmatrix}$ and $\begin{pmatrix}\widehat{\alpha} \\ \widehat{X} \\ \widehat{H}\end{pmatrix}$, with respect to the decomposition $\mathfrak{d} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$.

Proof. The Lie algebra $\mathfrak{g}$ is abelian which implies that $\mathrm{ad}_{\mathfrak{g}} \equiv 0$ and hence $K_{\mathfrak{g}} \equiv 0$. This gives the form of the Ricci tensor. Then we see that Equation (4.3.1) and Equation (4.3.2) are both satisfied, hence Proposition 4.10 implies the Ricci tensor of $\mathfrak{d}$ is 2-step nilpotent. □

Note that the Ricci tensor being 2-step nilpotent is a special case of Corollary 4.7, where we note that abelian Lie algebras are nilpotent. We can also apply Remark 4.3 to obtain this result.

Theorem 4.13. *If $\mathfrak{d}(\mathfrak{g}, \mathbb{R}) = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$ is a double extension of $\mathfrak{g}$, an abelian Lie algebra, then $\mathfrak{d}$ is conformal to Einstein. In fact, $\mathfrak{d}$ is conformally Ricci-flat.*

Proof. From Corollary 4.7 we have that $\mathfrak{d}$ the Bach tensor is identically 0, implying that second obstruction is satisfied. If we consider the first obstruction, from Corollary 4.7 we must find a gradient Υ^a such that

$$\Upsilon^a C_{abcd} = \Upsilon^a\Big(- R_{cdab} + \frac{1}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac})\Big) = 0.$$

Note that $\begin{pmatrix}\alpha\\0\\0\end{pmatrix}$, written with respect to the vector space decomposition $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$, is in the centre of $\mathfrak{d}$, so that

$$C(\begin{pmatrix}\alpha\\0\\0\end{pmatrix}, A_1, A_2, A_3) = \frac{1}{n-2}(\langle\alpha, A_3\rangle \operatorname{Ric}(A_1, A_2) - \langle\alpha, A_2\rangle \operatorname{Ric}(A_3, A_1))$$

for any $A_1, A_2, A_3 \in \mathfrak{d}$. With respect to the vector space decomposition, $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$, write $A_i = (\alpha_i, X_i, H_i)^t$ with $\alpha_i \in \mathbb{R}\alpha_0$ and $H_i \in \mathbb{R}H_0$ and $X_i \in \mathfrak{g}$. From Lemma 4.12, the Ricci tensor of $\mathfrak{d}$ is

$$\operatorname{Ric}\left(\begin{pmatrix}\alpha_i\\X_i\\H_i\end{pmatrix}, \begin{pmatrix}\alpha_j\\X_j\\H_j\end{pmatrix}\right) = -\frac{1}{4}H_iH_j \operatorname{tr} A^2.$$

This implies that the scalar curvature is 0. Hence

$$\begin{aligned}C(\alpha, A_1, A_2, A_3) &= \frac{1}{n-2}(\langle\alpha, A_3\rangle \operatorname{Ric}(A_1, A_2) - \langle\alpha, A_2\rangle \operatorname{Ric}(A_3, A_1))\\ &= \frac{1}{n-2}(-\alpha H_3 \frac{1}{4} H_1 H_2 \operatorname{tr} A^2 + \alpha H_2 \frac{1}{4} H_3, H_1 \operatorname{tr} A^2)\\ &= 0.\end{aligned}$$

This means both obstructions are satisfied for $\mathfrak{d}$ for vectors of the form $\begin{pmatrix}\alpha\\0\\0\end{pmatrix}$. We suspect a gradient of this form will be the gradient needed to satisfy Equation (1.5.4) and hence make $\mathfrak{d}$ conformal to Einstein.

Let $\mathfrak{g}$ have dimension n. With respect to the decomposition $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$, we use the basis $X_0 = (1, 0, 0)$, $X_i = (0, e_i, 0)$ and $X_{n+1} = (0, 0, 1)$ for $i = 1, \ldots, n$. Let G be any connected Lie group with Lie algebra $\mathfrak{d}$. The bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ induces a metric, g on G.

Let $\psi(\cdot) = g(X_0, \cdot)$. Now $\nabla\psi = 0$, which implies $d\psi = 0$. Hence locally we can find a $u \in \mathfrak{F}(G)$ such that $du = \psi$. Our approach is to try a $h : \mathbb{R} \to \mathbb{R}$ and define $f = h \circ u$ and then $\rho = f\psi$, and to show that this ρ satisfies Equation (1.5.4).

Consider that

$$\begin{aligned}(\nabla_V \rho)Y = (\nabla_V f\psi)Y = (\nabla_V f)\psi(Y) + 0 &= df(V)\psi(Y)\\ &= d(h \circ u)(V)\psi(Y)\\ &= (dhdu)(V)\psi(Y)\\ &= (h'(u)\psi(V))\psi(Y).\end{aligned}$$

This implies $\nabla\rho = h'\psi \otimes \psi$. Now $\operatorname{div}(\rho) = g(E_i, E_j)\nabla_{E_i}E_j = h'g(E_i, E_j)\psi(E_i) \circ \psi(E_j) = h'g(\psi(E_i)E_i, \psi(E_j)E_j) = h'g(X_{n+1}, X_{n+1}) = 0$. Here E_i is a coordinate basis for G.

Finally consider that the Schouten tensor P is a multiple of $\psi \otimes \psi$ as P_{ab} is zero unless $a = b = n+1$. We note that $\psi(X_a)\psi(X_b)$ is also zero unless $a = b = n+1$. Hence we have $P = -\frac{1}{4n} \operatorname{tr}\{A^2\}\psi \otimes \psi$.

Consider now Equation (1.5.4), with ρ inserted into the Υ position. That is,

$$\mathsf{P}_{ab} - \nabla_a \rho_b + \rho_a \rho_b - \frac{1}{n}(\mathsf{J} - \nabla^d \rho_d + \rho^d \rho_d) g_{ab} = 0$$

for indices running from 0 to $n+1$. We know the scalar curvature is 0 so $J = 0$. We also know that $\rho^d \rho_d = g^{-1}(\rho, \rho) = 0$ and $\nabla^d \rho_d = \operatorname{div}(\rho) = 0$. Hence we are left with

$$-\frac{1}{4n}\operatorname{tr}\{A^2\}\psi \otimes \psi - h'\psi \circ \psi + h^2 \psi \circ \psi = 0.$$

To solve this, we must find $h : \mathbb{R} \to \mathbb{R}$ such that

$$-\frac{1}{4n}\operatorname{tr}\{A^2\} - h' + h^2 = 0.$$

This is a separable ODE for which we can solve to find

$$\begin{aligned}
h &= \frac{-1}{x+b} && \text{when } \operatorname{tr}\{A^2\} = 0,\\
h &= \sqrt{-\frac{1}{4n}\operatorname{tr}\{A^2\}}\tan\left(\sqrt{-\frac{1}{4n}\operatorname{tr}\{A^2\}}x + b\right) && \text{when } \operatorname{tr}\{A^2\} < 0,\\
h &= \sqrt{\frac{1}{4n}\operatorname{tr}\{A^2\}}\left(\frac{2}{1 - be^{2x\sqrt{\frac{1}{4n}\operatorname{tr}\{A^2\}}}} - 1\right) && \text{when } \operatorname{tr}\{A^2\} > 0
\end{aligned}$$

for some $b \in \mathbb{R}$. The constant b can be chosen to suit the initial conditions of the ODE. Note that we only need the solutions to exist in a neighbourhood, as we are considering locally conformally flat. We also note that when $\operatorname{tr}\{A^2\} = 0$, the metric is Ricci-flat hence is already Einstein. The case when $\operatorname{tr}\{A^2\} > 0$ can also be written using the hyperbolic tangent function.

This solution implies we have found a $\rho = f\psi$ that satisfies Equation (1.5.4). Note that the initial condition will determine b in each case. Now if we consider that

$$d(\rho) = d(f\psi) = df \wedge \psi + f d\psi = df \wedge \psi = h'\psi \wedge \psi = 0.$$

This implies locally we can find $\Upsilon \in \mathfrak{F}(G)$ such that $\operatorname{grad}(\Upsilon) = \rho$. Hence this Υ satisfies Equation (1.5.4), where $\nabla_a \Upsilon = \Upsilon_a$ and $\nabla^a \Upsilon = \Upsilon^a$. Hence G is conformally Einstein.

Recall from Proposition 1.23 that the conformal change of the Ricci tensor is Equation (1.5.3)

$$\widehat{R}_{ab} = R_{ab} - (\nabla_c \Upsilon^c + (n-2)\Upsilon^c \Upsilon_c) g_{ab} + (n-2)(\Upsilon_a \Upsilon_b - \nabla_a \Upsilon_b).$$

Then on the Lie algebra with $d(\Upsilon) = \rho$ as above, we have

$$\widehat{\mathrm{Ric}} = -\frac{1}{4}\operatorname{tr} A^2 \psi \otimes \psi + n(h^2 \psi \otimes \psi - nh'\psi \otimes \psi) = n(-\frac{1}{4n}\operatorname{tr}\{A^2\} + h^2 - h')\psi \otimes \psi = 0.$$

Hence $\mathfrak{d}$ is conformally Ricci-flat.

□

Remark 4.4. As $\mathfrak{d}$ is not semisimple, then by Lemma 3.3 it can be Einstein if and only if it is Ricci-flat. However, as Ricci tensor restricted to $\mathbb{R}H$ is proportional to $\operatorname{tr}\{A^2\}$ and zero in all other components, this is if and only if $\operatorname{tr}\{A^2\} = 0$. Hence if $\operatorname{tr}\{A^2\} \neq 0$, $\mathfrak{d}$ is not Einstein but it is conformal to Einstein.

It is quite possible for $\operatorname{tr}\{A^2\} = 0$ and for the Lie algebra to be indecomposable, as this can occur whenever the metric on $\mathfrak{g}$ is not Riemannian. In this case, we can write the derivation on $\mathfrak{g}$ as

$$A = \begin{pmatrix} D & B \\ B^t & C \end{pmatrix}$$

where $\mathfrak{g}$ is has metric with signature (p, q) and $D \in \mathfrak{so}(p)$, B is a $q \times p$ matrix and $C \in \mathfrak{so}(q)$. Then the trace of A^2 is

$$\operatorname{tr}\{A^2\} = \operatorname{tr}\{D^2\} + \operatorname{tr}\{C^2\} + \operatorname{tr}\{B^t B\}.$$

However, by Lemma A.14 any matrix in $P \in \mathfrak{so}(n)$, has imaginary eigenvalues, hence squaring these gives non-positive numbers of which the trace P^2 of is the sum. This means $\operatorname{tr}\{D^2\} + \operatorname{tr}\{C^2\}$ is non-positive. On the other hand, the trace of $\operatorname{tr}\{B^t B\}$ is always non-negative as $B^t B$ is a non-negative definite matrix. One can fix D, C and adjust B so that the trace is zero, negative or positive.

Note that in the case that the metric on $\mathfrak{g}$ is Riemannian, the matrix B above does not exist so the trace is then negative whenever $A \neq 0$.

We now consider an example of Theorem 4.13 when $\mathfrak{g}$ is equipped with a Riemannian metric by introducing the Heisenberg and Oscillator algebras.

Definition 4.2. The *Heisenberg algebra*, $\mathfrak{he}_{2m+1}$ is the Lie algebra of dimension $2m + 1$ with basis elements $\{p_1, \ldots, p_m, q_1, \ldots, q_m, z\}$ such that

$$[p_i, p_j] = [q_i, q_j] = 0, \qquad [p_i, q_j] = \delta_{ij} z, \qquad [p_i, z] = [q_i, z] = 0.$$

An element $a_i q_i + b_j q_j + cz$ can be represented by the matrix

$$\begin{bmatrix} 0 & \begin{pmatrix} a_1 & \ldots & a_m \end{pmatrix} & c \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ 0 & \ldots & 0 \end{bmatrix}.$$

The *Oscillator algebra* $\mathfrak{os}_{2m+2}$ is the Lie algebra defined as the semi-direct product $\mathfrak{os}_{2m+1} = \mathfrak{he}_{2m+1} \rtimes \mathbb{R}$. If we let $r = (0, 1) \in \mathfrak{he}_{2m+1} \rtimes \mathbb{R}$, then the commutator relations are

$$[r, p_i] = q_i, \qquad [r, q_i] = -p_i, \qquad [r, z] = 0.$$

The *Oscillator algebra over* λ, $\mathfrak{os}(\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ is defined as with the Oscillator algebra but with commutator relations

$$[r, p_i] = \lambda_i q_i, \qquad [r, q_i] = -\lambda_i p_i, \qquad [r, z] = 0.$$

We aim to consider Theorem 4.13 when $\mathfrak{g}$ is equipped with a Riemannian metric, and hence the double extension has Lorentzian metric. Note that from the classification of simple Lie groups, the only real simply connected simple Lie group with Lorentzian signature Killing form is $\mathrm{SL}_2(\mathbb{R})$.

Theorem 4.14 (Medina [25])**.** *Each indecomposable non-simple metric Lie algebra of Lorentzian signature* $(1, n-1)$ *is an Oscillator algebra.*

The following proof is similar to the proof in Baum and Kath [3].

Proof. There is no Lorentzian manifolds in dimension 1. For dimension $n \geq 2$ we are in the position of a double extension, by Theorem 4.1. From Equation (4.1.2),

$$\text{signature}(\langle \cdot, \cdot \rangle_{\mathfrak{d}}) = \text{signature}(\langle \cdot, \cdot \rangle_{\mathfrak{g}}) + (\dim(\mathfrak{h}), \dim(\mathfrak{h}))$$

hence we need to consider $\mathfrak{h}$ as a 1-dimensional Lie algebra, that is $\mathfrak{h} = \mathbb{R}$, and $\mathfrak{g}$ to be equipped with a Riemannian metric. By Proposition 2.3, $\mathfrak{g}$ is reductive, and splits into ideals $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{z}$ is abelian, $\mathfrak{g}'$ semi-simple. By Theorem 4.5, $\mathfrak{g}$ cannot have a semisimple factor, so $\mathfrak{g}$ must be abelian.

Now $\pi : \mathfrak{h} \to \mathrm{der}(\mathfrak{g})$ is a Lie algebra homomorphism. As $\mathfrak{h}$ is one dimensional, and $\mathfrak{g}$ is abelian, then for all $x \in \mathfrak{h}$, $\pi(x) = xA$ for some $A \in \mathfrak{so}(\mathfrak{g})$. If A has a kernel, this is an ideal of $\mathfrak{g}$ and then A acts trivially on this ideal. Hence the double extension can be decomposed using the map in Proposition 4.4 which contradicts the assumption that the double extension is indecomposable. Hence A has no kernel, implying that $A \in \mathrm{GL}(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g})$. By Proposition A.15, $\mathfrak{g}$ is even dimensional.

Picking a basis for $\mathfrak{g}$ we can write A as a matrix and compute the eigenvalues and eigenvectors. As in Proposition A.15, the eigenvalues will be imaginary and the eigenvectors complex. Let the eigenvectors be $x_1, , \ldots, x_{\frac{n}{2}}, \overline{x_1}, \ldots, \overline{x_{\frac{n}{2}}}$, with corresponding eigenvalues $\lambda_1 i, \ldots, \lambda_{\frac{n}{2}} i, -\lambda_1 i, \ldots, -\lambda_{\frac{n}{2}}$. Then consider the basis $q_j = x_j + \overline{x_j}, p_j = -i(x_j - \overline{x_j})$, which are the real and imaginary parts of the x_j's. Then

$$Aq_j = A(x_j + \overline{x_j}) = \lambda_j i x_j + \overline{\lambda_j i x_j} = -\lambda_j p_j.$$

Similarly, $Ap_j = \lambda_j q_j$. Hence, we can write

$$A = \begin{pmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_{\frac{n}{2}} \end{pmatrix} \quad \text{where} \quad \Lambda_i = \begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}.$$

Define $r = 1 \in \mathbb{R}$ and $z \in \mathbb{R}^*$ the identity function.

If we take $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ the Riemannian metric such that $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle = \delta_{ij}$ and zero otherwise, and $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ a scalar multiple of the dot product, such that $\langle r, r \rangle_{\mathfrak{h}} = c \in \mathbb{R}$. Then in the basis $\{z, p_1, q_1, p_2, q_2 \ldots, p_{\frac{n}{2}}, q_{\frac{n}{2}}, r\}$ we have

$$\langle \cdot, \cdot \rangle_{\mathfrak{d}} = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & I & 0 \\ 1 & \cdots & c \end{pmatrix}$$

where I is the $n \times n$ identity matrix. As in Remark 4.2, one can show that different values of c produce isometric Lie algebras, so we will take $c = 0$. Considering the bracket operations, we find that

$$[p_i, p_j] = [q_i, q_j] = 0,\ [p_i, q_j] = \delta_{ij} z,\ [p_i, z] = [q_i, z] = [r, z] = 0,\ [r, p_i] = \lambda_i q_i,\ [r, q_i] = -\lambda_i p_i,$$

which is exactly the Oscillator algebra commutators.

Finally, we note that we made the choice of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. However, by Proposition 4.9, we have that if $A, \widehat{A} \in \mathfrak{so}(\mathfrak{g})$ then there is an isometric isomorphism $\mathfrak{d}_A \to \mathfrak{d}_{\widehat{A}}$ if and only if there is a $\lambda_0 \in \mathbb{R} \setminus \{0\}$, and a $\phi \in \mathrm{Aut}(\mathfrak{g}, \langle \cdot, \cdot, \rangle)$ such that

$$\phi^{-1} \widehat{A} \phi = \lambda_0 A.$$

Then for any Riemannian metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and derivation $\widehat{A}$ on $\mathfrak{g}$ such that the double extension is indecomposable, we can pick an orthonormal basis for $\mathfrak{g}$ and then using Proposition 4.9 we can find a ϕ such that $\mathfrak{d}_{\widehat{A}}$ is isometric to $\mathfrak{d}_A$ with A in the same form as above. This shows that all indecomposable non-simple Lorentzian metric Lie algebras are Oscillator algebras. $\square$

Remark 4.5. In Theorem 4.14 we have shown that the Oscillator algebra of dimension $n+2$ is a 1-dimensional double extension $\mathfrak{os} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R} = \mathfrak{d}_A$. Here, $\mathfrak{g}$ is an abelian metric Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ of dimension n equipped with a Riemannian metric, and $A \in \mathfrak{so}(\mathfrak{g})$, an anti-symmetric non-degenerate derivation of $\mathfrak{g}$, is used in the double extension.

Corollary 4.15. *If we apply Theorem 4.13 to Theorem 4.14 this means that all Oscillator algebras can be equipped with a Lorentzian metric, and that this metric is conformally Einstein. In fact it is conformally Ricci-flat.*

We note here that $\mathrm{tr}\{A^2\} < 0$ for the Oscillator algebra, hence it is not an Einstein manifold but it is conformally Einstein. This result has been seen in the literature previously. For example, Lorentzian groups with bi-invariant metrics are special cases of Lorentzian symmetric spaces, which are special cases of Lorentzian plane waves. These are shown to be conformally Einstein and conformally Ricci-flat using tractor calculus in Proposition 8.1 in Leistner [23, pg. 476], whereas our proof follows the direct computation from Theorem 4.13.

4.4 Bach tensor for double extensions of signature $(2, n-2)$

Kath and Olbrich [20] have presented a classification result and proof concerning indecomposable metric Lie algebras of signature $(2, n-2)$. Baum and Kath [3] present the related structure theorem, showing that indecomposable metric Lie algebras of signature $(2, n-2)$ fall into only three cases. We consider the three cases in Baum and Kath's structure theorem to determine whether each case is Einstein or conformal to Einstein.

The first case they present is a 1-dimensional double extension of an abelian Lorentzian Lie algebra. In Theorem 4.13 we showed this was conformal to Einstein.

The second case is a 1-dimensional double extension of an Oscillator algebra. The third is a 1-dimensional double extension of an Oscillator algebra with an additional abelian component. These two cases are considered below. To consider these cases, we first find the structure of anti-symmetric derivations of the Oscillator algebra, and we use Proposition 4.9 to simplify the different structures up to isometry.

Lemma 4.16. *Anti-symmetric derivations of the Oscillator algebra are of the form*

$$A = \begin{pmatrix} 0 & b & 0 \\ 0 & U & -b^t \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{os} = \mathbb{R}\alpha \oplus \mathfrak{g} \oplus \mathbb{R}H$*. If* A_0 *is the anti-symmetric derivation used, as in Remark 4.5, then* $[A_0, U] = 0$ *and* $U^t = -U$*.*

Proof. As A is a derivation, it is a linear map between vector spaces and hence we write A as matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

with respect to the $\mathfrak{os}$ decomposition. To be antisymmetric, the matrix must satisfy

$$\langle AZ, Y\rangle_{\mathfrak{os}} = -\langle Z, AY\rangle$$

for all $Z, Y \in \mathfrak{os}$ which is equlivalent to

$$A^t g = -gA \text{ where } g = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & I & 0 \\ 1 & \cdots & 0 \end{pmatrix}.$$

We find that

$$\begin{pmatrix} g & d & a \\ h & e & b \\ i & f & c \end{pmatrix} = -\begin{pmatrix} g & h^t & i \\ d^t & e^t & f^t \\ a & b^t & c \end{pmatrix}$$

hence $c = -c = 0 = g = -g$ and $i = -a$, $f = -b^t$, $i = -a$ and $e = -e^t$. We will relable

$$A = \begin{pmatrix} a & b & 0 \\ c & U & -b^t \\ 0 & -c^t & -a \end{pmatrix}.$$

To satisfy the Leibniz rule, we require

$$A[Z, Y]_{\mathfrak{os}} = [AZ, Y]_{\mathfrak{os}} + [Z, AY]_{\mathfrak{os}}$$

for all $Z, Y \in \mathfrak{os}$. As any derivation must send the centre of a Lie algebra to the centre, see Lemma A.36 for details, then $c = 0$. We can also see this directly by considering

$$\left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix}\right]_{\mathfrak{os}} = \begin{pmatrix} \beta(X, \widehat{X}) \\ \pi(H)\widehat{X} - \pi(\widehat{H})X \\ 0 \end{pmatrix} = \begin{pmatrix} X^t A_0^t \widehat{X} \\ HA_0\widehat{X} - \widehat{H}A_0 X \\ 0 \end{pmatrix}.$$

Hence

$$A\left[\begin{pmatrix}\alpha\\X\\H\end{pmatrix},\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}} = \begin{pmatrix}a & b & 0\\c & U & -b^t\\0 & -c^t & -a\end{pmatrix}\begin{pmatrix}X^tA_0^t\widehat{X}\\HA_0\widehat{X}-\widehat{H}A_0X\\0\end{pmatrix}$$
$$= \begin{pmatrix}aX^tA_0^t\widehat{X}+b(HA_0\widehat{X}-\widehat{H}A_0X)\\cX^tA_0^t\widehat{X}+U(HA_0\widehat{X}-\widehat{H}A_0X)\\-c^t(HA_0\widehat{X}-\widehat{H}A_0X)\end{pmatrix}$$

and

$$\left[A\begin{pmatrix}\alpha\\X\\H\end{pmatrix},\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}} + \left[\begin{pmatrix}\alpha\\X\\H\end{pmatrix},A\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}$$
$$= \left[A\begin{pmatrix}a\alpha+bX\\c\alpha+UX-b^tH\\-c^tX-aH\end{pmatrix},\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}} + \left[\begin{pmatrix}\alpha\\X\\H\end{pmatrix},A\begin{pmatrix}a\widehat{\alpha}+b\widehat{X}\\c\widehat{\alpha}+U\widehat{X}-b^t\widehat{H}\\-c^t\widehat{X}-a\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}$$
$$= \begin{pmatrix}(\alpha c^t+X^tU^t-Hb)A_0^t\widehat{X}+X^tA_0^t(\widehat{\alpha}c+U\widehat{X}-\widehat{H}b^t)\\-(c^tX+aH)A_0\widehat{X}-\widehat{H}A_0(c\alpha+UX-b^tH)+HA_0(\widehat{\alpha}c+U\widehat{X}-\widehat{H}b^t)+(c^t\widehat{X}+a\widehat{H})A_0X\\0\end{pmatrix}.$$

Equating the two, we immediately see that $c=0$. We are left with

$$\begin{pmatrix}aX^tA_0^t\widehat{X}+b(HA_0\widehat{X}-\widehat{H}A_0X)\\U(HA_0\widehat{X}-\widehat{H}A_0X)\\0\end{pmatrix}$$
$$= \begin{pmatrix}(X^tU^t-Hb)A_0^t\widehat{X}+X^tA_0^t(U\widehat{X}-\widehat{H}b^t)\\-aHA_0\widehat{X}-\widehat{H}A_0(UX-b^tH)+HA_0(U\widehat{X}-\widehat{H}b^t)+a\widehat{H}A_0X\\0\end{pmatrix}.$$

Cancelling any repeated terms we have

$$\begin{pmatrix}aX^tA_0^t\widehat{X}+b(HA_0\widehat{X}-\widehat{H}A_0X)\\U(HA_0\widehat{X}-\widehat{H}A_0X)\\0\end{pmatrix} = \begin{pmatrix}(X^tU^t-Hb)A_0^t\widehat{X}+X^tA_0^t(U\widehat{X}-\widehat{H}b^t)\\-aHA_0\widehat{X}-\widehat{H}A_0UX+HA_0U\widehat{X}+a\widehat{H}A_0X\\0\end{pmatrix}.$$

If we pick $H=\widehat{H}=0$ then we have

$$\begin{aligned}aX^tA_0^t\widehat{X} =& X^tU^tA_0^t\widehat{X}+X^tA_0^tU\widehat{X}\\ =& X^t(U^tA_0^t+A_0^tU)\widehat{X}\\ \Rightarrow aA_0 =& -UA_0+A_0U. \qquad (4.4.1)\end{aligned}$$

By Lemma A.38, $a=0$ and $A_0U=UA_0$. One can check that A now satisfies both the derivation and anti-symmetry properties required. □

Remark 4.6. The following lemma can be shown using a proof similar to Lemma 4.16. This shows the derivations of double extension of an abelian Lie algebra with metric (p, q). However, the results in the rest of the chapter do not hold in the generality seen here.

Lemma 4.17. *Let $\mathfrak{a}$ be an abelian Lie algebra with scalar product g of signature (p, q). Let $\mathfrak{d}_{A_0}(\mathfrak{a})$ be a double extension by a 1-dimensional Lie algebra, where A_0 is a anti-symmetric derivation of $\mathfrak{a}$. Then the anti-symmetric derivations of $\mathfrak{d}_{A_0}(\mathfrak{a})$ are of the form*

$$A = \begin{pmatrix} a & b & 0 \\ 0 & U & \tilde{b} \\ 0 & 0 & -a \end{pmatrix}$$

such that $b \in \mathbb{R}^n$, $\tilde{b} = b^t I_{p,q}$, and $U \in \mathfrak{so}(p, q)$, $A_0 U - U A_0 = a A_0$.

For the proof, see Remark A.9.

A corollary to Theorem 4.6 is as follows.

Corollary 4.18. *Any 1-dimensional double extension of the Oscillator algebra, $\mathfrak{d}(\mathfrak{os})$, is solvable.*

Proof. If one considers the case of the double extension of the Oscillator algebra in Remark 4.5, we have that the form of ad is

$$\mathrm{ad}\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix} = \begin{pmatrix} 0 & X^t A_0^t & 0 \\ 0 & H A_0 & -A_0 X \\ 0 & 0 & 0 \end{pmatrix}$$

where A_0 is invertible. Note then that as A_0 is invertible, $[\mathfrak{os}, \mathfrak{os}] = \mathfrak{z} \oplus \mathfrak{g}$ where $\mathfrak{z} = \mathbb{R}^*$ is the centre of $\mathfrak{g}$. However, from Lemma 4.16 the derivations of $\mathfrak{os}$ are of the form

$$A = \begin{pmatrix} 0 & b & 0 \\ 0 & U & -b^t \\ 0 & 0 & 0 \end{pmatrix}.$$

The image of A is contained in $\mathfrak{z} \oplus \mathfrak{g}$, hence the image of A is precisely contained in $[\mathfrak{os}, \mathfrak{os}]$ regardless of the choice of U or b. Applying Theorem 4.6 gives the result. □

Remark 4.7. One can also see that $\mathfrak{d}(\mathfrak{os})$ is solvable by directly calculating elements of $[\mathfrak{d}, \mathfrak{d}]^2$ and seeing that these are all 0.

Lemma 4.19. *Let $U = -U^t$ and let $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & b & 0 \\ 0 & U & -b^t \\ 0 & 0 & 0 \end{pmatrix}$ be two anti-symmetric derivations of the Oscillator algebra, $\mathfrak{os}$, with respect to the decomposition $\mathfrak{os} = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$. Then the double extensions of $\mathfrak{os}$, $\mathfrak{d}_1$ and $\mathfrak{d}_2$, defined by A_1 and A_2 respectively, are isometric.*

Proof. By Proposition 4.9, if $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are two 1-dimensional double extensions of a metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ using antisymmetric derivations $A_1, A_2 \in \mathrm{Der}(\mathfrak{g})$ then the double extensions are isometric if and only if there is a $\lambda \in \mathbb{R}\setminus\{0\}$, $X_0 \in \mathfrak{g}$ and $\psi_0 \in \mathfrak{gl}(\mathfrak{g})$ such that $\psi_0^{-1} A_2 \psi_0 = \lambda A_1 + \mathrm{ad}(X_0)$. Consider that

$$\mathrm{ad}_{\mathfrak{os}} \begin{pmatrix} 0 \\ A_0^{-1} b^t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & -b^t \\ 0 & 0 & 0 \end{pmatrix}.$$

Then if we let $\psi_0 = id, \lambda = 1, X_0 = \begin{pmatrix} 0 \\ A_0^{-1} b^t \\ 0 \end{pmatrix}$ we conclude that A_1 and A_2 produce isometric double extensions of $\mathfrak{os}$. $\square$

Proposition 4.20. *If $\mathfrak{os}$ is an Oscillator algebra with Lorentzian metric (as described in Remark 4.5), then the Bach tensor of the double extension $\mathfrak{d}(\mathfrak{os}, \mathbb{R}) = \mathbb{R}\alpha \oplus \mathfrak{os} \oplus \mathbb{R}H$ is identically 0.*

Proof. One can show this using Corollary 4.18, that $\mathfrak{d}$ is solvable, and then applying Proposition 3.12, which states that solvable Lie groups have 2-step nilpotent Ricci tensor and are hence Bach-flat. The direct proof by showing that the Ricci tensor of $\mathfrak{d}$ is 2-step nilpotent follows. We include it as it shows the formula for the Killing form used in Theorem 4.24.

With respect to the decomposition, $\mathfrak{d}(\mathfrak{os}) = \mathbb{R} \oplus \mathfrak{os} \oplus \mathbb{R} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R} \oplus \mathbb{R}$, we use the basis $X_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_i = \begin{pmatrix} 0 \\ 0 \\ e_i \\ 0 \\ 0 \end{pmatrix}$, $X_{n+1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $X_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, where e_i is the standard basis in $\mathbb{R}^n$ and $i = 1, \ldots, n$. In this basis, we have

$$g_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \mathrm{tr}(\mathrm{ad}_{\mathfrak{os}}(\cdot)A) \\ 0 & 0 & 0 & \mathrm{tr}(A_0^2) & \\ 0 & \multicolumn{3}{c}{\mathrm{tr}(\mathrm{ad}_{\mathfrak{os}}(\cdot)A)} & \mathrm{tr}\, A^2 \end{pmatrix}.$$

Note that $\mathrm{tr}\{A^2\} = \mathrm{tr}\{U^2\}$. We have that $\mathrm{ad}_{\mathfrak{os}}(X_0) = 0$,

$$\mathrm{ad}_{\mathfrak{os}}(X_i) = \begin{pmatrix} 0 & g(A_0 X_i, \cdot) & 0 \\ 0 & 0 & -A_0 X_i \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_{\mathfrak{os}}(X_{n+1}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have that $\operatorname{tr}(A \operatorname{ad}_{os} X_0) = 0$,

$$\operatorname{tr}(A \operatorname{ad}_{\mathfrak{os}} X_i) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & g(A_0X_i, \cdot) & 0 \\ 0 & 0 & -A_0X_i \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -UA_0X_i \\ 0 & 0 & 0 \end{pmatrix} = 0$$

and

$$\operatorname{tr}(A \operatorname{ad}_{\mathfrak{os}} X_{n+1}) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & UA_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{tr}(UA_0).$$

This gives that

$$K_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{tr}(A_0^2) & \operatorname{tr}(UA_0) \\ 0 & 0 & 0 & \operatorname{tr}(UA_0) & \operatorname{tr}(A^2) \end{pmatrix}.$$

Finally we have

$$\begin{aligned} \operatorname{Ric}_{\mathfrak{d}}^2 &= \frac{1}{16} K_{\mathfrak{d}} g_{\mathfrak{d}}^{-1} K_{\mathfrak{d}} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{tr}(A_0^2) & \operatorname{tr}(UA_0) \\ 0 & 0 & 0 & \operatorname{tr}(UA_0) & \operatorname{tr}(A^2) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{tr}(A_0^2) & \operatorname{tr}(UA_0) \\ 0 & 0 & 0 & \operatorname{tr}(UA_0) & \operatorname{tr}(A^2) \end{pmatrix} \\ &= 0. \end{aligned}$$

Hence $\mathfrak{d}$ has 2-step nilpotent Ricci tensor and hence has trivial Bach tensor. □

Lemma 4.21. *Consider a Lie algebra, $\mathfrak{g} = \mathbb{R}\beta \oplus \mathfrak{os}$, as an orthogonal decomposition into ideals. Equip this with metric $I \oplus g$ where g is the metric on $\mathfrak{os}$ as described in Remark 4.5 and I is the dot product metric on $\mathbb{R}$. Then the anti-symmetric derivations of $\mathfrak{g}$ are of the form*

$$A = \begin{pmatrix} 0 & 0 & 0 & c \\ -c & 0 & g & 0 \\ 0 & 0 & B & -g^t \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{g} = \mathbb{R} \oplus (\mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R})$, where $B^t = -B$. If A_0 is the anti-symmetric derivation used for $\mathfrak{os}$, as in Remark 4.5, then $[B, A_0] = 0$.

Proof. We follow the proof in Lemma 4.16. If we apply the anti-symmetry condition, we have that $A = \begin{pmatrix} 0 & a & b & c \\ -c & d & g & 0 \\ -b^t & -j^t & B & -g^t \\ -a & 0 & j & -d \end{pmatrix}$. Applying the derivation condition, we immediately find that $a = b = j = 0$. Using Lemma A.38, it can be shown that $d = 0$ and $[B, A_0] = 0$. □

Lemma 4.22. *On the Lie algebra,* $\mathfrak{g} = \mathbb{R}\beta \oplus \mathfrak{os}$*, as defined in Lemma 4.21, the two derivations* $A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 0 & 0 & 0 & c \\ -c & 0 & g & 0 \\ 0 & 0 & cB & -g^t \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *produce isometric 1-dimensional double extensions whenever* $c \neq 0$.

Proof. By Proposition 4.9, if $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are two 1-dimensional double extensions of a metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ using antisymmetric derivations $A_1, A_2 \in \mathrm{Der}(\mathfrak{g})$ then the double extensions are isometric if and only if there is a $\lambda \in \mathbb{R} \setminus \{0\}$, $X_0 \in \mathfrak{g}$ and $\psi_0 \in \mathfrak{gl}(\mathfrak{g})$ such that $\psi_0^{-1} A_2 \psi_0 = \lambda A_1 + \mathrm{ad}(X_0)$. Consider that

$$\mathrm{ad}_{\mathfrak{g}} \begin{pmatrix} 0 \\ 0 \\ A_0^{-1} b^t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b^t \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then if we let $\psi_0 = id, \lambda = c, X_0 = \begin{pmatrix} 0 \\ 0 \\ A_0^{-1} b^t \\ 0 \end{pmatrix}$ we conclude that A_1 and A_2 produce isometric double extensions of $\mathfrak{os}$. □

Henceforth we will double extend $\mathbb{R}\beta \oplus \mathfrak{os}$ with derivations of the form of A_1 only. Note that if $c = 0$, the double extension $\mathbb{R} \oplus (\mathbb{R}\beta \oplus \mathfrak{os}) \oplus \mathbb{R}$ is directly decomposable to $\beta\mathbb{R} \oplus \mathfrak{d}_A(\mathfrak{os})$ where $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and the metric on $\beta\mathbb{R} \oplus \mathfrak{d}_A(\mathfrak{os})$ is $I \oplus g_{\mathfrak{d}_A}$. Here I is the standard dot product on $\mathbb{R}$.

Theorem 4.23. *If* $\mathfrak{g} = \mathbb{R}\beta \oplus \mathfrak{os}$ *is the Lie algebra in Lemma* 4.21*, then any one-dimensional double extension,* $\mathfrak{d}$*, of* $\mathfrak{g}$ *has Bach tensor identically 0.*

Proof. We show that the Ricci tensor of $\mathfrak{d}$ is 2-step nilpotent.

Firstly, consider that we have extended the Oscillator algebra by an abelian 1-dimensional ideal. Hence $\mathrm{Ric}_{\mathfrak{g}}$ and $\mathrm{ad}_{\mathfrak{g}}$ are identically 0 when either component is in $\mathbb{R}\beta$. Hence computing the Killing form of $\mathfrak{d}$ will be the same as the Killing form of Proposition 4.20 with an additional zero row and zero column. As the metric also has a similar form, then the same proof shows that $\mathrm{Ric}^2 = 0$ and the result follows by Proposition 3.11. □

Remark 4.8. As with double extensions of the Oscillator algebra, one can also show that $\mathfrak{d}(\mathbb{R} \oplus \mathfrak{os})$ is solvable by directly calculating elements of $[\mathfrak{d}, \mathfrak{d}]^2$ and seeing that these are all 0. This implies that $[\mathfrak{d}, \mathfrak{d}]$ is nilpotent, which implies $\mathfrak{d}$ must be solvable by Corollary A.25. This also implies that the Ricci tensor is 2-step nilpotent by Proposition 3.12 and hence that the Bach tensor is 0.

Baum and Kath [3] give the structure of all non-simple indecomposable metric Lie algebras with signature $(2, n-2)$. The three cases are included in Proposition 4.20, Theorem 4.23 and Theorem 4.13. They all solvable, all have Bach tensor identically zero, hence they satisfy the second obstruction to be conformal to Einstein. In the following section we show they do not always satisfy the first obstruction, and hence may not be conformal to Einstein.

4.5 First obstruction for double extensions of signature $(2, n-2)$

Theorem 4.24. *If $\mathfrak{os}$ is an Oscillator algebra as described in Remark 4.5, then any indecomposable double extension $\mathfrak{d}(\mathfrak{os})$ by a 1-dimensional Lie algebra does not satisfy the first obstruction and is hence not conformally Einstein.*

Proof. As in Proposition 4.20, with respect to the decomposition, $\mathfrak{d}(\mathfrak{os}) = \mathbb{R} \oplus \mathfrak{os} \oplus \mathbb{R} = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R} \oplus \mathbb{R}$, we use the basis $X_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_i = \begin{pmatrix} 0 \\ 0 \\ e_i \\ 0 \\ 0 \end{pmatrix}$, $X_{n+1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $X_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, where e_i is the standard basis on $\mathfrak{g} = \mathbb{R}^n$ and $i = 1, \ldots, n$. In this basis, we have

$$g_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } K_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{tr}(A_0^2) & \mathrm{tr}(UA_0) \\ 0 & 0 & 0 & \mathrm{tr}(UA_0) & \mathrm{tr}(A^2) \end{pmatrix},$$

where $\mathrm{tr}\{A^2\} = \mathrm{tr}\{U^2\}$.

We have that $\mathrm{ad}_{\mathfrak{d}}(X_-) = 0$, $\mathrm{ad}_{\mathfrak{d}}(X_0) = 0$,

$$\mathrm{ad}_{\mathfrak{d}}(X_i) = \begin{pmatrix} 0 & 0 & -e_i^t U & 0 & 0 \\ 0 & 0 & -e_i^t A_0 & 0 & 0 \\ 0 & 0 & 0 & -A_0 e_i & -Ue_i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_{\mathfrak{d}}(X_{n+1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{ad}_{\mathfrak{os}}(X_+) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Proposition 4.20, the Ricci tensor is 2-step nilpotent, and so by Proposition 3.11 the first obstruction reduces to

$$\Upsilon^a C_{abcd} = -\Upsilon^a R_{cdab} + \frac{\Upsilon^a}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) = 0$$

where $a, b, c, d = -, 0, 1, \ldots, n, n+1, +$. Here, $\Upsilon \in \mathfrak{F}(G)$ where G is a metric Lie group with $\mathfrak{d}$ as its Lie algebra. As the Lie algebra is the set of all left invariant vector fields, for any $\Upsilon \in \mathfrak{F}(G)$ we must have $\operatorname{grad} \Upsilon = f^a X_a$ for X_a a basis of $\mathfrak{d}$ and for some functions $f^a \in \mathfrak{F}(G)$. When evaluating at a point $p \in G$, $\operatorname{grad} \Upsilon|_p = f^a(p) X_a|_p$ is an $\mathbb{R}$-linear combination of Lie algebra elements so must be an element of $\mathfrak{d}$. Define $f^a(p) = \Upsilon^a \in \mathbb{R}$.

We now proceed to show that Υ^a must be zero. As the point was chosen arbitrarily, this means that $f^a = 0$, which implies $\operatorname{grad}(\Upsilon) = 0$. This implies $\Upsilon \in \mathfrak{F}(G)$ is constant on G. However, if it is constant, then $e^{2\Upsilon}\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ must be a positive constant multiple of $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, and $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ must then be Einstein. However, we know $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ is not Einstein as it not Ricci-flat, as shown in Theorem 4.23. Hence $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ cannot be conformally Einstein.

If we consider $d = -$, then $R_{ad} = 0 = R_{bd}$ and, as the Riemann curvature tensor depends on $\operatorname{ad}_{X_-} = 0$ then $R_{cdab} = 0$ and the first obstruction becomes

$$\Upsilon^a C_{abc-} = \frac{\Upsilon^a}{n-2}(g_{-a}R_{bc} - g_{-b}R_{ac}) = 0.$$

If we put $b = +$, then this reduces further to

$$\Upsilon^a C_{a+c-} = \frac{\Upsilon^a}{n-2}(g_{-a}R_{+c} - g_{-+}R_{ac}) = -\frac{\Upsilon^a}{n-2}R_{ac} = \frac{\Upsilon^a}{n-2}R_{ac} = 0$$

and hence

$$\Upsilon^+ R_{c+} + \Upsilon^{n+1} R_{c(n+1)} = 0. \tag{4.5.1}$$

If instead we put $b = n+1$ then

$$\Upsilon^a C_{a(n+1)c-} = \frac{\Upsilon^a}{n-2} g_{-a} R_{(n+1)c} = \frac{\Upsilon^+}{n-2} R_{(n+1)c} = 0.$$

For $c = n+1$ this reduces to $\Upsilon^+ R_{(n+1)(n+1)} = \frac{-\Upsilon^+}{4} \operatorname{tr} A_0^2 = 0$. As $\operatorname{tr} A_0^2 < 0$, then $\Upsilon^+ = 0$. Then by Equation (4.5.1)above, we have $\Upsilon^{n+1} = 0$ also.

Consider now $d = i$ where $i = 1, \ldots, n$. Then $R_{bi} = 0 = R_{ai}$ and the first obstruction reduces to

$$\Upsilon^a C_{abci} = -\Upsilon^a R_{ciab} + \frac{\Upsilon^a}{n-2}(g_{ia}R_{bc} - g_{ib}R_{ac}) = 0.$$

Letting $b = c = +$, this reduces further to

$$\Upsilon^a R_{+ia+} = \frac{\Upsilon^i}{n-2} R_{++} = \frac{-\Upsilon^i}{(n-2)} \operatorname{tr}(U^2)$$

Now $\Upsilon^a R_{+ia+} = -\frac{1}{4} g([X_+, X_i], [\nabla\Upsilon, X_+]) = -\frac{1}{4} e_i U^2 \nabla\Upsilon$ for $i = 1, \ldots, n$. Hence we require that Υ satisfy

$$U^2 \nabla \Upsilon = -\operatorname{tr}(U^2) \nabla \Upsilon$$

where we take only the $i = 1, \ldots, n$ components of $\nabla\Upsilon$. This implies that $\nabla\Upsilon$ is an eigenvector of U^2 with eigenvalue $-\operatorname{tr}(U^2)$. As $U \in \mathfrak{so}(n)$, then $U^2 = -UU^t$ is non-positive definite, with all zero eigenvalues if and only if $U = 0$. As $U \neq 0$, then $-\operatorname{tr}(U^2) > 0$, however then this cannot be an eigenvalue of U^2. Hence the only solution is $\nabla\Upsilon = 0$ and all components $\Upsilon^i = 0$ for $i = 1, \ldots, n$.

We have that $\nabla\Upsilon \in \operatorname{span}\{X_-, X_0\}$. This reduces the obstruction to

$$\Upsilon^a C_{abcd} = \frac{\Upsilon^-}{n-2}(-g_{c-}R_{bd} + g_{d-}R_{bc}) + \frac{\Upsilon^0}{n-2}(-g_{c0}R_{bd} + g_{d0}R_{bc}).$$

As Weyl tensor is skew in c, d, we pick $c = +, d = n+1$ to get the only non-zero equations from the obstruction as follows

$$\Upsilon^a C_{ab+(n+1)} = \frac{\Upsilon^-}{n-2}R_{b(n+1)} + \frac{\Upsilon^0}{n-2}R_{b+}.$$

Hence we need that $\begin{pmatrix}\Upsilon^- & \Upsilon^0\end{pmatrix}^t$ is in the kernel of the matrix

$$\begin{pmatrix} -\operatorname{tr}(UA_0) & \operatorname{tr}(U^2) \\ -\operatorname{tr}(A_0^2) & \operatorname{tr}(UA_0) \end{pmatrix}.$$

This matrix has non-zero kernel vectors if and only if $\operatorname{tr}(UA_0)^2 = \operatorname{tr}(U^2)\operatorname{tr}(A_0)^2$. Note that the bilinear form $B(A,B) = \operatorname{tr}(AB)$, $B : \operatorname{span}\{U, A_0\} \to \mathbb{R}$, in the basis $\{U, A_0\}$ has matrix representation

$$\begin{pmatrix} \operatorname{tr}(U^2) & \operatorname{tr}(UA_0) \\ \operatorname{tr}(UA_0) & \operatorname{tr}(A_0^2) \end{pmatrix}.$$

This is clearly non-degenerate provided U, A_0 are linearly independent, so no non-zero kernel vectors exist. However, if U, A_0 are instead linearly dependent, then the double extension would be decomposable. Hence $\Upsilon^a = 0$ for all a. This completes the proof. □

Theorem 4.25. *As in Lemma 4.21, consider a Lie algebra, $\mathfrak{g} = \mathbb{R}\beta \oplus \mathfrak{os}$, as an orthogonal decomposition into ideals. Equip this with metric $I \oplus g$ where g is the metric on $\mathfrak{os}$ as described in Remark 4.5 and I is the dot product on $\mathbb{R}$. Then any indecomposable 1-dimensional double extension of $\mathfrak{g}$ is Bach-flat but not conformally Einstein.*

Proof. From Theorem 4.23, we know the double extension is Bach-flat. This proof closely follows the proof of Theorem 4.24, however we additionally show that any possible candidate for a solution to the conformal to Einstein equation, Equation (1.5.4), must be zero in the $\mathbb{R}\beta$ component. We leave the proof here for clarity.

With respect to the decomposition, $\mathfrak{d}(\mathfrak{os}) = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R} = \mathbb{R} \oplus (\mathbb{R}\beta \oplus \mathbb{R} \oplus \mathfrak{h} \oplus \mathbb{R}) \oplus \mathbb{R}$, we use the basis $X_- = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$, $X_{-1} = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$, $X_0 = \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$ $X_i = \begin{pmatrix}0\\0\\0\\e_i\\0\\0\end{pmatrix}$, $X_{n+1} = \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$ and $X_+ = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$, where e_i is the standard basis on $\mathfrak{h} = \mathbb{R}^n$ and $i = 1, \ldots, n$. In this basis, we have

$$g_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } K_{\mathfrak{d}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{tr}(A_0^2) & \mathrm{tr}(UA_0) \\ 0 & 0 & 0 & 0 & \mathrm{tr}(UA_0) & \mathrm{tr}(U^2) \end{pmatrix}$$

where I is the n-dimensional identity matrix.

We have that $\mathrm{ad}_{\mathfrak{d}}(X_-) = 0$, $\mathrm{ad}_{\mathfrak{d}}(X_0) = 0$,

$$\mathrm{ad}_{\mathfrak{d}}(X_{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{\mathfrak{d}}(X_i) = \begin{pmatrix} 0 & 0 & 0 & -e_i^t U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_i^t A_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_0 e_i & -U e_i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{\mathfrak{d}}(X_{n+1}) = \begin{pmatrix} 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{ad}_{\mathfrak{os}}(X_+) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Theorem 4.23, the Ricci tensor is 2-step nilpotent, and so by Proposition 3.11 the first obstruction reduces to

$$\Upsilon^a C_{abcd} = -\Upsilon^a R_{cdab} + \frac{\Upsilon^a}{n-2}(g_{cb}R_{ad} - g_{ca}R_{bd} + g_{da}R_{bc} - g_{db}R_{ac}) = 0$$

where $a, b, c, d = -, -1, 0, 1, \ldots, n, n+1, +$. Here, $\Upsilon \in \mathfrak{F}(G)$ where G is a metric Lie group with $\mathfrak{d}$ as its Lie algebra. As the Lie algebra is the set of all left invariant vector fields, for any $\Upsilon \in \mathfrak{F}(G)$ we must have $\mathrm{grad}\,\Upsilon = f^a X_a$ for X_a a basis for $\mathfrak{d}$ and for some functions $f^a \in \mathfrak{F}(G)$. When evaluating at a point $p \in G$, $\mathrm{grad}\,\Upsilon|_p = f^a(p) X_a|_p$ is an $\mathbb{R}$-linear combination of Lie algebra elements so must be an element of $\mathfrak{d}$. Define $f^a(p) = \Upsilon^a \in \mathbb{R}$.

We now proceed to show that Υ^a must be zero. As the point was chosen arbitrarily, this means that $f^a = 0$, which implies $\mathrm{grad}(\Upsilon) = 0$. This implies $\Upsilon \in \mathfrak{F}(G)$ is constant on G. However, if it is constant, then $e^{2\Upsilon}\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ must be a positive constant multiple of $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, and $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ must then be Einstein. However, we know $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ is not Einstein as it not Ricci-flat, as shown in Theorem 4.23. Hence $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ cannot be conformally Einstein.

If we consider $d = -$, then $R_{ad} = 0 = R_{bd}$ and, as the Riemann curvature tensor depends on $\mathrm{ad}_{X_-} = 0$ then $R_{cdab} = 0$ and the first obstruction becomes

$$\Upsilon^a C_{abc-} = \frac{\Upsilon^a}{n-2}(g_{-a}R_{bc} - g_{-b}R_{ac}) = 0.$$

If we put $b = +$, then this reduces further to

$$\Upsilon^a C_{a+c-} = \frac{\Upsilon^a}{n-2}(g_{-a}R_{+c} - g_{-+}R_{ac}) = -\frac{\Upsilon^a}{n-2}R_{ac} = \frac{\Upsilon^a}{n-2}R_{ac} = 0$$

and hence

$$\Upsilon^+ R_{c+} + \Upsilon^{n+1}R_{c(n+1)} = 0. \tag{4.5.2}$$

If instead we put $b = n+1$ then

$$\Upsilon^a C_{a(n+1)c-} = \frac{\Upsilon^a}{n-2}g_{-a}R_{(n+1)c} = \frac{\Upsilon^+}{n-2}R_{(n+1)c} = 0.$$

For $c = n+1$ this reduces to $\Upsilon^+ R_{(n+1)(n+1)} = \frac{-\Upsilon^+}{4}\operatorname{tr} A_0^2 = 0$. As $\operatorname{tr} A_0^2 < 0$, then $\Upsilon^+ = 0$. Then by Equation (4.5.2) above, we have $\Upsilon^{n+1} = 0$ also.

Consider now $d = i$ where $i = 1, \dots, n$. Then $R_{bi} = 0 = R_{ai}$ and the first obstruction reduces to

$$\Upsilon^a C_{abci} = -\Upsilon^a R_{ciab} + \frac{\Upsilon^a}{n-2}(g_{ia}R_{bc} - g_{ib}R_{ac}) = 0.$$

Letting $b = c = +$, this reduces further to

$$\Upsilon^a R_{+ia+} = \frac{\Upsilon^i}{n-2}R_{++} = \frac{-\Upsilon^i}{(n-2)}\operatorname{tr}(U^2).$$

Now $\Upsilon^a R_{+ia+} = -\frac{1}{4}g([X_+, X_i], [\nabla\Upsilon, X_+]) = -\frac{1}{4}e_i U^2 \nabla\Upsilon$ for $i = 1, \dots, n$. Hence we require that Υ satisfy

$$U^2\nabla\Upsilon = -\operatorname{tr}(U^2)\nabla\Upsilon$$

where we take only the $i = 1, \dots, n$ components of $\nabla\Upsilon$. This implies that $\nabla\Upsilon$ is an eigenvector of U^2 with eigenvalue $-\operatorname{tr}(U^2)$. As $U \in \mathfrak{so}(n)$, then $U^2 = -UU^t$ is non-positive definite, with all zero eigenvalues if and only if $U = 0$. As $U \neq 0$, then $-\operatorname{tr}(U^2) > 0$, however then this cannot be an eigenvalue of U^2. Hence the only solution is $\nabla\Upsilon = 0$ and all components $\Upsilon^i = 0$ for $i = 1, \dots, n$.

We have that $\nabla\Upsilon \in \operatorname{span}\{X_-, X_{-1}, X_0\}$. This reduces the obstruction to

$$\begin{aligned}\Upsilon^a C_{abcd} =& \frac{\Upsilon^-}{n-2}(-g_{c-}R_{bd} + g_{d-}R_{bc}) + \frac{\Upsilon^{(-1)}}{n-2}(-g_{c-1}R_{bd} + g_{d-1}R_{bc}) \\ &- \Upsilon^{(-1)}R_{cd(-1)b} + \frac{\Upsilon^0}{n-2}(-g_{c0}R_{bd} + g_{d0}R_{bc}) = 0.\end{aligned}$$

Picking $c = -1$, we have

$$\Upsilon^{(-1)}R_{(-1)d(-1)b} = \frac{-\Upsilon^{(-1)}}{n-2}R_{bd}.$$

Now picking $d = b = +$ then $R_{(-1)d(-1)b} = \frac{1}{4}g([X_{-1}, X_+], [X_{-1}, X_+]) = g(X_0, X_0) = 0$ however $R_{++} = -\frac{1}{4}\operatorname{tr}(U^2) \neq 0$ so the only solution is $\Upsilon^{(-1)} = 0$.

Finally, we have that $\nabla\Upsilon \in \operatorname{span}\{X_-, X_0\}$. This reduces the obstruction to

$$\Upsilon^a C_{abcd} = \frac{\Upsilon^-}{n-2}(-g_{c-}R_{bd} + g_{d-}R_{bc}) + \frac{\Upsilon^0}{n-2}(-g_{c0}R_{bd} + g_{d0}R_{bc}).$$

As Weyl tensor is skew in c, d, we pick $c = +, d = n+1$ to get the only non-zero equations from the obstruction as follows

$$\Upsilon^a C_{ab+(n+1)} = \frac{\Upsilon^-}{n-2}R_{b(n+1)} + \frac{\Upsilon^0}{n-2}R_{b+}.$$

Hence we need that $\begin{pmatrix}\Upsilon^- & \Upsilon^0\end{pmatrix}^t$ is in the kernel of the matrix

$$\begin{pmatrix} -\operatorname{tr}(UA_0) & \operatorname{tr}(U^2) \\ -\operatorname{tr}(A_0^2) & \operatorname{tr}(UA_0) \end{pmatrix}.$$

This matrix has non-zero kernel vectors if and only if $\operatorname{tr}(UA_0)^2 = \operatorname{tr}(U^2)\operatorname{tr}(A_0)^2$. Note that the bilinear form $B(A, B) = \operatorname{tr}(AB)$, $B : \operatorname{span}\{U, A_0\} \to \mathbb{R}$, in the basis $\{U, A_0\}$ has matrix representation

$$\begin{pmatrix} \operatorname{tr}(U^2) & \operatorname{tr}(UA_0) \\ \operatorname{tr}(UA_0) & \operatorname{tr}(A_0^2) \end{pmatrix}.$$

This is clearly non-degenerate provided U, A_0 are linearly independent, so no non-zero kernel vectors exist. However, if U, A_0 are instead linearly dependent, then the double extension would be decomposable. Hence $\Upsilon^a = 0$ for all a. This completes the proof. $\square$

4.6 Conclusion

In this chapter, we have described the double extensions of Medina [25] and Medina and Revoy [26] and shown further geometric results. Medina and Revoy [26] presented a theorem describing indecomposable metric Lie algebras as either simple, 1-dimensional or extensions of metric Lie algebras by 1-dimensional or simple Lie algebras. We first showed that double extensions by simple Lie algebras cannot be Ricci-flat, hence are not Einstein.

We then showed that all nilpotent Lie algebras are 1-dimensional double extensions of nilpotent metric Lie algebras using nilpotent derivations. We showed that solvable metric Lie algebras must be 1-dimensional double extensions, and that it must double extend a solvable Lie algebra with further conditions on this Lie algebra or on the derivation used. Using results from Chapter 3, we concluded that these solvable and nilpotent metric Lie algebras are Bach-flat and that the first obstruction to having a conformally Einstein metric is simplified.

We showed that the Lorentzian signature double extensions were always an Oscillator algebra, which is a 1-dimensional double extension of an abelian Riemannian metric Lie algebra. These were shown to be conformally Einstein, and that the conformally changed metric is Ricci-flat. We also showed in general that any 1-dimensional double extension of an abelian metric Lie algebra, which are solvable, are conformally Einstein and that the conformally changed metrics are Ricci-flat, regardless of the signature of the metric.

We considered two further classes of examples with metric signature $(2, n-2)$. Both classes are solvable, shown to have 2-step nilpotent Ricci tensor and are hence Bach-flat. However, neither class is conformally Einstein.

Conclusion and future research

This thesis has investigated bi-invariant metrics on Lie groups and considered their geometric properties. It begins with an introduction to semi-Riemannian geometry and outlines basic definitions of metrics and curvature. It defines important tensors that are used to show geometric conditions on a metric to be Einstein or conformally Einstein. It concludes by showing necessary algebraic conditions on a metric to be conformally Einstein, which can also be found in Gover and Nurowski [13].

The following chapter shows considers the algebraic properties of bi-invariant metrics on connected Lie groups, which we call metric Lie groups. It connects bi-invariant metrics on Lie groups to symmetric, non-degenerate, ad-invariant bilinear forms on the Lie algebra, which allow algebraic techniques on the Lie algebra to be used to classify the metrics on the Lie group. We define Lie algebras with such a bilinear form to be metric Lie algebras. In this chapter, we show that reductive metric Lie algebras can be orthogonally decomposed into simple and abelian metric Lie algebras. We describe bi-invariant metrics on abelian Lie algebras, and then show that all simple Lie groups can be equipped with a bi-invariant metric induced by the Killing form. We also show cases of simple real Lie algebras that may have a bi-invariant metric from a two-dimensional space of bilinear forms. We show that this space is spanned by the real and imaginary parts of the Killing form of a complex Lie algebra and that this occurs if and only if the Lie algebra is of complex type. In this case, the metric has signature $(\frac{n}{2}, \frac{n}{2})$. We prove that there are no other possible bi-invariant metrics on simple connected Lie groups. We also show that if the metric is Riemannian that the Lie algebra must then be reductive, hence we classified the Riemannian bi-invariant metrics.

The third chapter uses the tensors and algebraic conditions from the first chapter to consider geometric properties bi-invariant metrics on Lie groups, particularly the metrics discussed in the second chapter. We first simplify the algebraic conditions in the case where the metric is bi-invariant, deducing that one of them is that the metric must be at least Bach-flat for the metric to be conformally Einstein. We then show that the Killing form is an Einstein metric, hence the well known fact that Riemannian metric Lie groups are products of Einstein manifolds. When the simple Lie algebra is of complex type, we show that both the real and imaginary part of the Killing form described above are both Bach-flat metrics. However, we show that linear combinations of these metrics are not Bach-flat and hence not conformally Einstein. We also show that when the metric is purely a multiple of the imaginary part of the Lie algebra, while it is Bach-flat, we also show that it is not conformally Einstein as it does not satisfy another algebraic condition from the first chapter. We also show that Lie algebras with two-step nilpotent Ricci tensor are Bach-flat, and that solvable Lie algebras always have two-step nilpotent Ricci tensor.

The final chapter explores metric Lie groups that are not reductive. To do this, we use a double extension procedure developed by Medina [25] and Medina and Revoy [26] and their important theorem that all indecomposable metric Lie algebras are either simple, 1-dimensional or double extensions of simple and 1-dimensional Lie algebras. We show that a double extension by a simple Lie algebra cannot be Einstein, however that 1-dimensional double extensions may be Einstein. We show that all indecomposable Lorentzian metrics Lie algebras are precisely the indecomposable 1-dimensional double extensions of Riemannian abelian metric Lie algebras and that these are solvable and isomorphic to Oscillator algebras. We then show that all 1-dimensional double extensions of abelian metric Lie algebras are conformally Einstein and that the conformally transformed metrics are Ricci-flat. We finally consider two classes of metric Lie algebra of with metrics of signature $(2, n-2)$, which are solvable, and show that their metric is Bach-flat but not conformally Einstein.

We hope that by describing metrics that are Bach-flat but not conformally Einstein that we have given some insight into open problems posed in Peterson [32]. We also note that while many aspects of Lie groups and Lie algebras are well known and completely classified, that the geometric behaviour of their bi-invariant metrics may not be well behaved, that is, not Einstein, nor conformally Einstein, nor even Bach-flat. The author found it fascinating that such behaviour of the metric is present on bi-invariant metrics of simple Lie groups, particularly as simple Lie groups have such rich and well known algebraic structure.

Future research

While we showed exact requirements for a double extension to be nilpotent, we only found some conditions under which a double extension is solvable. It may be useful to also find exact requirements here. For instance, we showed that 1-dimensional double extensions of 1-dimensional double extensions of abelian Lie algebras equipped with Riemannian metrics are solvable, but it may be the case that equipping them with non-Riemannian metrics that they are not solvable. This would require considering the general form of derivations on 1-dimensional double extensions of abelian Lie algebras, which we calculated in Lemma 4.17. These were of the form

$$A = \begin{pmatrix} a & b & 0 \\ 0 & U & \tilde{b} \\ 0 & 0 & -a \end{pmatrix}$$

such that $b \in \mathbb{R}^n$, $\tilde{b} = b^t I_{p,q}$, and $U \in \mathfrak{so}(p, q)$. The important thing we showed in the Riemannian case was that $a = 0$ and hence $\mathrm{Im}(A) \subset [\mathfrak{g}, \mathfrak{g}]$ which satisfied one of our solvability conditions, however we have not shown that $a = 0$ in the general case. This would be interesting in general because if a metric Lie algebra is not solvable, it may not be Bach-flat, and this would mean it is not conformally Einstein.

Further examples of non Bach-flat bi-invariant metrics may be found when considering double extensions by simple Lie algebras. We showed that these cannot be solvable in Chapter 4. In this case, one can show that the square of the Ricci tensor is dependent upon the Killing form of the simple Lie algebra, and as this is non-degenerate on a simple Lie algebra, it will not be zero. This may mean that the Ricci tensor is not two-step nilpotent, and may also show that the Bach tensor is not flat. However, further investigation is necessary.

A different attack could be to consider further results in Baum and Kath [3]. They have included a classification of all indecomposable simple connected metric Lie groups up to dimension 6, which could be used to consider metrics of type $(3, n-3)$ or $(\frac{n}{2}, \frac{n}{2})$ in general and their Einstein and conformally Einstein properties. Baum and Kath [3] also have a section on solvable metric Lie algebras with maximal isotropic centre. These are constructed using repeated 1-dimensional double extensions of abelian Riemannian metric Lie algebra. As they are solvable, they are Bach-flat, so the first obstruction would also need to be satisfied for them to be conformally Einstein. From our results, we would postulate that this first obstruction would not be satisfied beyond the second double extension, but would need further reasoning to show this is true. One may also postulate this more generally: that double extensions of metric Lie algebras that are not conformally Einstein are also not conformally Einstein. This may be easy to show if the metric fails one of the obstructions, for instance if the metric is not Bach-flat, however more research is necessary.

Appendix A

APPENDICES

A.1 Vector fields and tensors

We can consider vector fields in different ways, including as derivations on $\mathfrak{F}(M)$. In this case, if $V \in \mathfrak{X}(M)$ is a vector field then it is a map $V\colon \mathfrak{F}(M) \to \mathfrak{F}(M)$ with the properties:

(1) it is $\mathbb{R}$-linear: $V(af + bg) = aV(f) + bV(g)$ for all $a, b \in \mathbb{R}$ and $f, g \in \mathfrak{F}(M)$.

(2) it has the Leibnizian property: $V(fg) = V(f)g + fV(g)$ for all $f, g \in \mathfrak{F}(M)$.

The bracket operation $[\cdot, \cdot]$ takes two vector fields and defines another vector field by the rule

$$[V, W] = VW - WV \qquad V, W, \in \mathfrak{X}(M). \tag{A.1.1}$$

The multiplication of vector fields is actually composition of the vector fields considered as derivations. Each $[V, W]$ is a function from $\mathfrak{F}(M)$ to $\mathfrak{F}(M)$:

$$f \in \mathfrak{F}(M) \text{ then } [V, W](f) = V(Wf) - W(Vf).$$

This bracket is $\mathbb{R}$-bilinear, anti-symmetric and satisfies the Jacobi Identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Consider how an inner product, for example the dot product, defines angles and distances in flat Euclidean space. To translate this idea to a manifold, we need a notion of an inner product. However, the manifold is not a vector space. So we consider the tangent spaces of the manifold and define an inner product on these instead. To do this, we use the notion of a metric tensor, which is defined in the main text. Here we give the definition of a tensor. Let $\Omega(M)$ be the set of smooth 1-forms on a manifold M.

Definition A.1. A function $A : (J^*)^r \times J^s \to \mathbb{K}$ that is $\mathbb{K}$-multilinear is called a *tensor of type* (r, s) over J, where J is a module over a field $\mathbb{K}$. We say a type $(0, 0)$ tensor is just an element of $\mathbb{K}$.
We denote $\mathfrak{T}^r_s(J)$ the set of all (r, s) tensors over J, which is again a module over $\mathbb{K}$.

Here J^* is the algebraic dual space of J.

We are interested in the tensors of T_pM and tensor fields, which are the sections of this. In this case, we abuse the notation and instead of $\mathfrak{T}^r_s(\mathfrak{X}(M))$ we denote the set $\mathfrak{T}^r_s(M)$ as the set of all tensors over the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$, which are also known as the set of all *tensor fields* over M. That is, if $A \in \mathfrak{T}^r_s(M)$, then A is called a tensor field and $A : \Omega(M)^r \times \mathfrak{X}(M)^s \to \mathfrak{F}(M)$. Another way of looking at A is as a smooth map from points in $p \in M$ to the tensor A_p over the $\mathbb{R}$ module T_pM. That is, $A_p : (T^*_pM)^r \times (T_pM)^s \to \mathbb{R}$ is a tensor of type (r, s) over T_pM.

We can consider any vector field, X, as a tensor field of type $(1, 0)$ (via the map $X(\phi) = \phi(X)$ where ϕ is any 1-form). Similarly, any 1-form is a $(0, 1)$ tensor.

In a coordinate neighbourhood, given any local basis of $\{dx^1, \dots, dx^n\}$ of $\Omega(M)$ and local basis $\{\partial_1, \dots, \partial_n\}$ of $\mathfrak{X}(M)$, then we can write $A = A^{i_1 \dots i_r}{}_{j_1,\dots,j_s} dx^{i_1} \otimes \dots \otimes dx^{i_r} \otimes \partial_{j_1} \otimes \dots \otimes \partial_{j_s}$. In this thesis, we will often use only the coefficients $A^{i_1 \dots i_r}{}_{j_1,\dots,j_s}$ to represent the tensor in calculations.

A.2 Contractions

There is a lot of information contained in the Riemann curvature tensor. Often it is easier to restrict our attention to a smaller set of information. Contractions of a tensor (also called traces) remove some information from the tensor. The general idea is to shrink an (r, s) tensor to $(r-1, s-1)$ tensor. In the main text, this notation is not used, however it is important in the definition of a covariant derivation and generalises the idea of a trace.

Definition A.2. The function $\boldsymbol{C} : \mathfrak{T}^1_1(M) \to \mathfrak{F}(M)$ called a (1,1) *contraction*, is the unique $\mathfrak{F}(M)$-linear function such that $\boldsymbol{C}(X \otimes \theta) = \theta(X)$, $\forall X \in \mathfrak{X}(M)$, $\forall \theta \in \Lambda(M)$.

For any (1,1) tensor A, $\boldsymbol{C}$ has the form

$$\boldsymbol{C}(A) = \boldsymbol{C}(A^i{}_j \partial_i \otimes dx^j) = A^i{}_j dx^j(\partial_i) = A^i{}_i.$$

One can show this is independent of coordinates.

If we consider any $A \in \mathfrak{T}^r_s$ and i, j any integers between r and s respectively. For any $\theta_1, \dots, \theta_{r-1} \in \Lambda(M)$ and $X^1, \dots, X^{s-1} \in \mathfrak{X}(M)$ then the tensor $B^i{}_j$ defined by

$$B^i_j(\phi, Y) = A(\theta_1, \dots, \theta_{i-1}, \phi, \theta^i, \dots, \theta^{r-1}, X^1, \dots, X^{j-1}, Y, X^j, \dots, X^{s-1})$$

is a (1,1) tensor. We define the $(r-1, s-1)$ tensor $\boldsymbol{C}^i_j A(\theta_1, \dots, \theta_{r-1}, X^1, \dots, X^{s-1}) := \boldsymbol{C}(B^i_j)$, as the *contraction of A over i, j.*

We may interpret (1,1) tensors as functions smoothly assigning to each point $p \in M$ a linear transformation on the tangent space at p, T_pM. We can then write the contraction as the trace at each point as follows.

If $A \in \mathfrak{T}^1_1(M)$ then $A = A^i{}_j \partial_i \otimes dx^j$. At any point $p \in M$, $\partial_i\big|_p$ forms a basis for the tangent space T_pM. Then, we can consider A at p and apply this to a single vector field only, say $\partial_i\big|_p$ by

$$A\big|_p(\partial_k\big|_p) = A^i_j\big|_p \partial_i\big|_p \otimes dx^j\big|_p(\partial_k\big|_p) = A^i_k\big|_p \partial_i\big|_p \in T_pM.$$

This is now a linear function from T_pM to T_pM, and can be represented by the matrix $[A^i{}_k|_p]$. The trace is $A^i{}_i|_p$ which is the effect of contracting A and evaluating at p.

Any change of basis does not change the eigenvalues of a linear transformation, hence the trace is invariant under change of basis. Thus contractions are independent of coordinates.

A related concept to contractions is that of type changing. This is the idea of raising or lowering the indices. Our metric, g, creates an isomorphism between the set of smooth vector fields, $\mathfrak{X}(M)$, and the set of smooth 1-forms, $\Omega(M)$, as we see in the following proposition.

Proposition A.1. *Let M be a semi-Riemannian manifold. If $V \in \mathfrak{X}(M)$, then define $V^\flat$ the 1-form on M such that*

$$V^\flat(X) = g(V, X) \text{ for all } X \in \mathfrak{X}(M).$$

Then the function $V \mapsto V^\flat$ is a $\mathfrak{F}(M)$-linear isomorphism from $\mathfrak{X}(M)$ to $\Omega(M)$.

Proof: Sketch. Following O'Neill [29, pg. 60], the proof uses the facts as follows:

- The non-degeneracy of the metric tensor implies that $g(V, X) = g(W, X)\ \forall X \in \mathfrak{X}(M)$ if and only if $V = W$.
- For all $\theta \in \Lambda(M)$ there is a unique $V \in \mathfrak{X}(M)$ such that $\theta(X) = g(V, X)$ for all $X \in \mathfrak{X}(M)$. The uniqueness follows from the first point. Existence follows from considering in a small neighbourhood, U, that if $\theta = \sum \theta_i dx^i$ on U then define $V = g^{ij}\theta_i\partial_j$. If we consider $g(V, \partial_k)$ and the fact that (g_{ij}) and (g^{ij}) are inverses, the result follows.

□

Now, let A be any element of $\mathfrak{T}^r_s$. We can change the type of A using this isomorphism. Assume we change A to be an element of $\mathfrak{T}^{r-1}_{s+1}$. Then

$$\begin{aligned} &A(\theta_{i_1}, \ldots, \theta_{i_{a-1}}, X^{i_a}, \theta_{i_{a+1}}, \ldots, \theta_{i_{r-1}}, X^{j_1}, \ldots, X^{j_s}) \\ &\quad = g(A(\theta_{i_1}, \ldots, \theta_{i_{a-1}}, \cdot, \theta_{i_a}, \ldots, \theta_{i_r}, X^1, \ldots, X^{j_s}), X^{i_a}). \end{aligned}$$

Here we consider $A(\theta_{i_1}, \ldots, \theta_{i_{a-1}}, \cdot, \theta_{i_a}, \ldots, \theta_{i_r}, X^1, \ldots, X^{j_s})$ as an element of $\mathfrak{T}^1_0$. This means it can be considered as a vector field. Hence the metric can be applied to it. In indices, this is

$$A^{i_1\ldots i_{a-1}}{}_{i_a}{}^{i_{a+1}\ldots i_r}{}_{j_1\ldots j_s} = g_{i_a c} A^{i_1\ldots i_{a-1} c i_{a+1}\ldots i_r}{}_{j_1\ldots j_s}.$$

Type changing A to an element of $\mathfrak{T}^{r+1}_s$ uses the reverse isomorphism. In indices, this is

$$A^{i_1\ldots i_r}{}_{j_1\ldots j_{b-1}}{}^{j_b}{}_{j_{b+1}\ldots j_s} = g^{j_a c} A^{i_1\ldots i_r}{}_{j_1\ldots j_{b-1} c j_{b+1}\ldots j_s}$$

Combining a type change with a contraction is called a *metric contraction*, and often written $\boldsymbol{C}_{ab}$ or $\boldsymbol{C}^{ab}$. These are often called traces over entries a, b. For further details see O'Neill [29, pg. 83].

A.3 Tensor derivations

This section extends the idea of a derivation to tensors and includes a way of extending the Levi-Civita connection to tensor fields.

Definition A.3. A tensor derivation on a smooth manifold M is a set of $\mathbb{R}$-linear functions

$$\mathscr{D} := \mathscr{D}^r_s : \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_s(M) \qquad (r, s \geq 0)$$

such that for any tensors A and B

1. $\mathscr{D}(A \otimes B) = \mathscr{D}A \otimes B + A \otimes \mathscr{D}B$
2. $\mathscr{D}(\boldsymbol{C}A) = \boldsymbol{C}(\mathscr{D}A)$ for any contraction $\boldsymbol{C}$.

Note that if $r = s = 0$ then $\mathscr{D}^0_0$ is a derivation, hence represented by a smooth vector field, V where $\mathscr{D}f = Vf$ for all $f \in \mathfrak{F}(M)$.

In O'Neill [29, pg. 44], it shows the following product rule for a tensor $A \in \mathfrak{T}^r_s$.

$$\begin{aligned}\mathscr{D}(A(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)) =& (\mathscr{D}A)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) \\ &+ \sum A(\theta^1, \ldots, \mathscr{D}\theta^i, \ldots, \theta^r, X_1, \ldots, X_s) \\ &+ \sum A(\theta^1, \ldots, \theta^r, X_1, \ldots, \mathscr{D}X_i, \ldots, X_s)\end{aligned} \tag{A.3.1}$$

Note that then $\mathscr{D}A$ is solely determined by what it does to functions, 1-forms and vector fields. However,

$$(\mathscr{D}\theta)(X) = \mathscr{D}(\theta X) - \theta(\mathscr{D}X) \tag{A.3.2}$$

for a 1-form θ, hence functions and vector fields suffice.

To generalise the Levi-Civita connection and the differential d we define the following:

Definition A.4. $V \in \mathfrak{X}(M)$. The *Levi-Civita covariant derivative* ∇_V is the unique tensor derivation on M such that

$$\nabla_V f = Vf \qquad f \in \mathfrak{F}(M)$$

and $\nabla_V W$ is the Levi-Civita connection for all $W \in \mathfrak{X}(M)$.

As mentioned in Remark 1.1, we define the covariant derivative of a tensor as follows.

Definition A.5. The *covariant differential* of an (r, s) tensor A on M is the $(r, s+1)$ tensor ∇A such that

$$(\nabla A)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, V) = (\nabla_V A)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)$$

Here $\nabla_V A$ is the Levi-Civita covariant derivative. When $r = s = 0$ then this is just the differential operator d, where $(\nabla f)V = \nabla_V f = Vf = df(V)$ for all functions $f \in \mathfrak{F}(M)$ and vector fields $V \in \mathfrak{X}(M)$. See O'Neill [29, pg. 64] for further details.

A.4 Geodesics, the exponential map and normal coordinate systems

For this section, assume I is an interval in $\mathbb{R}$.

Definition A.6. A tensor field is *parallel* provided its covariant differential is 0, i.e. $D_V A = 0\ \forall A \in \mathfrak{X}(M)$.

A vector field is a smooth section of TM; it is a function $X : M \to TM$ such that $\pi \circ X = id$. This next definiton extends this.

Definition A.7. A *vector field on a smooth map* $\phi : P \to M$ is a mapping $Z : P \to TM$ such that $\pi \circ Z = \phi$, where π is the projection $\pi : TM \to M$. Then we define $\mathfrak{X}(\phi)$ as the set of all smooth vector fields on ϕ.

Note that for any $p \in P$, then $Z(p) \in T_{\phi(p)}M$. An example of this is on the curve $\gamma : I \to M$, we have that γ' is a vector field on γ.

Now want to extend the definition of covariant derivative by defining a vector rate of change, Z', of a vector field $Z \in \mathfrak{X}(\gamma)$.

Proposition A.2. *Let M be a semi-Riemannian manifold and $\gamma : I \to M$ a curve on M. There is a unique map from $\mathfrak{X}(\gamma)$ to $\mathfrak{X}(\gamma)$ called the* induced covariant derivative*, where a vector field Z is mapped to $Z' = DZ/dt$ and the function has the properties*

1. *$(aZ_1 + bZ_2)' = aZ_1' + bZ_2'$ for scalars $a, b \in \mathbb{R}$ and vector fields $Z_1, Z_2 \in \mathfrak{X}(\gamma)$.*
2. *$(hZ)' = \frac{dh}{dt}Z + hZ'$ for $h \in \mathfrak{F}(I)$ and $Z \in \mathfrak{X}(\gamma)$.*
3. *$V_\gamma'(t) = \nabla_{\gamma'(t)}(V)$ for $t \in I$, $V \in \mathfrak{X}(M)$ where V_γ is the restriction of $V \in \mathfrak{X}(M)$ to the range of γ, and ∇ is Levi-Civita connection.*
4. *The above three points also imply that*

$$\frac{d}{dt} g(Z_1, Z_2) = g(Z_1', Z_2) + g(Z_1, Z_2')$$

for vector fields $Z_1, Z_2 \in \mathfrak{X}(\gamma)$, where g is a metric on M.

Proof: Sketch. Uniqueness: Take a single coordinate system, and assume a map exists with first 3 properties satisfied. Take $Z \in \mathfrak{X}(\gamma)$. Using properties 1 and 2 we have

$$Z' = \sum \frac{dZ^i}{dt} \partial_i\big|_\gamma + \sum Z^i (\partial_i\big|_\gamma)'$$

Using property 3,

$$Z' = \sum \frac{dZ^i}{dt} \partial_i\big|_\gamma + \sum Z^i D_{\gamma'}(\partial_i). \tag{A.4.1}$$

Hence Z' is uniquely determined.

Existence: Assume J is a subinterval of I, with $\gamma(J)$ defined in a coordinate neighbourhood. Define Z' by Equation (A.4.1). Straightforward computations show all four properties hold, then uniqueness implies this is a single vector field in $\mathfrak{X}(\gamma)$. □

See O'Neill [29, pg. 65] for the full proof.

Definition A.8. If $Z = \gamma'$ then $Z' = \gamma''$ is called the *acceleration* of the curve.

There is an important notion of a geodesic which generalises the notion of a straight line in Euclidean space as follows.

Definition A.9. A *geodesic* in a semi-Riemannian manifold M is a curve $\gamma : I \to M$ whose vector field γ' is parallel. That is, $\gamma'' = 0$.

Proposition A.3. *Given any tangent vector $v \in T_pM$ there is a unique geodesic γ_v in M such that*

1. $\gamma_v'(0) = v$
2. *The domain I_v of γ_v is as large as possible, so that any other geodesic with same initial velocity has domain $J \subset I$ and the curve is the restriction of γ_v to J.*

Definition A.10. If the unique geodesic, γ_v from Proposition A.3, has maximal domain, then γ_v is known as *maximal* or *geodesically inextensible.* If every maximal geodesic in M is defined on the entire real line, then M is called *geodesically complete* or *complete.*

See O'Neill [29, pg. 68] for further details.

The idea now is to collect the geodesics at a point into a single mapping.

Definition A.11. If $o \in M$, let $\mathscr{D}_o$ be the set of vectors v in T_oM such that the inextensible geodesic γ_v is defined at least on $[0, 1]$. The *exponential map* of M at o is the function $\exp_o : \mathscr{D}_o \to M$ such that $\exp_o(v) = \gamma_v(1)$ for $v \in \mathscr{D}_o$.

Definition A.12. A subset of a vector space is called *starshaped* of there is a point that can be joined to any other point by a straight line.

Proposition A.4. *For each point $o \in M$ there is a nghd $\overline{U}$ of* 0 *in T_oM on which exp_o is a diffeomorphism onto a neighbourhood of U of o in M.*

This proof uses inverse function theorem, and can be found in O'Neill [29, pg. 71]

Definition A.13. If $\overline{U}$ is starshaped, and $\overline{U}, U$ are as in the previous proposition, then U is called a *normal neighbourhood* about o. Let $e_1, \ldots, e_n$ be an orthonormal basis for T_oM so that $g(e_i, e_j) = \delta_{ij}\epsilon_j$, then the *normal coordinate system* $\zeta = (x^1, \ldots, x^n)$ determined by $e_1, \ldots, e_n$ assigns to each point $p \in U$ the vector coordinates relative to $e_1, \ldots, e_2$ of the corresponding point $\exp_o^{-1}(p) \in \overline{U} \subset T_oM$. Hence,

$$exp_o^{-1} = \sum x^i(p)e_i$$

O'Neill [29, pg. 73] gives an important result which is that the Christoffel symbols are identically 0 at the point o when a normal coordinate system about o is used, and this fact is part of the proof of Proposition 1.7.

Remark A.1. When exp is used in the context of Lie groups and Lie algebras, normally only $\exp_e$ is considered, where e is the identity element of the Lie group. Henceforth exp will be used instead of $\exp_e$. By Proposition A.4, exp is a local diffeomorphism from a neighbourhood of the identity of $\mathfrak{g}$ to a neighbourhood of the identity of G. Rossman [33, pg. 153, 156] gives a gentle introduction to the exponential map on Lie groups.

A.5 Proof of the Weyl tensor symmetries

This section contains the proof of Properties 1.2, which states that the Weyl tensor has the following symmetries

1. $$C_{abcd} = C_{[ab][cd]} = C_{cdab}.$$

2. $$C_{[abc]d} = 0.$$

Proof. Let $G_{abcd} = 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c}$ so that $C_{abcd} = -R_{abcd} - G_{abcd}$. We will use Equations (1.1.1) to (1.1.4) for the curvature tensor term and the symmetry of g and P for the remaining two terms.

1. Firstly, using the symmetries of the Riemann tensor we have

$$R_{cdab} = -R_{cdba} = -R_{abdc} = R_{abcd}.$$

Then consider

$$\begin{aligned} G_{abcd} = 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c} &= g_{ca}\mathsf{P}_{bd} - g_{cb}\mathsf{P}_{ad} + g_{db}\mathsf{P}_{ac} - g_{da}\mathsf{P}_{bc} \\ &= g_{ac}\mathsf{P}_{db} - g_{bc}\mathsf{P}_{da} + g_{bd}\mathsf{P}_{ca} - g_{ad}\mathsf{P}_{cb} \\ &= g_{ac}\mathsf{P}_{db} - g_{ad}\mathsf{P}_{cb} + g_{bd}\mathsf{P}_{ca} - g_{bc}\mathsf{P}_{da} \\ &= 2g_{a[c}\mathsf{P}_{d]b} + 2g_{b[d}\mathsf{P}_{c]a} = G_{cdab}. \end{aligned}$$

Hence we have

$$\begin{aligned} C_{abcd} &= -R_{abcd} - G_{abcd} \\ &= -R_{cdab} - G_{cdab} \\ &= C_{cdab}. \end{aligned}$$

Secondly, consider

$$C_{[ab][cd]} = \frac{1}{4}\left(C_{abcd} - C_{bacd} - C_{abdc} + C_{badc}\right).$$

For the Riemannian curvature term, consider that $R_{[cd][ab]} = R_{cdab}$ because of its symmetries.

For the remaining terms, consider

$$G_{abcb} = 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c} = -2g_{c[b}\mathsf{P}_{a]d} - 2g_{d[a}\mathsf{P}_{b]c} = -G_{bacb}$$

and

$$\begin{aligned} G_{abcb} = 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c} =& 2g_{d[a}\mathsf{P}_{b]c} + 2g_{c[b}\mathsf{P}_{a]d} \\ =& -2g_{d[b}\mathsf{P}_{a]c} - 2g_{c[a}\mathsf{P}_{b]d} = -G_{abdc}. \end{aligned}$$

Hence

$$\frac{1}{4}\left(G_{abcd} - G_{bacd} - G_{abdc} + G_{badc}\right) = \frac{1}{4}\left(G_{abcd} + G_{abcd} + G_{abcd} + G_{abcd}\right) = G_{abcd}.$$

So finally,

$$\begin{aligned}
C_{[ab][cd]} &= -\frac{1}{4}R_{cdab} - \frac{1}{4}\left(G_{abcd} - G_{bacd} - G_{abdc} + G_{badc}\right) \\
&= -R_{cdab} - G_{abcd} \\
&= C_{abcd}.
\end{aligned}$$

2. Consider that

$$C_{[abc]d} = \frac{1}{6}\left(C_{abcd} + C_{cabd} + C_{bcad} - C_{acbd} - C_{cbad} - C_{bacd}.\right)$$

Then

$$\begin{aligned}
&\frac{1}{6}\left(R_{cdab} + R_{bdca} + R_{adbc} - R_{bdac} - R_{adcb} - R_{cdba}\right) \\
&= \frac{1}{6}\left(R_{cdab} + R_{bdca} + R_{adbc} + R_{bdca} + R_{adbc} + R_{cdab}\right) \\
&= \frac{1}{3}\left(R_{cdabc} + R_{bdca} + R_{adbc}\right) \\
&= \frac{1}{3}\left(-R_{dcab} - R_{dbca} - R_{dabc}\right) \\
&= 0.
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{1}{6}\left(G_{abcd} + G_{cabd} + G_{bcad} - G_{acbd} - G_{cbad} - G_{bacd}\right) \\
&= \frac{1}{6}\left(G_{abcd} + G_{cabd} + G_{bcad} + G_{cabd} + G_{bcad} + G_{abcd}\right) \\
&= \frac{1}{6}\left(G_{abcd} + G_{cabd} + G_{bcad}\right)
\end{aligned}$$

and

$$\begin{aligned}
G_{abcd} &= g_{ca}\mathsf{P}_{bd} - g_{cb}\mathsf{P}_{ad} + g_{db}\mathsf{P}_{ac} - g_{da}\mathsf{P}_{bc} \\
&= g_{ac}\mathsf{P}_{bd} - g_{da}\mathsf{P}_{cb} - g_{bc}\mathsf{P}_{ad} + g_{dc}\mathsf{P}_{ab} \\
G_{cabd} &= -g_{ab}\mathsf{P}_{cd} + g_{db}\mathsf{P}_{ca} + g_{bc}\mathsf{P}_{ad} - g_{dc}\mathsf{P}_{ab} \\
G_{bcad} &= g_{ab}\mathsf{P}_{cd} - g_{db}\mathsf{P}_{ca} - g_{ca}\mathsf{P}_{bd} + g_{da}\mathsf{P}_{bc}.
\end{aligned}$$

Summing the three terms gives 0 as required.

Hence $C_{[abc]d} = \frac{1}{6}\left(C_{abcd} + C_{cabd} + C_{bcad} - C_{acbd} - C_{cbad} - C_{bacd}\right) = 0.$ □

The following is the proof of Properties 1.3, which states that the Weyl tensor is tracefree.

Proof. Using Properties 1.2, we have that

$$C_{a\ \ cd}^{\ a} = C^{a}{}_{acd} = g^{ab}C_{[ba][cd]} = \frac{1}{2}(g^{ab}C_{ba[cd]} - g^{ab}C_{ab[cd]}) = 0$$

and that

$$C_{abc}{}^{c} = C_{ab\ c}^{\ \ c} = 0 \qquad \text{similarly.}$$

Now consider that

$$\begin{aligned} C_{a\ cb}^{\ b} &= C_{abc}{}^{b} \\ &= -R_{c\ ab}^{\ b} - \left(g_{ca}\mathsf{P}^{b}{}_{b} - g_{c}^{b}\mathsf{P}_{ab} + g_{b}^{b}\mathsf{P}_{ac} - g_{ba}\mathsf{P}^{b}{}_{c}\right) \\ &= R^{b}{}_{acb} - \left(g_{ca}\mathsf{J} - \mathsf{P}_{ac} + n\mathsf{P}_{ac} - g_{a}^{b}\mathsf{P}_{bc}\right) \\ &= R_{ac} - (g_{ca}\mathsf{J} - \mathsf{P}_{ac} + n\mathsf{P}_{ac} - \mathsf{P}_{ac}) \\ &= R_{ac} - (g_{ca}\mathsf{J} + (n-2)\mathsf{P}_{ac}) \\ &= R_{ac} - R_{ac} \\ &= 0. \end{aligned}$$

In fact, we have shown that $G_{abc}{}^{b} = G_{a\ cb}^{\ b} = -G^{b}{}_{acb} = G^{b}{}_{abc} = R_{ac}$ using skewness of G. Finally,

$$\begin{aligned} C^{a}{}_{bca} &= -R_{ca\ b}^{\ \ a} - G^{a}{}_{bca} \\ &= R_{ac\ b}^{\ \ a} - G^{a}{}_{bca} \\ &= -R^{a}{}_{cba} - G^{a}{}_{bca} \\ &= -R_{cb} + R_{bc} = 0 \end{aligned}$$

and the result follows. □

A.6 Lie groups and Lie algebras

For a fantastic introduction to Lie groups and Lie algebras, the reader is directed to Rossman [33]. Here the main results used in the thesis are given. As in the main chapters, $\mathfrak{g}$ will be used to denote a Lie algebra and G to denote a Lie group.

Lemma A.5. *Let G be a Lie group. Any connected neighbourhood, U, of the identity e generates the connected component of the group.*

Proof. Let $S = \langle U \rangle$ be the subgroup generated by U. Any element h of U is contained in hU which is open as L_h is an open map. Now $h = h_1 \dots h_n$ where $h_i \in U$. Then $h_n \subset h_nU \cap U$ and $h_{n-1}h_n \subset h_{n-1}h_nU \cap h_nU$ etc so S is connected. Finally, take any $b \in S^c$. Then bU is an open subset of h. Moreover, if $a \in bU \cap S$ then there is $h, h_i \in U$ such that $bh = h_1 \dots h_n$ so $b = h^{-1}h_1 \dots h_n \in S$ which is a contradiction. So $bU \subset S^c$. Hence S is closed. So S is the connected component of the identity. □

Note that G need only be a topological group for this theorem to hold - that is, a topological manifold where the inverse and multiplication operations are continuous.

A.6.1 Conjugation and adjoint representations

In this section we consider G a Lie group and show how to equip the tangent space at the identity, $\mathfrak{g} = T_eG$, with a Lie bracket.

Define $\mathrm{End}(G)$ as the set of endomorphisms of G, that is, the set of homomorphisms from G to G. Define $\mathrm{Aut}(G)$ as the set of automorphisms of G, that is the set of isomorphisms from G to G.

For each $h \in G$ define the automorphism $c_h : G \to G$ by $c_h p = hph^{-1}$ for any $p \in G$. This is called *conjugation* of p by h. We can also write $c_h = L_h \circ R_{h^{-1}}$. Now the map $\phi : G \to \mathrm{Aut}(G)$ sending h to c_h is well-defined smooth map between manifolds. We have that $c_{h_1h_2}(p) = h_1h_2ph_2^{-1}h_1^{-1} = c_{h_1} \circ c_{h_2}(p)$ so that ϕ is a group homomorphism between G and $\mathrm{Aut}(G)$.

Differentiating c_h at the identity we define the *Adjoint representation of* G as the map $\mathrm{Ad} : G \to \mathfrak{gl}(\mathfrak{g})$ by $\mathrm{Ad}(h)X = dc_h|_e X$ where $X \in \mathfrak{g}$. That is, $\mathrm{Ad}(h) = dc_h|_e = d(L_h \circ R_{h^{-1}})|_e = dL_h|_{h^{-1}} \circ dR_{h^{-1}}|_e$.

If we are dealing with linear groups, this is just $\mathrm{Ad}(h)X = hXh^{-1}$. Using that ϕ is a homomorphism and $c_h(e) = e$ for any $h \in G$, we have $\mathrm{Ad}(h_1h_2) = dc_{h_1h_2}|_e = dc_{h_1}|_e dc_{h_2}|_e = \mathrm{Ad}(h_1)\,\mathrm{Ad}(h_2)$. That is, Ad is a homomorphism.

Differentiating Ad at the identity, we define the *adjoint representation of* $\mathfrak{g}$ as the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$. We have $\mathrm{ad}(X)Y = d\,\mathrm{Ad}_e(X)(Y)$. If G is a linear group, this is just $\mathrm{ad}(X)Y = XY - YX$. We define the *Lie bracket of* $\mathfrak{g}$ as the map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ where $[X, Y] = \mathrm{ad}(X)Y$.

A beautiful result connects the exponential map and the two adjoint representations, which we use in Lemma 2.2.

Proposition A.6. *For* $\mathfrak{g}$ *a Lie algebra of a Lie group* G *we have*

$$\mathrm{Ad} \circ \exp(X) = \exp \circ \,\mathrm{ad}(X) \qquad \forall X \in \mathfrak{g}.$$

See Rossman [33, pg. 14, 78] for a proof of this well-known fact. It can also be found in Warner [39, pgs. 104, 114]. Note that the exponential map on the right hand side is the matrix exponential map,

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i \quad \text{for any square matrix } A.$$

Note that one may define a Lie algebra independently from a Lie group. Here, a Lie algebra, $\mathfrak{g}$, is a vector space, equipped with a bracket operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that the bracket is bilinear, antisymmetric and satisfies the Jacobi identity. Note that $\mathrm{ad}(X)Y$ satisfies these conditions for a bracket on the Lie algebra of a Lie group, identified with the tangent space at the identity. The following theorem shows that all finite dimensional Lie algebras (including those defined independently from a Lie group) are isomorphic to Lie subalgebras of $\mathfrak{gl}_n\mathbb{R}$.

Theorem A.7 (Ado's Theorem)**.** *Every finite dimensional Lie algebra,* $\mathfrak{g}$*, admits a faithful finite-dimensional representation. That is, there is an injective Lie algebra homomorphism from* $\mathfrak{g}$ *to* $\mathfrak{gl}_n\mathbb{R}$*.*

The proof of this theorem can be found in Jacobson [17, pg. 202]. There is also a connection with any Lie subgroups of a Lie group with Lie subalgebras of a Lie groups Lie algebra as follows.

Theorem A.8. *Let G be Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there is a unique connected Lie subgroup $H \hookrightarrow G$ such that $\mathfrak{h}$ is the Lie subalgebra of H.*

The proof of this theorem can be found in Warner [39, pg. 94]. These theorems imply that any Lie algebra (including those defined independently from a Lie group) is the Lie algebra of some Lie group. It is in fact true that connected Lie groups are in one-to-one correspondence with Lie algebras, up to universal cover, and several theorems that convey this fact can be found in Warner [39] and in Lee [22].

Lemma A.9. *Up to isomorphism, there is only one non-abelian Lie algebra of dimension 2. This Lie algebra has a basis $\{X, Y\}$ such that $[X, Y] = X$, and hence the Lie algebra is solvable.*

Proof. Assume $\{U, V\}$ is any basis for $\mathfrak{g}$, 2 dimensional Lie algebra. Then $[U, V] = aU + bV$ for some a, b in the field. Case 1: If $a = b = 0$ the Lie algebra is abelian. Case 2: Assume $a = 0, b \neq 0$ then pick $Y = \frac{-1}{b}U, X = V$. Then $[X, Y] = bV = X$ as required. Case 3: Assume $a \neq 0$, then pick $X = aU + bV, Y = \frac{1}{a}V$, then $[X, Y] = [U, V] = X$ as required.

If $[X, Y] = X$, then $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = [\operatorname{span} X, \operatorname{span} X] = \{0\}$. Hence the derived series terminates and the Lie algebra is solvable. See Definition A.20 for the definition of a solvable Lie algebra. □

A.7 Simple, semisimple and reductive Lie algebras

To make a classification of bi-invariant metrics easier, it would be effective to break the Lie algebra into smaller pieces, then study the metric on these pieces. These pieces are precisely the ideals of the Lie algebra. Here, ideals are introduced and terminology that is used in the main chapters.

Definition A.14. $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$ if it is a subspace and $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Definition A.15. A Lie algebra is *simple* if it is non-abelian and if its only ideals are trivial or itself.

Definition A.16. The *centre* of $\mathfrak{g}$ is the set $\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid [X, Y] = 0\ \forall Y \in \mathfrak{g}\}$. This is clearly the kernel of the adjoint representation.

Definition A.17. The *radical* of a Lie algebra is the largest solvable ideal.

See Definition A.20 for the definition of solvable.

Theorem A.10. *The following are equivalent for a Lie algebra $\mathfrak{g}$.*

1. *The Killing form is non-degenerate on $\mathfrak{g}$. This is known as the* The Cartan Criterion*.*
2. *$\mathfrak{g}$ has no non-zero abelian ideals.*

3. $\mathfrak{g}$ *has no non-zero solvable ideals.*

4. The radical of $\mathfrak{g}$ *is zero.*

5. $\mathfrak{g}$ *is the direct sum of simple Lie algebras.*

When these conditions hold, $\mathfrak{g}$ *is called* semisimple.

Proof: Sketch. Clearly 3 and 4 are equivalent by definition of the radical. 3 implies 2, as abelian ideals are solvable, and 2 implies 3 as eventually there must be an abelian ideal to have a terminating derived series. 5 implies 2, as simple ideals are never abelian.

1 implies 2 as if $\mathfrak{h}$ is an abelian ideal, then $[X, Y] = 0$ for all $X, Y \in \mathfrak{h}$ and $[X, Z] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$. Hence $0 = [X, [Y, Z]] = \mathrm{ad}(X) \circ \mathrm{ad}(Y)(Z) \forall X, Y \in \mathfrak{h}$, $Z \in \mathfrak{g}$, and it follows that $\mathrm{ad}(X) \circ \mathrm{ad}(Y) \equiv 0$ on $\mathfrak{h}$. Then $\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\big|_{\mathfrak{h}}) = \mathrm{tr}(0) = 0 \ \forall Y \in \mathfrak{g}$ and hence the Killing form is degenerate.

Note that if I is an ideal in $\mathfrak{g}$ then there is an orthogonal complement with respect to an non-degenerate form, I' which is also an ideal. Now if I is simple, K is non-degenerate, then $I \cap I' \subset I$ is an ideal in I so must be trivial, so $\mathfrak{g} = I \oplus I'$. Using induction, this also shows that 1 implies 5.

The final implication that 2 implies 1 is quite technical and the reader is directed, for example, to Varadarajan [37, pg. 210] or Fulton and Harris [12, pgs. 479-480]. □

Definition A.18. A *reductive* Lie algebra $\mathfrak{g}$ is a Lie algebra that that is isomorphic to the direct sum of simple and abelian ideals. That is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$$

where $\mathfrak{h}$ is semisimple and $\mathfrak{s}$ is abelian.

Reductive is equilvalent to the property that the complement of any ideal is an ideal.

Proposition A.11. *If* $\mathfrak{g}$ *is a semi-simple Lie algebra then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof. By Theorem A.10, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \oplus_i \mathfrak{h}_i$ where $\mathfrak{h}_i$ are simple ideals. Now $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$. It cannot be trivial as $\mathfrak{g}$ is semi-simple. Note that $[\mathfrak{h}_i, \mathfrak{g}] \subset \mathfrak{h}_i$ is an ideal, which is also non-trivial, hence $[\mathfrak{h}_i, \mathfrak{g}] = \mathfrak{h}_i$. We then have that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. □

Remark A.2. If $\mathfrak{g}$ is reductive, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where $\mathfrak{h}$ is semisimple and $\mathfrak{s}$ is abelian, then it follows from the definitions that $\mathfrak{s}$ is the centre of $\mathfrak{g}$ and, by Proposition A.11 that $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$.

Denote $\mathrm{der}(\mathfrak{g})$ the space of derivations on $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{g}) := \{\mathrm{ad}(X) \mid X \in \mathfrak{g}\}$, a subset of $\mathrm{der}(\mathfrak{g})$ known as the *inner derivations of* $\mathfrak{g}$.

Proposition A.12. *Every derivation on a semi-simple Lie algebra is inner.*

Proof. Take $D \in \mathrm{der}(\mathfrak{g})$. Define the Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \rtimes \mathbb{K}D$ with the bracket

$$[X, Y]_{\tilde{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} \qquad \forall X, Y \in \mathfrak{g}$$

and

$$[D, X]_{\tilde{\mathfrak{g}}} = D(X) \in \mathfrak{g} \qquad \forall X \in \mathfrak{g}$$

Notice that $\mathfrak{g}$ is a semi-simple ideal in $\tilde{\mathfrak{g}}$. If the Killing form is non-degenerate overall, then $\tilde{\mathfrak{g}}$ is semisimple and can hence be decomposed into a direct product of ideals $\mathfrak{g} \oplus \mathfrak{s}$ where $\mathfrak{s}$ is one dimensiona. Then $\mathfrak{s}$ must be abelian, on which the Killing form must be degenerate, contradicting the orginal assumpiton. Hence the Killing form must be degenerate on $\tilde{\mathfrak{g}}$, which gives an $W \in \tilde{\mathfrak{g}}$ such that $K(W, Y) = 0$ for all $Y \in \tilde{\mathfrak{g}}$. As the Killing form is non-degernate on $\mathfrak{g}$, this vector cannot be in $\mathfrak{g}$, hence must be of the form $W = Z + D$ for some $Z \in \mathfrak{g}$. Then the span of W must be contained in $\mathfrak{g}^{\perp} = \{X \in \tilde{\mathfrak{g}} \mid K(X, Y) = 0, \forall Y \in \mathfrak{g}\}$.

$\mathfrak{g}^{\perp}$ is also an ideal as if $X \in \tilde{\mathfrak{g}}$, $Y \in \mathfrak{g}$ and $Z \in \mathfrak{g}^{\perp}$, then $[X, Y] \in \mathfrak{g}$ and hence $K(Y, [X, Z]) = K(-[X, Y], Z) = 0$, giving that $[\tilde{\mathfrak{g}}, \mathfrak{g}^{\perp}] \subset \mathfrak{g}^{\perp}$. Note that if $X \in \mathfrak{g} \cap \mathfrak{g}^{\perp}$ then as the Killing form is non-degenerate on $\mathfrak{g}$, $X = 0$ and therefore $\mathfrak{g} \cap \mathfrak{g}^{\perp} = \{0\}$. Now finally, for any $X \in \mathfrak{g}$, we have $[W, X] = [Z + D, X] = [Z, X] + D(X) \in \mathfrak{g} \cap \mathfrak{g}^{\perp} = \{0\}$ hence $D = [\cdot, -Z] \in \operatorname{ad}(\mathfrak{g})$. □

Note that in the above, we in fact showed that $\tilde{\mathfrak{g}} := \mathfrak{g} \rtimes \mathbb{K}D \cong \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, and hence $\mathfrak{g}^{\perp} = \operatorname{span}(W)$ is 1-dimensional and therefore abelian. This is hence an example of a reductive Lie algebra. This result and proof can be found in the literature, for instance in Helgason [16, pg. 132].

Proposition A.13. *If $\mathfrak{g}$ is a Lie algebra that decomposes into ideals as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ where $\mathfrak{h}$ is semisimple, then* $\operatorname{der}(\mathfrak{g}) = \operatorname{der}(\mathfrak{h}) \oplus \operatorname{der}(\mathfrak{s}) \cong \mathfrak{h} \oplus \operatorname{der}(\mathfrak{s})$.

Proof. Take $D \in \operatorname{der}(\mathfrak{g})$. Define D_1, D_2 such that for any $X \in \mathfrak{g}$ $D(X) = D_1(X) + D_2(X)$ where $D_1(X) \in \mathfrak{h}$ and $D_2(X) \in \mathfrak{s}$.

Consider $X_1, X_2 \in \mathfrak{h}$. Then

$$\begin{aligned} D_1([X_1, X_2]) + D_2([X_1, X_2]) &= D[X_1, X_2] \\ &= [D(X_1), X_2] + [X_1, D(X_2)] = [D_1(X_1), X_2] + [X_1, D_1(X_2)] \end{aligned}$$

As $D_2 \in \mathfrak{s}$ then $D_2([X_1, X_2]) = 0$ and $D_1([X_1, X_2]) = [D_1(X_1), X_2] + [X_1, D_1(X_2)]$. As $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, then $D_2 \equiv 0$ on $\mathfrak{h}$.

Consider $X_1 \in \mathfrak{h}, X_2 \in \mathfrak{s}$. Then

$$0 = D([X_1, X_2]) = [D_2(X_1), X_2] + [X_1, D_1(X_2)] = [X_1, D_1(X_2)]$$

as $D_2 \equiv 0$ on $\mathfrak{h}$. This implies $D_1(X_2)$ is in the centre of $\mathfrak{h}$, but as $\mathfrak{h}$ is semisimple, the centre is trivial and $D_1 \equiv 0$ on $\mathfrak{s}$. Hence D_1 is a derivation on $\mathfrak{h}$ and $D_1 = [Z, \cdot]$ for some $Z \in \mathfrak{h}$ by Proposition A.12.

Finally consider $X_1, X_2 \in \mathfrak{s}$. Then

$$D_2([X_1, X_2]) = D[X_1, X_2] = [D(X_1), X_2] + [X_1, D(X_2)] = [D_2(X_1), X_2] + [X_1, D_2(X_2)]$$

and hence $D_2|_{\mathfrak{s}} \in \operatorname{der}(\mathfrak{s})$. This implies the result. □

A.8 Elements of $\mathfrak{so}(\mathfrak{g}) \cap \mathrm{GL}(\mathfrak{g})$

Definition A.19. For a metric Lie algebra $\mathfrak{g}$ we define the special orthogonal Lie algebra of $\mathfrak{g}$ as $\mathfrak{so}(\mathfrak{g})$, the anti-symmetric endomorphisms of $\mathfrak{g}$. That is

$$\mathfrak{so}(\mathfrak{g}) = \{A \in \operatorname{End}(\mathfrak{g}) \mid g(Ax, y) = -g(x, Ay)\}$$

where g is the corresponding metric of $\mathfrak{g}$.

Note that on an n-dimensional vector space, we normally write $\mathfrak{so}(n)$ for the anti-symmetric endomorphisms, that is

$$\mathfrak{so}(\mathfrak{g}) = \{A \in \mathrm{End}(\mathfrak{g}) \mid A^t = -A\}.$$

This is a subalgebra of the general linear algebra, which is the Lie algebra of endomorphisms. The two Lie algebras $\mathfrak{so}(\mathfrak{g})$ and $\mathfrak{so}(n)$ are isomorphic when the metric is Riemannian.

Lemma A.14. *Elements of $\mathfrak{so}(n)$ have eigenvalues which are either purely imaginary or 0.*

Proof. Take $A \in \mathfrak{so}(n)$. Let $x \in \mathbb{C}^n$ be an eigenvector for A such that $Ax = \lambda x$ some $\lambda \in \mathbb{C}$.

$$\begin{aligned} Ax &= \lambda x \\ \overline{Ax} &= \overline{\lambda x} \\ -A^t\overline{x} &= \overline{\lambda}\overline{x} \\ -\overline{x}^t A &= \overline{\lambda}\overline{x}^t \\ -\overline{x}^t Ax &= \overline{\lambda}|x|^2 \\ -\overline{x}^t \lambda x &= \overline{\lambda}|x|^2 \\ -\lambda|x|^2 &= \overline{\lambda}|x|^2 \\ \Rightarrow \quad \lambda + \overline{\lambda} &= 0 \end{aligned}$$

hence λ is either imaginary or zero. □

Proposition A.15. *Let $\mathfrak{g}$ be a real metric Lie algebra with Riemannian metric, then $\mathfrak{so}(\mathfrak{g}) \cap \mathrm{GL}(\mathfrak{g})$ is empty or $\mathfrak{g}$ is even dimensional.*

Proof. Assume there is an $A \in \mathfrak{so}(\mathfrak{g}) \cap \mathrm{GL}(\mathfrak{g})$. As $A \in \mathrm{GL}(\mathfrak{g})$, it has no zero eigenvalues, so by Lemma A.14 all its eigenvalues are non-zero and imaginary. As all complex eigenvalues come in conjugate pairs, then there must be an even number of them, hence $\mathfrak{g}$ is even dimensional. □

A.9 Nilpotent and solvable Lie Algebras

For a comprehensive introduction to nilpotent and solvable Lie algebras, see the book Goze and Khakimdjanov [14, pgs. 11-21] or see Helgason [16, Ch. 3], which contain the results in the following section.

We define the descending series (or lower central series) as the series of ideals

$$\mathfrak{g}^0 = \mathfrak{g}, \ldots, \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}].$$

The derived series is defined as the series of ideals

$$g^{(0)} = \mathfrak{g}, \ldots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}].$$

By the definition, $g^k \subset g^{k-1}$ and using induction and the Jacobi identity it can be shown that $\mathfrak{g}^{(k)} \subset \mathfrak{g}^{(k-1)}$.

Definition A.20. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there is an $n \in \mathbb{N}$ such that $\mathfrak{g}^n = 0$. It is called *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$.

One can show $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$ and hence that any nilpotent Lie algebra is solvable.

For any solvable Lie algebra there is a $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n)} = 0$ and $\mathfrak{g}^{(n-1)} \neq 0$. Hence $\mathfrak{g}^{(n-1)}$ is a non-trivial abelian ideal of $\mathfrak{g}$.

A canonical example of a nilpotent Lie algebra is the strictly upper triangular $n \times n$ matrices; a canonical example of a solvable Lie algebra is the upper triangular $n \times n$ matrices. We will show that any nilpotent Lie algebra can be represented as a subalgebra of the strictly upper triangular matrixes, and any solvable Lie algebra can be represented as a subalgebra of the upper triangular matrices.

A.9.1 Properties of nilpotent Lie algebras

Proposition A.16. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then the following hold.*

1. *Homomorphic images and subalgebras of $\mathfrak{g}$ are nilpotent.*
2. *The centre of $\mathfrak{g}$ is non-trivial.*
3. *For any $X \in \mathfrak{g}$, $\mathrm{ad}_x^n = 0$ for some $n \in \mathbb{N}$.*

Proof: Sketch. Assume $\mathfrak{g}$ is solvable, then there is an $n \in \mathbb{N}$ such that $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}] = 0$ but $\mathfrak{g}^{n-1} \neq 0$.

1. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{h}^k \subset \mathfrak{g}^k$ and $\phi(\mathfrak{g}^k) = \phi(\mathfrak{g})^k$, for any homomorphism ϕ.
2. The centre must contain $\mathfrak{g}^{n-1}$.
3. Note that $\mathrm{ad}_x^n(y) \in \mathfrak{g}^n = 0$ for all $x, y \in \mathfrak{g}$. □

Define the Lie algebra $\mathfrak{g}/\mathfrak{i} := \{x + \mathfrak{i} \mid x \in \mathfrak{g}\}$ where $\mathfrak{i}$ is an ideal in a Lie algebra $\mathfrak{g}$.

Proposition A.17. *The following hold.*

1. *The sum of two nilpotent ideals is nilpotent.*
2. *For any ideal in the centre of a Lie algebra, $\mathfrak{i} \subset \mathfrak{z}(\mathfrak{g})$, if $\mathfrak{g}/\mathfrak{i}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*
3. *$\mathfrak{i}$ is an ideal then $\mathfrak{i}^n$ is an ideal.*

Proof: Sketch.
1. Using induction, one can show that for any two ideals, $\mathfrak{i}, \mathfrak{j}$, $(\mathfrak{i}+\mathfrak{j})^{2n} \subset \mathfrak{i}^n + \mathfrak{j}^n$ and the result follows.
2. Define the canonicla Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$. Then $\phi(\mathfrak{g}^n) = (\mathfrak{g}/\mathfrak{i})^n = 0$. Then $\mathfrak{g}^n = \mathfrak{i} \subset \mathfrak{z}(\mathfrak{g})$ which implies $\mathfrak{g}^{n+1} = 0$.
3. The result follows from induction and the Jacobi identity. □

A.9.2 Properties of Solvable Lie algebras

Proposition A.18. *The following properties hold for solvable Lie algebras.*

1. *Subalgebras and homomorphic images of solvable Lie algebras are solvable.*
2. *If* $\mathfrak{i}$ *is a solvable solvable ideal of* $\mathfrak{g}$ *and* $\mathfrak{g}/\mathfrak{i}$ *solvable, then* $\mathfrak{g}$ *solvable.*
3. *If* $\mathfrak{i} \subset \mathfrak{g}$ *is an ideal, then* $\mathfrak{i}^{(k)}$ *is an ideal in* $\mathfrak{g}$.
4. *The sum of solvable ideals are solvable.*

Proof: Sketch. The proof of the first and second points are analogous to the nilpotent case. The third point follows from induction. The fourth point can be proved using induction to show $(\mathfrak{i} + \mathfrak{j})^{(2n)} \subset \mathfrak{i}^{(n)} + \mathfrak{j}^{(n)}$. □

Definition A.21. The maximal solvable ideal is called the *radical* of $\mathfrak{g}$, denoted $\mathfrak{rad}(\mathfrak{g})$.

The following two subsections do not contain any proofs, instead the reader is referred to the appropriate page of Goze and Khakimdjanov [14].

A.9.3 Engel's Theorem

Lemma A.19. *Let* A *be an endomorphism of a complex vector space* V*, then there is a basis such that* A *is upper triangular.*

See Goze and Khakimdjanov [14, pg. 15] for the proof.

Definition A.22. A linear map $A : V \to V$ over a vector space V is called *nilpotent* if there is an $n \in \mathbb{N}$ such that $A^n \equiv 0$.

Remark A.3. If V is a complex vector space, then the above lemma implies that for any nilpotent endomorphism A, there is a basis of V in which A is strictly upper triangular.

Lemma A.20 (Engel's Lemma)**.** *If* V *is a vector space over* $\mathbb{R}$ *or* $\mathbb{C}$ *and let* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *be a subalgebra with every element a nilpotent endomorphism. Then there is a* $v \in V \setminus \{0\}$ *such that* $X(v) = 0$ *for all* $X \in \mathfrak{g}$.

See the proof in Goze and Khakimdjanov [14, pgs. 15-16].

Theorem A.21 (Engel's Theorem)**.** *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* $\mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g})$ *is nilpotent endomorphism of* $\mathfrak{g}$ *for all* $X \in \mathfrak{g}$.

The proof follows from Proposition A.16, Engel's lemma and Proposition A.17, see Goze and Khakimdjanov [14, pg. 16].

Corollary A.22. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *a representation over a finite dimensional vector space. If* $\rho(x)$ *are all nilpotent, then* $\rho(\mathfrak{g})$ *is simultaneously strictly upper triangulable. That is, we can find a basis for* V *that makes all the endomorphisms in* $\rho(\mathfrak{g})$ *strictly upper triangular in this basis. And hence* $\rho(\mathfrak{g})$ *is nilpotent.*

This follows using induction on $m = \dim(V)$, as in Leistner [24, pg. 60].

A.9.4 Lie's Theorem

Lemma A.23. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ for V complex vector space, then there is a common eigenvalue for all elements in $\mathfrak{g}$.*

See Goze and Khakimdjanov [14, pgs. 17-18].

Theorem A.24 (Lie's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}_(V)$ be a solvable Lie algebra and V a complex vector space. Then $\mathfrak{g}$ is simultaneously upper triangular. That is: there is a basis of V such that all $X \in \mathfrak{g}$ are upper triangular matrices.*

The proof uses induction over $m = \dim(V)$ and Lemma A.23. See Goze and Khakimdjanov [14, pgs. 17-19].

Corollary A.25. *A Lie algebra $\mathfrak{g}$ is solvable over $\mathbb{R}$ or $\mathbb{C}$ if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Particularly, $\mathfrak{g}$ is solvable if and only if ad_X is nilpotent for any $X \in [\mathfrak{g}, \mathfrak{g}]$.*

See the proof in Goze and Khakimdjanov [14, pgs. 19-20].

Theorem A.26 (Cartan's Solvability Criterion)**.** *Let V be a real or complex vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ subalgebra be such that $\mathrm{tr}(x \circ y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and for all $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

See the proof in Goze and Khakimdjanov [14, pgs. 20].

Corollary A.27. *Let $\mathfrak{g}$ be a real or complex Lie algebra, then $\mathfrak{g}$ is solvable if and only if the Killing form of $\mathfrak{g}$, $K_{\mathfrak{g}}$ satisfies $K_{\mathfrak{g}}(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$.*

Proof. Using the definition of the Killing form in Definition 2.2, this theorem follows from Theorem A.26 and Corollary A.25. □

A.10 Representations

Schur's lemma plays an important role in determining what type of metrics exist on different spaces. However, the proofs are normally considered in the language of representations, which we introduce now. Hall [15], Fulton and Harris [12] and Arvanitoyeorgos [2] are excellent references for the material in this section. Notably, Arvanitoyeorgos [2] focuses on the geometry of Lie groups and also includes a description of bi-invariant metrics.

Definition A.23. The definitions of representations on Lie groups and Lie algebras are as follows.

1. A *representation* of a Lie group G is a smooth group homomorphism $\rho : G \to \mathrm{GL}(V)$ from G to the general linear group of a (finite dimensional complex) vector space, V. We use the notation $gv = \rho(g)(v)$ for $g \in G$, $v \in V$.

2. A *representation* of a Lie algebra, $\mathfrak{g}$, on a complex vector space, V, is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

It is will know that representations of a connected and simply connected Lie group are in 1-1 correspondence with representations of the Lie algebra.

An example of a Lie group representation is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$; $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an example of a Lie algebra representation.

Definition A.24. If $\rho : G \to \mathrm{GL}(V)$ is a Lie group representation, then a subspace U of V is called *invariant* or *G-invariant* or *ρ-invariant* if $gU \subset U \ \forall g \in G$. This is analogously defined for a Lie algebra.

With this definition, an ideal is precisely an ad-invariant subspace of $\mathfrak{g}$. With reductive Lie algebras, the complement of an ideal is an ideal. This translates to the property that the complement of an ad-invariant subspace is another ad-invariant subspace on a reductive Lie algebra. However, some Lie algebras may contain ideals that do not have ideals as complements, and hence their representations have distinct properties.

Definition A.25.
1. A representation is called *irreducible* if the only invariant subspaces are $\{0\}$ and V.
2. A representation is called *completely reducible* if every subspace that is G-invariant has a complement in the vector space that is also G-invariant.

Using these definitions, a Lie algebra is reductive if and only if the adjoint representation is completely reducible. If the adjoint representation is completely reducible and without kernel, then $\mathfrak{g}$ is semisimple. See also Hall [15, pgs. 157-158].

Remark A.4. The adjoint representation is irreducible if and only if $\{0\}$ and $\mathfrak{g}$ are the only ad-invariant subspaces which is if and only if only $\{0\}$ and $\mathfrak{g}$ are the only ideals, which is if and only if $\mathfrak{g}$ is one-dimensional or simple. Hence, a Lie algebra $\mathfrak{g}$ is simple if and only if its adjoint representation is irreducible and without kernel.

Definition A.26. Let G be a group and ρ, ϕ representations of G acting on the vector spaces V, W. A linear map $\psi : V \to W$ is called an *intertwining map* or *intertwiner* of representations if

$$\psi(\rho(g)(v)) = \phi(g)(\psi(v)) \quad \text{or more succinctly} \quad \psi(gv) = g\psi(v)$$

See also Hall [15, pg. 92] for further results.

Lemma A.28 (Schur's Lemma)**.** *Let $\mathfrak{g}$ be a Lie algebra.*

1. *Let $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$, $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ be irreducible representations of $\mathfrak{g}$. Then any intertwiner is an isomorphism or identically 0.*
2. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation, where V is a complex vector space. Then any intertwiner from V to V is a complex multiple of the identity.*

Proof.
1. $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$, $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ are irreducible representations of $\mathfrak{g}$. Assume $A : V_1 \to V_2$ is an intertwiner. Then $\ker(A)$ is a subspace of V_1. This is G-invariant. As the representations are irreducible, then either $\ker(A)$=V_1, in which case $A \equiv 0$, or $\ker(A)$=$\{0\}$, in which case the map is injective. In the second case, $\mathrm{im}(A)$ is also G-invariant, so this must be equal to V_1. Hence A is an isomorphism.
2. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation, where V is a complex vector space. Let A be an intertwiner. Then either $A \equiv 0$, in which case $A = 0Id$, or A is an isomorphism. If V is one dimensional, we are done. Else, we know A can be represented as a matrix in $\mathrm{M}_n(\mathbb{C})$, which has at least one eigenvalue in $\mathbb{C}$. The

e'space corresponding to this eigenvalue is a subspace of V which is not trivial. It is G-invariant ($Agx = gAx = g\lambda x = \lambda gx$ so $gx \in$ subspace). So it must be equal to V. We can pick the standard basis to diagonalise to $A = \lambda Id$ as required.

□

See also Hall [15, pg. 113] for the proof and related results.

A.10.1 Complexifications, realifications and real forms

Real simple Lie algebras are closely related to their complex counterparts. To classify Riemannian bi-invariant metrics on reductive Lie groups, we introduce the following terminology surrounding complexifications. Comprehensive references for this section are Helgason [16, pgs. 180-182] and Onishchik [31, pgs 12-19].

Definition A.27. V is a vector space over the real numbers of dimension n, then we can consider it as a vector space, $V^{\mathbb{C}}$, over the complex numbers of dimension n by tensoring it with $\mathbb{C}$: $V^{\mathbb{C}} = V \otimes \mathbb{C}$. This is called *complexification* of V and is just a result of changing the scalar field. One may also interpret $V^{\mathbb{C}} = V \oplus \mathrm{i}V$ or just $V^{\mathbb{C}} = V \oplus V$.

If V is a vector space over the complex numbers of dimension n, then we can consider it as a real vector space, $V^{\mathbb{R}}$, of dimension $2n$ by identifying $v \in V$, with $v = a + bi$ where a, b are real vectors. We call $V^{\mathbb{R}}$ the *realification* of V.

If V is a vector space over the complex numbers and W is any real vector space such that the complexification of W is isomorphic to V, then W is called a *real form* of V.

Remark A.5. One can find a real form by taking the real span of any basis of W. In the case where V is a Lie algebra, the isomorphism is required to be a Lie algebra isomorphism. In this case, the real forms may not be isomorphic (over the real numbers), however they can be characterised as the fixed point set of a conjugate-linear involution, that is the fixed points of a map $J : V \to V$ such that $J^2 = -Id$, and $J(cx) = \bar{c}x$ for $c \in \mathbb{C}$ and $x \in V$. See Arvanitoyeorgos [2, pg. 46] or Helgason [16, pgs. 181] for an explicit example, which shows that a real form of a Lie algebra always exists.

Theorem A.29. *Let V be a real Lie algebra, then V is abelian, nilpotent, solvable or semisimple if and only if $V^{\mathbb{C}}$ is respectively.*

Proof: Sketch. For all, note that: $[V \oplus V, V \oplus V] = [V, V] \oplus [V, V]$. As each definition involves the use of the brackets, the result follows using induction. For semisimple, one can refer to the Cartan criterion (Theorem A.10): V is semisimple if and only the Killing form is non-degenerate. Then note that the Killing form on V is exactly the Killing form of $V^{\mathbb{C}}$, as the same basis elements have the same brackets. This fact completes the proof. □

See Serre [34, pg. 9] or Onishchik [31, pg. 16] for further details.

Theorem A.30. *Let V be a Lie algebra then V is simple if and only if $V^{\mathbb{C}}$ is simple or $V^{\mathbb{C}}$ is of the form $W \oplus \overline{W}$, with W and $\overline{W}$ simple and mutually conjugate with respect to V.*

Proof. We know from Theorem A.29 that $V^{\mathbb{C}}$ is at least semi-simple. Consider any non-trivial ideal, W, in $V^{\mathbb{C}}$. Then $\overline{W}$, under conjugation with respect to V, is also an ideal in $V^{\mathbb{C}}$. If we consider $W+\overline{W}$ and $W\cap\overline{W}$, these are also ideals and are equal to their conjugations. This implies they are complexifications of real ideals of V, say E_1 and E_2. That is, we have $W+\overline{W}=E_1^{\mathbb{C}}$ and $W\cap\overline{W}=E_2^{\mathbb{C}}$. As V is simple, $E_1=V$. Then E_2 is either $\{0\}$ or V, the first case implies $V^{\mathbb{C}}=W\oplus\overline{W}$, the second that $V^{\mathbb{C}}=W=\overline{W}=E_2$ and that $V^{\mathbb{C}}$ is simple. □

A similar result can be found in Onishchik [31, pg. 16] and in Di Scala et al. [7, pg. 633].

Remark A.6. In the above proof, note that

- In the first case, $V^{\mathbb{C}}$ has an irreducible adjoint representation and in the second case it has a reducible adjoint representation.
- *Conjugation with respect to* V means $\overline{W}=\{x-iy \mid x+iy\in W \text{with } x,\, y\in V\}$. For instance, we define $\mathbb{C}$ as $\mathbb{R}^2$ with conjugation with respect to $\mathbb{R}(1,0)$, where as we can easily change that to conjugation with respect to $\mathbb{R}(a,b)$ where $(a,b)\in S^1$, in this case $i=(-b,a)$. As they are canonically isomorphic, we rarely consider this case.
- If $V^{\mathbb{C}}=W\oplus\overline{W}$, then we have an isomorphism $V\cong\overline{W}_{\mathbb{R}}\cong W_{\mathbb{R}}$ via the map $\psi: W_{\mathbb{R}}\to V$, $v\mapsto\frac{1}{2}(v+\overline{v})$. For the proof, see Di Scala et al. [7, pg. 633].

Definition A.28. Let $\mathfrak{h}$ is a simple Lie algebra over $\mathbb{R}$. If $\mathfrak{h}^{\mathbb{C}}$ is also simple, we say $\mathfrak{h}$ is of *real type.* If $\mathfrak{h}^{\mathbb{C}}$ is not simple, then it splits as in Theorem A.30, and we say $\mathfrak{h}$ is of *complex type.*

Remark A.7 (Real type and complex type)*.* This definition is consistent with the definition of real, complex and hermitian type given in Proposition A.32, as we show in Corollary 2.6 that only real or complex types of simple Lie algebras are possible and that they correspond to this definition above.

Lemma A.31 (The Schur-lemma for bilinear bi-invariant forms)**.** *Continuing from Lemma A.28*

3. *Consider any two non-degenerate bilinear bi-invariant forms, B_1 and B_2, on a complex simple Lie algebra, $\mathfrak{h}$. Then they are complex multiples of each other.*

Proof. Let S be the automorphism of $\mathfrak{h}$ such that $B_1(x,y)=B_2(S(x),y)$ for all $x,\, y\in\mathfrak{h}$. Then

$$\begin{aligned} B_2(S(\mathrm{ad}(z)x),y) = B_1(\mathrm{ad}(z)x,y) &= -B_1(x,\mathrm{ad}(z)y)\\ &= -B_2(S(x),\mathrm{ad}(z)y) = B_2(\mathrm{ad}(z)S(x),y)\end{aligned}$$

$$\Rightarrow B_2(S(\mathrm{ad}(z)x)-\mathrm{ad}(z)S(x),y)=0$$

$$\forall x,\, y,\, z\in\mathfrak{h}$$

So S is $ad(x)$ invariant for all x, $\mathfrak{h}$. That is, S an intertwiner for the representation $\mathrm{ad}:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{h})$ and we can apply Lemma A.28 point 2, and, using the bilinearity, we get the result. □

A.11 The space of endomorphisms and the space of bilinear forms

This section collects some results from representation theory that can be used to classify the bilinear forms on simple Lie algebras. The main reference for this section is Di Scala et al. [7]. The notation here aims to align with this paper.

For a group G, we can define a G-module as a vector space of which there is a group representation $\phi : G \to \mathrm{GL}_n(V)$. When applying $\phi(A)x$ for $x \in V$ and $A \in G$ we will often drop the ϕ and instead write $\phi(A)x = Ax$.

If G is a Lie group and V, W two (real or complex) G-modules, the *algebra of invariant homomorphisms* is defined as

$$\mathrm{Hom}_G(V, W) := \{X \in \mathrm{Hom}(V, W) | A \circ X = X \circ A \text{ for all } A \in G\}.$$

Note this is precisely the set of intertwiners defined for groups in Definition A.26. The space of invariant endomorphisms of V, $\mathrm{End}_G(V)$, is such that $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$.

Proposition A.32. *Let G be a group and V a real irreducible G-module. Then* $\mathrm{End}_G(V)$ *is isomorphic to one of the real algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We label V as of* real, complex *or* quaterionic *type accordingly.*

Proof. One can just deduce from Schur's lemma, Lemma A.28 point 1, that $\mathrm{End}_G(V) \cong \mathrm{Aut}_G(V) \cup \{0\}$ which has the structure of a real division algebra. Then Frobenius's theorem (that the only real division algebras are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$) implies the result. However, the explicit isomorphisms are more informative than this.

Let us consider two cases: Firstly, $V^{\mathbb{C}}$, the complexification of V, is irreducible. Then $\mathrm{End}_G(V^{\mathbb{C}}) = \mathbb{C} \cdot Id$ by Schur's lemma, Lemma A.28 point 2. If $A \in \mathrm{End}_G(V)$ then $A^{\mathbb{C}} \in \mathrm{End}_G(V^{\mathbb{C}})$ so $A^{\mathbb{C}} = \lambda \cdot Id$ for some $\lambda \in \mathbb{C}$. Since $A^{\mathbb{C}}$ still leaves V invariant and V itself is invariant under conjugation, then for any $v \in V$, $v = \overline{v}$ and

$$\overline{\lambda} v = \overline{\lambda} \overline{v} = \overline{A^{\mathbb{C}} v} = A^{\mathbb{C}} v = \lambda v$$

so $\lambda = \overline{\lambda} \in \mathbb{R}$, hence $\mathrm{End}_G(V) = \mathbb{R} \cdot Id$.

Secondly, assume $V^{\mathbb{C}}$ is reducible, so $V^{\mathbb{C}} = W \oplus \overline{W}$, where $V \cong W_{\mathbb{R}}$ via an isomorphism ψ, and hence $\mathrm{End}_G(V) \cong \mathrm{End}_G(W_{\mathbb{R}})$. Any real endomorphism then decomposes uniquely into a complex linear and complex anti-linear part:

$$\mathrm{End}(W_{\mathbb{R}}) \cong \mathrm{End}(W) \oplus \mathrm{Hom}(W, \overline{W})$$

$$A = \frac{1}{2}(A + iAi) + \frac{1}{2}(A - iAi).$$

This descends to intertwiners:

$$\mathrm{End}_G(W_{\mathbb{R}}) \cong \mathrm{End}_G(W) \oplus \mathrm{Hom}_G(W, \overline{W}).$$

The Schur lemma then implies $\mathrm{End}_G(W) = \mathbb{C} \cdot Id$. Since both $W, \overline{W}$ are irreducible, $\mathrm{Hom}_g(W, \overline{W}) \subset \mathrm{Iso}(W, \overline{W}) \cup \{0\}$. If this is just $\{0\}$ then $\mathbb{C} \cdot Id = \mathrm{End}_G(V) \cong \mathrm{End}_G(W_{\mathbb{R}}) = \mathrm{span}_{\mathbb{R}}\{id, I\}$. Here $I := \psi \circ (i \cdot Id) \circ \psi^{-1}$, where ψ is the isomorphism from $W_{\mathbb{R}} \cong V$.

Otherwise, let j be a non-zero element of $\mathrm{Hom}_G(W, \overline{W})$. Schur's lemma implies j is an isomorphism. Then $j^2 \in \mathrm{End}_G(W)$ so $j^2 = \lambda \cdot Id$ for a non-zero $\lambda \in \mathbb{C}$. Since

$$\overline{\lambda} j(w) = j(\lambda w) = j(j^2(w)) = j^2(j(w)) = \lambda j(w)$$

then $\lambda = \overline{\lambda} \in \mathbb{R}$. If $\lambda > 0$ then $W_{\mathbb{R}}$ would decompose into the two $\pm\sqrt{\lambda}$-eigenspaces of $j_{\mathbb{R}}$. Thus assume $j^2 = -1$, rescaled as appropriate. For any other $A \in \mathrm{Hom}_G(W, \overline{W})$ we know $j \circ A \in \mathrm{End}_G(W)$ and hence $j \circ A = c \cdot Id$ for some $c \in \mathbb{C}$. However $j \circ (-\bar{c}j) = c \cdot Id$. As j is an isomorphism, then $A = -\bar{c}j$.

Hence finally

$$\mathrm{End}_G(W_{\mathbb{R}}) \cong \mathrm{End}_G(W) \oplus \mathrm{Hom}_G(W, \overline{W}) = \mathbb{C} \cdot Id \oplus \mathbb{C} \cdot j$$

which gives

$$\mathrm{End}_G(V) = \mathrm{span}_{\mathbb{R}}\{Id, I, J, I \circ J\}$$

with $I := \psi \circ (Id \cdot i) \circ \psi^{-1}$ and $J := \psi \circ (Id \cdot j) \circ \psi^{-1}$ both anti-commuting complex structures. $\square$

Note that we may now write any $A \in \mathrm{End}_G(V)$ as $A = \alpha Id + \beta J$ where J is a G-invariant complex structure (depending on A) and $\alpha, \beta \in \mathbb{R}$. See also Di Scala et al. [7, pgs. 633-634].

Now consider a group G acting irreducibly on V, a real vector space. Let $B_G(V)$ be the space of all G-invariant bilinear forms on V. If it is non-trivial, then via the Riesz representation theorem, we have a 1-1 correspondence between $B_G(V)$ and $\mathrm{End}_G(V)$, where $B \mapsto b = a(B(\cdot), \cdot)$ for $B \in \mathrm{End}_G(V)$ and any non-zero $a \in B_G(V)$. Note that a must be non-degenerate, otherwise it would have G-invariant kernel, which cannot happen as G acts irreducibly. This implies that $B_G(V)$ is also isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. However, one can also decompose $B_G(V)$ to symmetric ($S_G(V)$) and skew-symmetric ($\Lambda_G(V)$) forms via

$$B_G(V) = S_G(V) \oplus \Lambda_G(V).$$

This induces decomposition of $\mathrm{End}_G(V) = S^a_G(V) \oplus \Lambda^a_G(V)$. Here $S^a_G(V)$ are the a-self-adjoint operators, that is $a(B(x), y) = a(x, B(y))$, and $\Lambda^a_G(V)$ are the a-skew-adjoint operators, that is $a(B(x), y) = -a(x, B(y))$.

Remark A.8. If a is symmetric, then $S_G(V)$ corresponds to $S^a_G(V)$ as $b(x, y) = a(B(x), y) = a(y, B(x)) = a(B(y), x) = b(y, x)$. If a is skew-symmetric, then $S_G(V)$ corresponds to $\Lambda^a_G(G)$ as $b(x, y) = a(B(x), y) = -a(y, B(x)) = -a(B(y), x) = -b(y, x)$.

Proposition A.33. *If a, b, are linearly independent elements of $B_G(V)$. Then there is a $B \in \mathrm{End}_G(V)$ such that $b = a(B(\cdot), \cdot)$ and $B = \alpha Id + \beta J$ and*

1. *If both are (skew-)symmetric, then J is an element of $S^a_G(V)$ and thus J is an anti-isometry with respect to both a and b. In the symmetric case, the signatures of both a and b are $(n/2, n/2)$.*

2. *If a is symmetric, b skew, then $B = \beta J \in \Lambda^a_G(V)$ and thus J is an isometry with respect to both a and b.*

Proof. First note that Id is always a-self-adjoint and that

$$B \circ J = \alpha J + \beta J^2 = J \circ B.$$

1. If a, b are symmetric, then B is also a-self-adjoint. If a, b are skew, then $a(B(x), y) = b(x, y) = -b(y, x) = -a(B(y), x) = a(x, B(y))$ so B is also a-self-adjoint. So J is also a-self-adjoint. Hence

$$a(J(x), J(y)) = a(J^2(x), y) = -a(x, y)$$

and

$$\begin{aligned} b(J(x), J(y)) &= a(B \circ J(x), J(y)) = a(J \circ B(x), J(y)) = a(B(x), J^2(y)) \\ &= -a(B(x), y) = -b(x, y) \end{aligned}$$

and J is an anti-isometry of both a and b. Note that if x is such that $a(x, x) \neq 0$, then, without loss of generality, assume $a(x, x) > 0$, this implies that $a(J(x), J(x)) < 0$. This implies the signature result for a and similar for b.

2. Assume a symmetric and b skew. This implies that $a(B(x), y) = b(x, y) = -b(y, x) = -a(B(y), x)$. That is, B is a-skew-adjoint and hence J is also. Then $\alpha = 0$ and $b = \beta J$. Hence

$$a(J(x), J(y)) = -a(J^2(x), y) = a(x, y)$$

and

$$\begin{aligned} b(J(x), J(y)) &= a(B \circ J(x), J(y)) = a(J \circ B(x), J(y)) = -a(B(x), J^2(y)) \\ &= a(B(x), y) = b(x, y). \end{aligned}$$

□

See also Di Scala et al. [7, pg. 643].

Lemma A.34. *Let $\kappa : G \to GL(n, \mathbb{R})$ be an irreducible non-trivial representation on V a real vector space. If V is of quaterionic type, then the dimensions of $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ are both odd.*

Proof. Let a be any non-zero element of $\mathrm{End}_{\kappa(G)}(V)$. Identify $\mathrm{End}_{\kappa(G)}(V) \cong \mathbb{H}$. Clearly, $\mathrm{Re}(\mathbb{H}) = \mathbb{R} \cdot Id \subset S^a_{\kappa(G)}(V)$. On the other hand, $\Lambda^a_{\kappa(G)}(V) \subset \mathrm{Im}(\mathbb{H})$ (although not necessarily equal as can be seen in Proposition A.33). Hence $\mathrm{Im}(\mathbb{H}) = \left(S^a_{\kappa(G)}(V) \cap \mathrm{Im}(\mathbb{H})\right) \oplus \Lambda^a_{\kappa(G)}(V)$. We know that $\mathrm{Im}(\mathbb{H})$ has dimension 3 so one of these spaces must have dimension of greater than or equal to 2, and it contains two linearly independent complex-structures I, J which are anti-commuting by the properties of $\mathbb{H}$. Regardless of whether I, J are both a-self- or a-skew-adjoint, their product is always a-skew-adjoint

$$a(I \circ J(x), y) = \pm a(J(x), I(y)) = a(x, I \circ J(y)) = -a(x, J \circ I(y))$$

Hence $S^a_{\kappa(G)}(V)$ and $\Lambda^a_{\kappa(G)}(V)$ both have odd dimension for any $a \in \mathrm{End}_{\kappa(G)}(V)$. As either $S_{\kappa(G)}(V)$ or $\Lambda_{\kappa(G)}(V)$ must be non-empty, then pick a from one of these. Using the correspondence from Remark A.8, the resulting $S^a_{\kappa(G)}(V)$ and $\Lambda^a_{\kappa(G)}(V)$ imply that $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ must have odd dimension. □

See also Di Scala et al. [7, pg. 646].

A.12 Some proofs for Chapter 4

Proof of Proposition 4.2, which states that Ricci curvature of a double extension is

$$\mathrm{Ric}_{\mathfrak{d}_\pi}(\cdot,\cdot) = -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} \\ 0 & \mathrm{tr}\{\pi(\cdot)\,\mathrm{ad}_{\mathfrak{g}}(\cdot)\} & 2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)\pi(\cdot)\} \end{pmatrix}$$

and the scalar curvature is $S = S_{\mathfrak{g}}$ where $S_{\mathfrak{g}}$ is the scalar curvature of $\mathfrak{g}$.

Proof.

$$\left[\begin{pmatrix} \tilde{\alpha} \\ \tilde{X} \\ \tilde{H} \end{pmatrix}, \left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ 0 \\ 0 \end{pmatrix}\right]\right] = \begin{pmatrix} \mathrm{ad}_{\mathfrak{h}}^*(\tilde{H})(\mathrm{ad}_{\mathfrak{h}}^*(H)\widehat{\alpha}) \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} \tilde{\alpha} \\ \tilde{X} \\ \tilde{H} \end{pmatrix}, \left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ \widehat{X} \\ 0 \end{pmatrix}\right]\right] = \begin{pmatrix} \beta(\tilde{X}, [X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X}) + \mathrm{ad}_{\mathfrak{h}}^*(\tilde{H})(\beta(X,\widehat{X}) \\ [\tilde{X}, [X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X}] + \pi(\tilde{H})([X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X}) \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} \tilde{\alpha} \\ \tilde{X} \\ \tilde{H} \end{pmatrix}, \left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \widehat{H} \end{pmatrix}\right]\right] = \begin{pmatrix} \beta(\tilde{X}, -\pi(\widehat{H})X) - \mathrm{ad}_{\mathfrak{h}}^*(\widehat{H})\alpha) \\ [\tilde{X}, -\pi(\widehat{H})X] + \pi(\tilde{H})(-\pi(\widehat{H})X) - \pi([H,\widehat{H}])(\tilde{X}) \\ [\tilde{H}, [H,\widehat{H}]] \end{pmatrix}$$

From this,

$$\begin{aligned} K_{\mathfrak{d}_\pi}\left(\begin{pmatrix} \tilde{\alpha} \\ \tilde{X} \\ \tilde{H} \end{pmatrix}, \begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}\right) =& \mathrm{tr}\{\widehat{\alpha} \mapsto \mathrm{ad}_{\mathfrak{h}}^*(\tilde{H})(\mathrm{ad}_{\mathfrak{h}}^*(H)\widehat{\alpha}\} \\ &+ \mathrm{tr}\{\widehat{X} \mapsto [\tilde{X}, [X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X}] + \pi(\tilde{H})([X,\widehat{X}]_{\mathfrak{g}} + \pi(H)\widehat{X})\} \\ &+ \mathrm{tr}\{\widehat{H} \mapsto [\tilde{H}, [H,\widehat{H}]]\} \\ =& 2K_{\mathfrak{h}}(H,\tilde{H}) + K_{\mathfrak{g}}(\tilde{X}, X) + \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\tilde{X})\pi(H) + \pi(\tilde{H})(\mathrm{ad}_{\mathfrak{g}}(X) + \pi(H))\} \end{aligned}$$

and we write

$$K_{\mathfrak{d}_\pi}(\cdot,\cdot) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} \\ 0 & \mathrm{tr}\{\pi(\cdot)(\mathrm{ad}_{\mathfrak{g}}(\cdot)\} & 2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)(\cdot)\} \end{pmatrix}.$$

Hence, by Proposition 3.1. we have

$$\mathrm{Ric}_{\mathfrak{d}\pi}(\cdot,\cdot) = -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \mathrm{tr}\{\mathrm{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} \\ 0 & \mathrm{tr}\{\pi(\cdot)(\mathrm{ad}_{\mathfrak{g}}(\cdot)\} & 2K_{\mathfrak{h}}(\cdot,\cdot) + \mathrm{tr}\{\pi(\cdot)(\cdot)\} \end{pmatrix}.$$

The scalar curvature is the trace of the map

$$
-\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,\cdot) & \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} \\ 0 & \operatorname{tr}\{\pi(\cdot)(\operatorname{ad}_{\mathfrak{g}}(\cdot)\} & 2K_{\mathfrak{h}}(\cdot,\cdot)+\operatorname{tr}\{\pi(\cdot)(\cdot)\} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & g_{\mathfrak{g}}^* & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$
$$
= -\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(\cdot)\pi(\cdot)\} & K_{\mathfrak{g}}(\cdot,\cdot)g_{\mathfrak{g}}^* & 0 \\ 2K_{\mathfrak{h}}(\cdot,\cdot)+\operatorname{tr}\{\pi(\cdot)(\cdot)\} & \operatorname{tr}\{\pi(\cdot)(\operatorname{ad}_{\mathfrak{g}}(\cdot)\}g_{\mathfrak{g}}^* & 0 \end{pmatrix}.
$$

where $g_{\mathfrak{g}}$ is the metric on $\mathfrak{g}$ and $g_{\mathfrak{g}}^*$ is the corresponding metric on the dual space, such that its matrix representation is the inverse of the matrix representation of $\mathfrak{g}$. Hence $S = S_{\mathfrak{g}}$, where $S_{\mathfrak{g}}$ is the scalar curvature of $\mathfrak{g}$. ☐

The following shows the forumla for Ric^2 for a 1-dimensional double extension. This is used in the proof of Proposition 4.10.

Lemma A.35. *Let $Y_1,\dots,Y_n$ be an orthonormal basis for $\mathfrak{g}$, and let* $X_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $X_i = \begin{pmatrix}0\\Y_i\\0\end{pmatrix}$, $X_{n+1} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ *be a basis for $\mathfrak{d} = \mathbb{R}\alpha_0 \oplus \mathfrak{g} \oplus \mathbb{R}H_0$ with respect to this decomposition. Then*

$$
\begin{aligned}
\operatorname{Ric}^2(X_0, Y) &= 0 \quad \textit{for all } Z \in \mathfrak{d} \\
\operatorname{Ric}^2(X_i, X_k) &= \frac{1}{16}K_{\mathfrak{g}}(Y_i,Y_j)\epsilon_j K_{\mathfrak{g}}(Y_j,Y_k) = \operatorname{Ric}^2_{\mathfrak{g}}(Y_i,Y_k) \\
\operatorname{Ric}^2(X_i, X_{n+1}) &= \frac{1}{16}K_{\mathfrak{g}}(Y_i,Y_j)\epsilon_j \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(Y_j)\pi(1)\} \\
\operatorname{Ric}^2(X_{n+1}, X_{n+1}) &= \frac{1}{16}\operatorname{tr}\{\pi(1)\operatorname{ad}_{\mathfrak{g}}(Y_j)\}\epsilon_j \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(Y_j)\pi(1)\}.
\end{aligned}
$$

Proof. Note that

$$
\operatorname{Ric}^2_{ik} = R_i^{\ j}R_{jk} = R_{il}g^{lj}R_{jk}
$$

hence

$$
\operatorname{Ric}^2\left(\begin{pmatrix}\alpha_i,\\X_i\\H_i\end{pmatrix},\begin{pmatrix}\alpha_k\\X_k\\H_k\end{pmatrix}\right) = \frac{1}{16}\begin{pmatrix}\alpha_i & X_i & H_i\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(\cdot,X_l) & \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(\cdot)\pi(H_l)\} \\ 0 & \operatorname{tr}\{\pi(\cdot)\operatorname{ad}_{\mathfrak{g}}(X_l)\} & \operatorname{tr}\{\pi(\cdot)\circ\pi(H_l)\}\end{pmatrix}
$$
$$
\cdot\begin{pmatrix} 0 & 0 & \alpha_l(H_j) \\ 0 & \langle X_l,X_j\rangle_{\mathfrak{g}}^{-1} & 0 \\ \alpha_j(H_l) & 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{\mathfrak{g}}(X_j,\cdot) & \operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(X_j)\pi(\cdot)\} \\ 0 & \operatorname{tr}\{\pi(H_j)(\operatorname{ad}_{\mathfrak{g}}(\cdot))\} & \operatorname{tr}\{\pi(H_j)\circ\pi(\cdot)\}\end{pmatrix}\begin{pmatrix}\alpha_k\\X_k\\H_k\end{pmatrix}
$$
$$
= \frac{1}{16}(a+b+c+d)
$$

where $a = K_{\mathfrak{g}}(X_i,X_l)\langle X_l,X_j\rangle_{\mathfrak{g}}^{-1}K_{\mathfrak{g}}(X_j,X_k)$, $b = K_{\mathfrak{g}}(X_i,X_l)\langle X_l,X_j\rangle_{\mathfrak{g}}^{-1}\operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(X_j)\pi(H_k)\}$, $c = \operatorname{tr}\{\pi(H_i)(\operatorname{ad}_{\mathfrak{g}}(X_l)\}\langle X_l,X_j\rangle_{\mathfrak{g}}^{-1}K_{\mathfrak{g}}(X_j,X_k)$ and $d = \operatorname{tr}\{\pi(\tilde{H}_i)(\operatorname{ad}_{\mathfrak{g}}(X_l)\}\langle X_l,X_j\rangle_{\mathfrak{g}}^{-1}\operatorname{tr}\{\operatorname{ad}_{\mathfrak{g}}(X_j)\pi(H_k)\}$. Hence the result follows. ☐

Lemma A.36. *A derivation, $D : \mathfrak{g} \to \mathfrak{g}$, of a Lie algebra, $\mathfrak{g}$, sends the centre of the Lie algebra to the centre.*

Proof. A derivation satisfies

$$D[X, Y] = [DX, Y] + [X, DY]$$

for all $X, Y \in \mathfrak{g}$. If $X \in \mathfrak{z}(\mathfrak{g})$, then $[X, Y] = 0$ and $[X, DY] = 0$ for all $Y \in \mathfrak{g}$. Hence $[DX, Y] = 0$ for all $Y \in \mathfrak{g}$, and the result follows. □

Lemma A.37. *A surjective homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of two Lie algebra, $\mathfrak{g}$ and $\mathfrak{h}$, sends the centre of the Lie algebra to the centre.*

Proof. If $X \in \mathfrak{z}(\mathfrak{g})$, then $[X, Y] = 0$ for all $Y \in \mathfrak{g}$, which implies $0 = \phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $Y \in \mathfrak{g}$. As ϕ is surjective, $\phi(X)$ must be in the centre of $\mathfrak{h}$. □

Lemma A.38. *For $a \in \mathbb{R}$ and matrices $A, U \in \mathfrak{so}(n)$, the only solutions to the equation $aA = AU - UA$ are: $a = 0$ and $AU = UA$, or $a \in \mathbb{R}$ and $A = 0$.*

Proof. From Jost [18, pg. 278-279], the Killing form on $\mathfrak{so}(n)$ is $K(X, Y) = (n-2)\operatorname{tr}(XY)$. Note that this is negative definite. Now $aK(A, A) = K(aA, A) = K(AU - UA, A) = K([A, U], A) = -K(U, [A, A]) = 0$, so either $K(A, A) = 0$ and hence $A = 0$, or $a = 0$ and $AU = UA$. □

Lemma A.39. *Let V be a complex vector space and $T : V^n \to \mathbb{C}$ a multi-complex-linear form on V, where $n > 1$. Then there is a $v \in V$ such that $Im(T(\cdot, \ldots, \cdot, v, \cdot, \ldots)) = 0$ if and only if $Re(T(\cdot, \ldots, \cdot, v, \cdot, \ldots)) = 0$ and, hence, if and only if $T(\cdot, \ldots, \cdot, v, \cdot, \ldots) = 0$, where v is inserted into the same entry slot for each equation.*

Proof. The proof of first implication will use the complex linearity of the form and the observation that $\operatorname{Im}(iz) = \operatorname{Re}(z)$. Assume that $\operatorname{Im}(T(\cdot, \ldots, \cdot, v, \cdot, \ldots)) = 0$. Then for any arbitrary $x_1, \ldots, x_{n-1} \in V$, we have $\operatorname{Im}(T(x_1, \ldots, x_{i-1}, v, x_i, x_{n-1})) = 0$. And hence

$$\begin{aligned} 0 =& \operatorname{Im}(T(ix_1, \ldots, x_{i-1}, v, x_i, x_{n-1})) \\ =& \operatorname{Im}(iT(x_1, \ldots, x_{i-1}, v, x_i, x_{n-1}) \\ =& \operatorname{Re}(T(x_1, \ldots, x_{i-1}, v, x_i, x_{n-1})) \qquad\qquad \forall x_1, \ldots, x_{n-1} \in V \end{aligned}$$

The converse follows similarly, using the complex linearity and the observation that $\operatorname{Re}(-iz) = \operatorname{Im}(z)$. □

Remark A.9. The following is the proof of Lemma 4.17, which is: Let $\mathfrak{a}$ be an abelian Lie algebra with scalar product g of signature (p, q) where $p + q = n$ and n is the dimension of $\mathfrak{g}$. Let $\mathfrak{d}_{A_0}(\mathfrak{a})$ be a double extension by a 1-dimensional Lie algebra, where A_0 is a anti-symmetric derivation of $\mathfrak{a}$. Then the anti-symmetric derivations of $\mathfrak{d}_{A_0}(\mathfrak{a})$ are of the form

$$A = \begin{pmatrix} a & b & 0 \\ 0 & U & \tilde{b} \\ 0 & 0 & -a \end{pmatrix}$$

such that $b \in \mathbb{R}^n$, $\tilde{b} = b^t I_{p,q}$, and $U \in \mathfrak{so}(p, q)$, $A_0 U - U A_0 = a A_0$.

Proof. As A is a derivation, it is a linear map between vector spaces and hence we write A as matrix

$$A = \begin{pmatrix} a & b_1 & b_2 & c \\ d_1 & e_1 & f_1 & g_1 \\ d_2 & e_2 & f_2 & g_2 \\ h & i_1 & i_2 & j \end{pmatrix}$$

with respect to the decomposition $\mathfrak{d}_{A_0}(\mathfrak{a}) = \mathbb{R} \oplus \mathbb{R}^p \oplus \mathbb{R}^q \oplus \mathbb{R}$. To be antisymmetric, the matrix must satisfy

$$\langle AZ, Y\rangle_{\mathfrak{d}(\mathfrak{a})} = -\langle Z, AY\rangle_{\mathfrak{d}(\mathfrak{a})}$$

for all $Z, Y \in \mathfrak{os}$ which is equlivalent to

$$-g^{-1}A^t g = A \text{ where } g = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & I_{p,q} & 0 \\ 1 & \cdots & 0 \end{pmatrix}.$$

We find that

$$\begin{pmatrix} a & b_1 & b_2 & c \\ d_1 & e_1 & f_1 & g_1 \\ d_2 & e_2 & f_2 & g_2 \\ h & i_1 & i_2 & j \end{pmatrix} = \begin{pmatrix} -j & g_1^t & -g_2^t & -c \\ i_1^t & -e_1^t & e_2^t & b_1^t \\ -i_2^t & f_1^t & -f_2^t & -b_2^t \\ -h & d_1^t & -d_2^t & -a \end{pmatrix}.$$

We will relabel

$$A = \begin{pmatrix} a & b & 0 \\ c & U & \tilde{b} \\ 0 & \tilde{c} & -a \end{pmatrix}.$$

Where $\tilde{b} = b^t I_{p,q}$ and $\tilde{c} = c^t I_{p,q}$, and $U \in \mathfrak{so}(p,q)$. To satisfy the Liebniz rule, we require

$$A[Z,Y]_{\mathfrak{os}} = [AZ, Y]_{\mathfrak{os}} + [Z, AY]_{\mathfrak{os}}$$

for all $Z, Y \in \mathfrak{os}$. Now

$$\left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix} \right]_{\mathfrak{os}} = \begin{pmatrix} \beta(X, \widehat{X}) \\ \pi(H)\widehat{X} - \pi(\widehat{H})X \\ 0 \end{pmatrix} = \begin{pmatrix} X^t A_0^t I_{p,q} \widehat{X} \\ HA_0\widehat{X} - \widehat{H}A_0 X \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} A\left[\begin{pmatrix} \alpha \\ X \\ H \end{pmatrix}, \begin{pmatrix} \widehat{\alpha} \\ \widehat{X} \\ \widehat{H} \end{pmatrix} \right]_{\mathfrak{os}} &= \begin{pmatrix} a & b & 0 \\ c & U & \tilde{b} \\ 0 & \tilde{c} & -a \end{pmatrix} \begin{pmatrix} X^t A_0^t I_{p,q} \widehat{X} \\ HA_0\widehat{X} - \widehat{H}A_0 X \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} aX^t A_0^t I_{p,q}\widehat{X} + b(HA_0\widehat{X} - \widehat{H}A_0X) \\ cX^t A_0^t \widehat{X} + U(HA_0\widehat{X} - \widehat{H}A_0X) \\ \tilde{c}(HA_0\widehat{X} - \widehat{H}A_0X) \end{pmatrix}, \end{aligned}$$

and

$$\left[A\begin{pmatrix}\alpha\\X\\H\end{pmatrix},\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}+\left[\begin{pmatrix}\alpha\\X\\H\end{pmatrix},A\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}$$

$$=\left[A\begin{pmatrix}a\alpha+bX\\c\alpha+UX+\tilde{b}H\\\tilde{c}X-aH\end{pmatrix},\begin{pmatrix}\widehat{\alpha}\\\widehat{X}\\\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}+\left[\begin{pmatrix}\alpha\\X\\H\end{pmatrix},A\begin{pmatrix}a\widehat{\alpha}+b\widehat{X}\\c\widehat{\alpha}+U\widehat{X}+\tilde{b}\widehat{H}\\\tilde{c}\widehat{X}-a\widehat{H}\end{pmatrix}\right]_{\mathfrak{os}}$$

$$=\begin{pmatrix}(\alpha c^t+X^tU^t+H\tilde{b}^t)A_0^tI_{p,q}\widehat{X}+X^tA_0^tI_{p,q}(\widehat{\alpha}c+U\widehat{X}+\widehat{H}\tilde{b})\\(\tilde{c}X-aH)A_0\widehat{X}-\widehat{H}A_0(c\alpha+UX+\tilde{b}H)+HA_0(\widehat{\alpha}c+U\widehat{X}+\widehat{H}\tilde{b})-(\tilde{c}\widehat{X}-a\widehat{H})A_0X\\0\end{pmatrix}.$$

Equating the two, we immediately see that $c=0=\tilde{c}$, which we note can also be seen as a derivation must map the centre of a Lie algebra into the centre. We are left with

$$\begin{pmatrix}aX^tA_0^tI_{p,q}\widehat{X}+b(HA_0\widehat{X}-\widehat{H}A_0X)\\U(HA_0\widehat{X}-\widehat{H}A_0X)\\0\end{pmatrix}$$
$$=\begin{pmatrix}(X^tU^t+H\tilde{b}^t)A_0^tI_{p,q}\widehat{X}+X^tA_0^tI_{p,q}(U\widehat{X}+\widehat{H}\tilde{b})\\-aHA_0\widehat{X}-\widehat{H}A_0(UX+\tilde{b}H)+HA_0(U\widehat{X}+\widehat{H}\tilde{b})+a\widehat{H}A_0X\\0\end{pmatrix}.$$

Cancelling any repeated terms we have

$$\begin{pmatrix}aX^tA_0^tI_{p,q}\widehat{X}+b(HA_0\widehat{X}-\widehat{H}A_0X)\\U(HA_0\widehat{X}-\widehat{H}A_0X)\\0\end{pmatrix}=\begin{pmatrix}(X^tU^t+H\tilde{b}^t)A_0^tI_{p,q}\widehat{X}+X^tA_0^tI_{p,q}(U\widehat{X}+\widehat{H}\tilde{b})\\-aHA_0\widehat{X}-\widehat{H}A_0UX+HA_0U\widehat{X}+a\widehat{H}A_0X\\0\end{pmatrix}.$$

If we pick $H=\widehat{H}=0$ then from the top entries we have

$$\begin{aligned}aX^tA_0^tI_{p,q}\widehat{X}=&X^tU^tA_0^tI_{p,q}\widehat{X}+X^tA_0^tI_{p,q}U\widehat{X}\\=&X^t(U^tA_0^tI_{p,q}+A_0^tI_{p,q}U)\widehat{X}\\\Rightarrow aA_0^tI_{p,q}=&U^tA_0^tI_{p,q}+A_0^tI_{p,q}U\\\Rightarrow aA_0=&A_0U-UA_0\qquad\qquad A_0,U\in\mathfrak{so}(p,q).\end{aligned}\tag{A.12.1}$$

Note that when picking any other variables and equating to 0, they either are already equal or reduce to the same condition as above. Hence the result follows. □

A.13 Notes on proof of Lemma 3.15

A.13.1 Abstract Jordan decomposition

For a complex Lie algebra $\mathfrak{g}$, a derivation $D\in\mathrm{der}(\mathfrak{g})$ and an element $\lambda\in\mathbb{C}$ we can define the *generalised eigenspace* as

$$\mathfrak{g}_\lambda(D):=\{x\in\mathfrak{g}\mid(D-\lambda)^mx=0\text{ for some }m\in\mathbb{N}\}$$

Lemma A.40. *For any $\lambda_1, \lambda_2 \in \mathbb{C}$ we have*

$$[\mathfrak{g}_{\lambda_1}(D), \mathfrak{g}_{\lambda_2}(D)] \subset \mathfrak{g}_{\lambda_1+\lambda_2}$$

Proof. We first show by induction that

$$(D - (\lambda_1 + \lambda_2))^p[x, y] = \sum_{j=0}^{p} \binom{p}{j} [(D - \lambda_1)^j x, (D - \lambda_2)^{p-j}]$$

where $x, y \in \mathfrak{g}$. Consider that $p = 0$ holds trivially. Assume true for $p = k$ and consider the $k + 1$ case:

$$\begin{aligned}
&[(D - (\lambda_1 + \lambda_2))^{k+1}[x, y] \\
&\quad = \sum_{j=0}^{k} \binom{k}{j} (D - \lambda_1 - \lambda_2)[(D - \lambda_1)^j x, (D - \lambda_2)^{k-j}] \\
&\quad = \sum_{j=0}^{k} \binom{k}{j} \Big([D(D - \lambda_1)^j x, (D - \lambda_2)^{k-j}] + [(D - \lambda_1)^j x, D(D - \lambda_2)^{k-j}] \\
&\qquad - \lambda_1[(D - \lambda_1)^j x, (D - \lambda_2)^{k-j}] - \lambda_2[(D - \lambda_1)^j x, (D - \lambda_2)^{k-j}] \Big) \\
&\quad = \sum_{j=0}^{k} \binom{k}{j} \Big([(D - \lambda_1)^{x+1}, (D - \lambda_2)^{k-j}] + [(D - \lambda_1)^j x, (D - \lambda_2)^{k+1-j}] \\
&\quad = \sum_{j=0}^{k+1} \binom{k+1}{j} [(D - \lambda_1)^j x, (D - \lambda_2)^{k+1-j}]
\end{aligned}$$

and the induction holds.

Now consider $x \in \mathfrak{g}_{\lambda_1}(D)$ and $y \in \mathfrak{g}_{\lambda_2}(D)$, such that $(D-\lambda_1)^m x = 0$ and $(D-\lambda_2)^n y = 0$ for some $n, m \in \mathbb{N}$. Then let $p = m + n$. If $i \in \{0, 1, \ldots, n\}$ then either $i \geq m$ or $i < m$ whcih implies that $-i > -m$ and hence that $k - i > n$. Then one of $(D - \lambda_1)^i x$ and $(D - \lambda_2)^{k-i}$ are 0. Consider then that

$$(D - (\lambda_1 + \lambda_2))^k[x, y] = \sum_{j=0}^{k} \binom{k}{j} [(D - \lambda_1)^j x, (D - \lambda_2)^{k-j}] = 0,$$

which implies that $[x, y] \in \mathfrak{g}_{\lambda_1+\lambda_2}(D)$ and hence the result follows. □

This standard result can be found in Serre [34] using only inner derivations however.

Lemma A.41. *If $\mathfrak{g}$ is a complex Lie algebra and $D \in \operatorname{der}(\mathfrak{g})$ has the Jordan decomposition into $D = S + N$ for S diagonal and N nilpotent, then S, N are derivations of $\mathfrak{g}$*

Proof. Let λ_1 and λ_2 be two eigenvalues of S. Note that these are also eigenvalues of D and that the eigenspace of S corresponding to each λ_i, which we denote $E_{\lambda_i}(S)$, is equal to $\mathfrak{g}_{\lambda_i}(S)$ and equal to $\mathfrak{g}_{\lambda_i}(D)$. (This is due to properties of the Jordan Decomposition from linear algebra). Then let $x_i \in E_{\lambda_i}(S)$ then by Lemma A.40, we have

$$S[x_1, x_2] = (\lambda_i + \lambda_j)[x_1, x_2] = [\lambda_1 x_1, x_2] + [x_1, \lambda_2 x_2] = [Sx_1, x_2] + [x_1, Sx_2].$$

As S is a derivation, $N = D - S$ must be too. □

Theorem A.42 (Abstract Jordan Decompostion)**.** *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $X \in \mathfrak{g}$. Then there is elements $S, N \in \mathfrak{g}$ such that*

- $X = S + N$
- ad_S *is diagonalisable and* ad_N *is nilpotent*
- $[S, N] = 0$

Further more, if $[X, Y] = 0$ for a $Y \in \mathfrak{g}$ then $[S, Y] = [N, Y] = 0$.

Proof. Using Lemma A.41, we can decompose $\mathrm{ad}_X = D_S + D_N$, such that D_S is diagonalisable derivation and N is nilpotent derivation. By Proposition A.12, these derivations are inner, hence there is $S, N \in \mathfrak{g}$ such that $D_S = \mathrm{ad}_S$ and $D_N = \mathrm{ad}_N$ and $\mathrm{ad}_X = \mathrm{ad}_S + \mathrm{ad}_N = \mathrm{ad}_{S+N}$. As ad is injective, then $X = S + N$. As ad is a homomorphism, we also have

$$\mathrm{ad}_{[S,N]} = [\mathrm{ad}_S, \mathrm{ad}_N] = [D_S, D_N] = 0$$

and hence $[S, N] = 0$ again by injectivity of ad.

From Jordan decomposition, there are polynomials $p(x), q(x)$ such that $\mathrm{ad}_S = p(\mathrm{ad}_X)$ and $\mathrm{ad}_N = q(\mathrm{ad}_X)$. As ad_N is nilpotent, the constant term of q must be $q_0 = 0$. Now if $Y \in \mathfrak{g}$ such that $[X, Y] = 0$ then

$$\mathrm{ad}_S(Y) = \mathrm{ad}_X(Y) - \mathrm{ad}_N(Y) = 0 - q(\mathrm{ad}_X(Y)) = -q(0) = -q_0 = 0$$

which proves the last statement. □

A reference for the Jordan decomposition can be found in §16.6 of Erdmann and Wildon [9, pg. 200]

A.13.2 Cartan subalgebras and toral subalgebras

Definition A.29. A *toral* subalgebra, $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an abelian subalgebra such that ad_X are diagonalisable for each $X \in \mathfrak{h}$. Such X are called *semisimple elements* of $\mathfrak{g}$. A *maximal toral subalgebra* is a toral subalgebra that is not properly contained in a larger toral subalgebra.

Definition A.30. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra such that it is equal to its own normaliser, that is

$$\mathfrak{h} = \mathfrak{n}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [X, H] \in \mathfrak{h} \forall H \in \mathfrak{h}\}$$

Theorem A.43. *Every complex semisimple Lie algebra $\mathfrak{g}$ admits a non-trivial toral subalgebra.*

Proof. By the Jordan decomposition theorem, for any abritary $X \in \mathfrak{g}$ we can write $X = S + N$ where ad_S is diagonalisable and ad_N is nilpotent, and $S, N \in \mathfrak{g}$. If $S = 0$ for all $X \in \mathfrak{g}$ then $\mathfrak{g}$ must be a nilpotent Lie algebra by Engel's theorem Theorem A.21. Hence there is at least one non-zero element of $\mathfrak{g}$ such that its adjoint is diagonisable. The span of this element is a non-trivial toral subalgebra. □

Proposition A.44. *Let $\mathfrak{h}$ be a sub algebra of a semi-simple Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra if and only if it is a maximal toral sub algebra of $\mathfrak{g}$. The Killing form is also non-degenerate on $\mathfrak{h}$.*

For the proof see Serre [34] or for more details, see Exercise 3 for §2 of Bourbaki [5, pg. 55].

Lemma A.45. *If X is a semisimple element of a semisimple Lie algebra $\mathfrak{g}$ then there is a Cartan subalgebra that contains it.*

Proof. This is essentially a corollary of Proposition A.44. As $\mathrm{span}(X)$ is a 1-dimensional total subalgebra, it is contained in a maximal toral sub algebra. This maximal toral subalgebra is thence a Cartan sub algebra that contains X. □

Let $\mathfrak{h}$ be a Cartan sub algebra of $\mathfrak{g}$. For each $\alpha \in \mathfrak{h}^*$, define $\mathfrak{h}^\alpha$ as the corresponding eigenspace, that is

$$\mathfrak{h}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)x \ \forall H \in \mathfrak{h}\}$$

$\alpha \in \mathfrak{h}$ is called a root of $\mathfrak{g}$ if it is non-zero and such that $\mathfrak{h}_\alpha$ is non-empty. We call Δ the set of roots of $\mathfrak{g}$. Serre [34, pg. 44] gives the following result, which can also be found in

Theorem A.46. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}$. Then, as a direct sum of vector spaces, one has*

$$\mathfrak{g} = \mathfrak{h}_0 \oplus^\perp \overset{\perp}{\bigoplus_{\alpha \in \Delta}} \{\mathfrak{h}_\alpha \oplus \mathfrak{h}_{-\alpha}\}$$

Here orthogonality is taken with respect to the Killing form, and the Killing form is non-degenerate on $\mathfrak{h}_0$.

Remark A.10. The only simple Lie algebra with 1-dimensional Catan subalgebra is $\mathfrak{sl}_2\mathbb{R}$. This is demonstrated in Serre [34, pg. 26]. All other Lie algebras have Cartan subalgebras of larger dimension.

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