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QCD evolution of the spin structure functions of the neutron and proton

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Models of the long-distance behavior of QCD can be used to make predictions for the twist-two piece of the current correlation function appearing in deep-inelastic scattering. The connection between the model predictions, valid at a four-momentum transfer squared Q^2 of the order of 1 $(\mathrm{GeV}/c)^2$, and the data, collected at much higher Q^2, is provided by perturbative QCD. We present here the details of a method, previously used for spin-averaged lepton scattering, applied to the spin-dependent case. We then use the method to make predictions for the spin structure functions g_1^{proton} and g_1^{neutron}.

I. INTRODUCTION

For some years now QCD has been the leading candidate for a theory of the strong interaction. Perturbative QCD (PQCD) has provided us with a unifying framework for studying all high-energy scattering phenomena involving hadrons and, because of its property of asymptotic freedom, has given us an *a posteriori* justification of the popular parton model.[1]

At finite Q^2 moreover PQCD predicts logarithmic deviations from exact scaling. These are usually derived using the renormalization group[2] which makes predictions for the nth moment of the structure functions, or equivalently (at least at leading order) the quark and gluon distributions, of the form

$$q(n,Q^2)=A_n C_n(Q^2/\mu^2,g_c)+\text{higher twists}\ , \qquad (1)$$

where

$$C_n(Q^2/\mu^2,g_c)=C_n(1,\bar{g})\exp\left[-\int_{g_c}^{\bar{g}} dg\,\gamma_n(g)/\beta(g)\right]\ . \qquad (2)$$

Here A_n is a matrix element which is dependent on the quark distribution under consideration and the C_n's are coefficient functions which give rise to the Q^2 dependence of the moments $q(n,Q^2)$. $\bar{g}$ is the running coupling constant while g_c is the coupling constant at the reference scale μ. γ_n and β are the usual anomalous dimensions and the Callan-Symanzik functions, respectively.[1]

In perturbation theory one typically expands C_n, γ_n, and β up to some power of the coupling constant and evaluates them to this order. This was done for the unpolarized case by Gross and Wilczek, Politzer, and others up to first order[3,4] and later up to second order by Floratos, Ross, and Sachrajda, and others.[5-7]

For polarized scattering the calculations to leading order have been done by Ahmed and Ross and by Ito and Sasaki.[8] At next-to-leading order the coefficients C_n as well as γ_1 have been calculated by Kodaira *et al.*[9] For general n, the anomalous dimensions have not been calculated beyond leading order.

The works mentioned above give direct predictions only for the moments of the distribution functions. This is somewhat unfortunate as the experimentally measured quantities are the actual distributions themselves which typically vary in accuracy as a function of x and are only measured down to some finite small x, with a subsequent extrapolation to $x=0$.[10,11] Hence the extraction of the moments is subject to some uncertainty and there may be considerable correlation between the errors on different moments. Moreover, in the case of polarized scattering a great deal of attention has recently been given to the QCD evolution of the first moment of the spin structure function of the proton,[12] while relatively little attention has been paid to the consistent evolution of the other moments.[13] As we have previously pointed out,[14] consideration of the latter in fact makes it unlikely that the small value of $\int_0^1 g_1^p(x,Q^2=10\ \mathrm{GeV}^2)dx$ measured by the European Muon Collaboration (EMC) can be reconciled with the picture that at low energies the proton consists exclusively of three nonstrange valence quarks.[15]

In the case of unpolarized scattering, the evolution of the relevant distributions has been calculated in a variety of ways. Originally, at first order, Gross and others[16] reconstructed the nonsinglet quark distribution using asymptotic expansions of the anomalous dimensions and the inverse Mellin transform technique. Other methods have relied on parametrizations of the structure functions which have then been evolved by numerically solving[17] the integrodifferential Altarelli-Parisi[18] equations, by using the method of Laguerre polynomials[19] or by using particular parametrizations of the quark and glue distributions.[20] In this paper we shall use the method of Gonzales-Arroyo *et al.*, the details of which can be found in a series of papers, starting with the second-order evolution of the unpolarized nonsinglet[6] and leading to the more complicated case of the singlet.[7]

In Sec. II we present their method applied to the polarized case. Because the anomalous dimensions for the polarized singlet have not been calculated at second order we restrict ourselves to leading order only. As is the case in Refs. 6 and 7, we expect the generalization to second order to be straightforward. Higher-order effects may be important, because of both the considerable distance between the model scale and the experimental momentum transfers, and because the model scale is rather close to the confinement scale $\Lambda_{\rm QCD}$. For these reasons our results should be viewed as indicators of general trends which may be subject to some higher-order modifications.

Because of its speed the method described in Sec. II is particularly useful for comparing model predictions of structure functions, which apply typically at some low scale $Q^2=\mu^2$, to data at higher Q^2 (in our case $Q^2=10$ GeV^2). This is because the model-dependent scale μ is *a priori* unknown and has to be obtained by fitting some characteristic prediction of the model to data. In Ref. 21, for example, the scale of a model where the quarks carry all the momentum of the proton is extracted from the known momentum fraction that they carry at $Q^2=10$ GeV^2. In Sec. III we shall present the results for a particular model where we use the shape of the spin-independent valence distribution to determine μ, enabling us to make predictions for $g_1(x,Q^2)$ of the proton and neutron.

II. THE RECONSTRUCTION VIA THE BERNSTEIN POLYNOMIALS

We describe here the method of Refs. 6 and 7, applied to the spin-dependent quark and glue distributions. To illustrate the method we take the case of the spin-dependent nonsinglet (which, because of parity invariance, is identical to that of the spin-independent nonsinglet) and then give the straightforward generalization to the singlet. The main result of this paper is the calculation of sets of coefficients, given in Eq. (23) and in the Appendix. The reader familiar with the method may want to proceed directly to these results.

We suppose that we have a nonsinglet quark distribution

$$\Delta q_{\rm NS}(x,Q^2)\equiv q^{\uparrow}_{\rm NS}(x,Q^2)-q^{\downarrow}_{\rm NS}(x,Q^2) \tag{3}$$

whose nth moment

$$\Delta q_{\rm NS}(n,Q^2)\equiv \int_0^1 x^{n-1}\Delta q_{\rm NS}(x,Q^2)dx \tag{4}$$

obeys the evolution equation (1). To first order this yields

$$\Delta q_{\rm NS}(n,Q^2)=\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{-A_{qq}(n)/(2\beta_0)}\Delta q_{\rm NS}(n,\mu^2)\ . \tag{5}$$

Here $q^{\uparrow(\downarrow)}_{\rm NS}(x,Q^2)$ is the distribution of quarks with helicity equal (opposite) to that of the parent hadron, $\alpha(Q^2)$ is the strong coupling constant at the scale Q^2, $A_{qq}(n)$ is the (lowest-order) anomalous dimension arising from the polarized quark-quark splitting function and β_0 is the lowest-order coefficient in the expansion of the Callan-Symanzik function and is given by, for SU(3) color,

$$\beta_0=\frac{33-2f}{12}\ , \tag{6}$$

where f is the number of flavors. In order to reconstruct the distribution $\Delta q_{\rm NS}(x,Q^2)$ from $\Delta q_{\rm NS}(n,Q^2)$ we write

$$\Delta q_{\rm NS}(x,Q^2)=\int_0^1 \Delta(y-x)\Delta q_{\rm NS}(y,Q^2)dy\ , \tag{7}$$

where $\Delta(y-x)$ must have the property

$$f(x)=\int_0^1 \Delta(y-x)f(y)dy\ . \tag{8}$$

A sufficient, but not necessary, choice for $\Delta(y-x)$ is the Dirac δ function. We however use functions related to the Bernstein polynomials

$$\begin{aligned}\Delta(y-x)&=\lim_{\substack{N,k\to\infty\\ k/N\to x}}\frac{(N+1)!}{(N-k)!k!}y^k(1-y)^{N-k}\\ &=\lim_{\substack{N,k\to\infty\\ k/N\to x}}\frac{(N+1)!}{k!}\sum_{l=0}^{N-k}\frac{(-1)^l y^{k+l}}{l!(N-k-l)!}\ .\end{aligned} \tag{9}$$

By integrating over y with a test function y^c, or otherwise,[22] it can readily be shown that $\Delta(y-x)$ exhibits the behavior of a δ function in the range $0\le y\le 1$ (it need not be symmetric in x and y for our purposes, nor does it have to behave like a δ function for y outside this interval). Using Eqs. (9) and (5) in (7), we obtain

$$\Delta q_{\rm NS}(x,Q^2)=\lim_{\substack{N,k\to\infty\\ k/N\to x}}\frac{(N+1)!}{k!}\sum_{l=0}^{N-k}\frac{(-1)^l}{l!(N-k-l)!}\int_0^1 y^{k+l}\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{-A_{qq}(l+k+1)/(2\beta_0)}\Delta q_{\rm NS}(y,\mu^2)dy\ . \tag{10}$$

As has been pointed out in Refs. 6 and 7, it is dangerous to sum Eq. (10) numerically because of the oscillatory nature due to the appearance of the $(-1)^l$. Because of the form of $A_{qq}(l+k+1)$ it is not possible to do the sum in Eq. (10) analytically. However, if one uses the expansion

$$r^{-A_{qq}(n)/(2\beta_0)}=\sum_{i,j}\frac{c(i,j)}{(p+n)^{i+\rho}[\ln a(n+p)]^j} \tag{11}$$

it is in fact possible to perform the sum analytically for each individual term in Eq. (11). In Eq. (11), $r=[\alpha(Q^2)/\alpha(\mu^2)]$, and a, ρ, and p are arbitrary constants; later on we shall use $a=e^{C-3/4}$ and $p=2$, where C is Euler's constant. ρ is dependent on the particular expansion and is a function of r. We then find that

$$\lim_{\substack{N,k\to\infty\\ k/N\to x}}\frac{(N+1)!}{k!}\sum_{l=0}^{N-k}\frac{(-1)^l}{l!(N-k-l)!}\frac{y^{k+l}}{(p+k+l)^{i+\rho}}=\frac{\theta(y-x)}{\Gamma(i+\rho)y}\left[\frac{x}{y}\right]^{p-1}\left[\ln\frac{y}{x}\right]^{i+\rho-1}\ , \tag{12}$$

and

$$\lim_{\substack{N,k\to\infty\\ k/N\to x}} \frac{(N+1)!}{k!}\sum_{l=0}^{N-k}\frac{(-1)^l}{l!(N-k-l)!}\frac{y^{k+l}}{(p+k+l)^{i+\rho}[\ln a(p+k+l)]^j}$$

$$=\frac{\theta(y-x)}{\Gamma(i+\rho)y}\left[\frac{x}{y}\right]^{p-1}\left[\ln\frac{y}{x}\right]^{i+\rho-1}\int_0^\infty ds\frac{s^{j-1}a^{-s}\left[\ln\frac{y}{x}\right]^s}{\Gamma(i+\rho+s)} . \tag{13}$$

Hence we obtain

$$\Delta q_{\rm NS}(x,Q^2)=\int_x^1\frac{dy}{y}b_{qq}(x/y,r)\Delta q_{\rm NS}(y,\mu^2) , \tag{14}$$

with

$$b_{qq}(x/y,r)=\left[\frac{x}{y}\right]^{(p-1)}\sum_i\left[\ln\frac{y}{x}\right]^{(i+\rho-1)}\left[\frac{c_{qq}(i,0)}{\Gamma(i+\rho)}+\sum_{j\neq 0}\frac{c_{qq}(i,j)}{\Gamma(j)}\int_0^\infty ds\frac{s^{j-1}a^{-s}\left[\ln\frac{y}{x}\right]^s}{\Gamma(i+\rho+s)}\right] . \tag{15}$$

By direct reevaluation of the moments of Eq. (14) it is easy to check that Eq. (5) is indeed satisfied. For the particular case of the nonsinglet shown here, the expression simplifies considerably because the coefficients $c_{qq}(i,j)$ are 0 for $j\neq 0$. The details of the QCD evolution are contained in the b coefficients and are independent of the input distributions $\Delta q_{\rm NS}(y,\mu^2)$. This makes the method fast as one only needs to evaluate the b's once for a suitable set of x/y and r values [this is of particular use for the singlet where the coefficients $c_{qq}(i,j)$ are not 0 for $j\neq 0$].

In the case of the singlet distributions the differential equation governing the evolution of its moments is given by

$$\frac{d}{d\ln Q^2}\begin{bmatrix}\Delta\Sigma(n,Q^2)\\ \Delta g(n,Q^2)\end{bmatrix}=\frac{\alpha(Q^2)}{2\pi}\begin{bmatrix}A_{qq}(n) & A_{qg}(n)\\ A_{gq}(n) & A_{gg}(n)\end{bmatrix}\begin{bmatrix}\Delta\Sigma(n,Q^2)\\ \Delta g(n,Q^2)\end{bmatrix} . \tag{16}$$

Here $\Delta g(n,Q^2)$ are the moments of the glue distributions analogous to Eqs. (3) and (4). The A's are proportional to the anomalous dimensions and for three flavors are given by

$$\begin{aligned}
A_{qq}(n)&=\frac{4}{3}\left[\frac{3}{2}+\frac{1}{n(n+1)}-\frac{2}{n}-2\sum_{j=1}^{n-1}\frac{1}{j}\right] ,\\
A_{qg}(n)&=3\left[\frac{n-1}{n(n+1)}\right] ,\\
A_{gq}(n)&=\frac{4}{3}\left[\frac{n+2}{n(n+1)}\right] ,\\
A_{gg}(n)&=3\left[\frac{3}{2}+\frac{2}{n}-\frac{4}{n+1}-2\sum_{j=1}^{n-1}\frac{1}{j}\right] .
\end{aligned} \tag{17}$$

Equation (16) has the solution

$$\begin{aligned}
\Delta\Sigma_n(Q^2)=&\left[w_1(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_+(n)}+w_2(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_-(n)}\right]\Delta\Sigma_n(\mu^2)\\
&+\left[-w_3(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_+(n)}+w_3(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_-(n)}\right]\Delta g_n(\mu^2) ,\\
\Delta g_n(Q^2)=&\left[-w_4(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_+(n)}+w_4(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_-(n)}\right]\Delta\Sigma_n(\mu^2)\\
&+\left[w_2(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_+(n)}+w_1(n)\left[\frac{\alpha(Q^2)}{\alpha(\mu^2)}\right]^{t_-(n)}\right]\Delta g_n(\mu^2) ,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&w_{1,2}(n)=\frac{1}{2}\left[1\mp\frac{A_-(n)}{R(n)}\right],\quad w_3(n)=\Delta A_{qg}(n)/R(n) ,\\
&w_4(n)=\Delta A_{gq}(n)/R(n),\quad t_\pm(n)=\tfrac{1}{9}[-A_+(n)\pm R(n)]
\end{aligned} \tag{19}$$

and

$$A_{\pm}(n)=\Delta A_{qq}(n)\pm\Delta A_{gg}(n),\quad R(n)=\sqrt{A_{-}^{2}(n)+4\Delta A_{qg}(n)\Delta A_{gq}(n)}\;. \tag{20}$$

Finally, for the singlets we obtain the following expressions analogous to (14):

$$\Delta\Sigma(x,Q^2)=\int_x^1\frac{dy}{y}[b_{qq}(x/y,r)\Delta\Sigma(y,\mu^2)+b_{qg}(x/y,r)\Delta g(y,\mu^2)] \tag{21}$$

and

$$\Delta g(x,Q^2)=\int_x^1\frac{dy}{y}[b_{gq}(x/y,r)\Delta\Sigma(y,\mu^2)+b_{gg}(x/y,r)\Delta g(y,\mu^2)]\;, \tag{22}$$

where $b_{qg}(x/y,r)$, $b_{gq}(x/y,r)$, and $b_{gg}(x/y,r)$ are equivalent to $b_{qq}(x/y,r)$ given in Eq. (15) with $c_{qqk}(i,j)$ replaced by $c_{qgk}(i,j)$, $c_{gqk}(i,j)$, and $c_{ggk}(i,j)$, respectively. Here $k=\pm$ distinguishes the two sets of expansions necessary for each series of coefficients corresponding to the two distinct powers t_+ and t_- appearing in Eq. (18).

Choosing $p=2$, and using

$$\sum_{j=1}^{n-1}\frac{1}{j}=C+\Psi(n)=C+\ln n-\frac{1}{2n}-\frac{1}{12n^2}+\cdots$$

one arrives at the expansions

$$\begin{aligned}
r^{-A_{qq}(n)/(2\beta_0)}&=(az)^{16u/27}\left[1-\frac{8u}{9z}+\left[\frac{-76u}{81}+\frac{32u^2}{81}\right]\frac{1}{z^2}+O(z^{-3})\right]\,,\\
w_1(n)r^{t_+(n)}&=(az)^{4u/3}\left[\frac{1}{(\ln az)^2}\left[\frac{9}{25z^2}\right]+O((z,\ln az)^{-3})\right]\,,\\
w_1(n)r^{t_-(n)}&=(az)^{16u/27}\left[\frac{1}{(\ln az)^2}\left[\frac{9}{25z^2}\right]+O((z,\ln az)^{-3})\right]\,,\\
w_2(n)r^{t_+(n)}&=(az)^{4u/3}\left[1-\frac{2u}{z}+\left[\frac{-37u}{9}+2u^2\right]\frac{1}{z^2}+\frac{1}{\ln az}\frac{4u}{15z^2}-\frac{1}{(\ln az)^2}\frac{9}{25z^2}+O((z,\ln az)^{-3})\right]\,,\\
w_2(n)r^{t_-(n)}&=(az)^{16u/27}\left[1-\frac{8u}{9z}+\left[\frac{-76u}{81}+\frac{32u^2}{81}\right]\frac{1}{z^2}-\frac{1}{\ln az}\frac{4u}{15z^2}-\frac{1}{(\ln az)^2}\frac{9}{25z^2}+O((z,\ln az)^{-3})\right]\,,\\
w_3(n)r^{t_+(n)}&=(az)^{4u/3}\left[\frac{1}{\ln az}\left[-\frac{9}{10z}+\frac{9u}{5z^2}\right]-\frac{1}{(\ln az)^2}\frac{27}{20z^2}+O((z,\ln az)^{-3})\right]\,,\\
w_3(n)r^{t_-(n)}&=(az)^{16u/27}\left[\left[\frac{1}{\ln az}\right]\left[\frac{9}{10z}-\frac{4u}{5z^2}\right]+\left[\frac{1}{(\ln az)^2}\right]\left[\frac{27}{20z^2}\right]+O((z,\ln az)^{-3})\right]\,,\\
w_4(n)r^{t_+(n)}&=(az)^{4u/3}\left\{\left[\frac{1}{\ln az}\right]\left[\frac{-2}{5z}+\left[\frac{-6}{5}+\frac{4u}{5}\right]\frac{1}{z^2}\right]+\left[\frac{1}{(\ln az)^2}\right]\left[\frac{3}{5z^2}\right]+O((z,\ln az)^{-3})\right\}\,,\\
w_4(n)r^{t_-(n)}&=(az)^{16u/27}\left\{\left[\frac{1}{\ln az}\right]\left[\frac{2}{5z}+\left[\frac{6}{5}-\frac{16u}{45}\right]\frac{1}{z^2}\right]+\left[\frac{1}{(\ln az)^2}\right]\left[\frac{3}{5z^2}\right]+O((z,\ln az)^{-3})\right\}\,.
\end{aligned} \tag{23}$$

Here $a=e^{C-3/4}$, $z=n+2$, $u=\ln r=\ln[\alpha(Q^2)/\alpha(\mu^2)]$, and $O((z,\ln az)^{-3})$ imply that terms of order $(\ln az)^{-3}$ have been omitted. Any such terms are also at least of order z^{-3}. Terms of higher order are listed in the Appendix. The rate of convergence of Eq. (23) is a function of $\ln r$; more terms need to be kept if the evolution is over a greater range of Q^2. For the value of r considered in the next section terms up to order $(z,\ln az)^{-6}$ are more than sufficient. Because the expansions (11) are in powers of $1/(n+p)$ and $1/\ln(n+p)$ the accuracy of the series (23) increases rapidly as n increases. Hence the method is most accurate at large x, and indeed the number of terms that are to be kept is primarily determined by the precision that is required as x approaches 0.

Finally, for the polarized singlet only, it turns out that the expansions do not converge for $n=1$. There are several ways to overcome this. Because all higher moments converge arbitrarily well one might choose to just add a δ function at $x=0$, this being the only function that influences the first moment exclusively. Alternatively, and this is the course taken by us, one may add to the expansions (23) a term such as z^{-c}, with c being some large number. Again only the first moment is affected significantly. Because of the arbitrariness in the choice of

c this should be seen purely as a check that the distributions for nonzero x are not sensitive to the problem with the singlet's first moment, i.e., that the higher moments are not affected by this additional term. No significance should be attached to the predictions in the region that is affected by the term z^{-a}. We will examine the effect of this term in the next section, after briefly outlining a model that we shall use to illustrate the method that has been developed in this section.

III. RESULTS

We now wish to apply the procedure developed in the previous section to a realistic problem. We shall start with a model prediction for the proton and neutron spin-dependent quark distributions at some, in principle unknown, scale μ^2. This scale is determined, within the model, by evolving the model prediction for the twist-two piece of the nonsinglet spin-independent valence distribution $xq_v(x,\mu^2)=xu_v(x,\mu^2)+xd_v(x,\mu^2)$, and comparing the resultant $xq_v(x,Q^2)$ with parametrizations of the data at $Q^2=10$ GeV^2.[23] This enables us to make predictions for $g_1^p(x,Q^2=10\text{ GeV}^2)$ and $g_1^n(x,Q^2=10\text{ GeV}^2)$.

The model we use should describe the low-energy properties of the nucleons, such as g_A, the magnetic moment, the N-Δ mass splitting, etc. For this reason we shall use MIT bag wave functions for the nucleons, with the center-of-mass momentum projected out through the use of the Peierls-Yoccoz procedure.[24] We shall assume that $\Delta s(x,\mu^2)=\Delta g(x,\mu^2)=0$.

$\Delta u(x,\mu^2)$ and $\Delta d(x,\mu^2)$ are given by

$$\Delta q_f(x,\mu^2)=2M\sum_m\int\frac{d^3p_2}{(2\pi)^3}\delta(M-xM-p_2^+)(|\langle \mathbf{p}_2|b_{m,f}\Psi_{m+}^{\uparrow}(0)|\mathbf{p}=\mathbf{0}\rangle|^2-|\langle \mathbf{p}_2|b_{m,f}\Psi_{m+}^{\downarrow}(0)|\mathbf{p}=\mathbf{0}\rangle|^2)\ . \tag{24}$$

Here $|\mathbf{p}=\mathbf{0}\rangle$ is the Peierls-Yoccoz projected wave function of the nucleon at rest while $\langle\mathbf{p}_2|$ is the wave function of the two-quark intermediate state with momentum $\mathbf{p}_2$. The destruction operator for a quark of flavor f and total angular momentum m is $b_{m,f}$ and Ψ_m refers to the corresponding spinor. The plus indicates the projection onto the "good" components and $\uparrow\downarrow$ refer to the two helicity projections: i.e.,

$$\Psi_{m+}^{\uparrow\downarrow}=\frac{\gamma^-\gamma^+}{2}\frac{1\pm\gamma^5}{2}\Psi_m\ .$$

(Note that while Ψ_m is an eigenstate of the total angular momentum operator, $\Psi_{m+}^{\uparrow\downarrow}$ are not.) Further details of the model can be found in Ref. 25.

The N-Δ mass splitting is incorporated into the model by adjusting the diquark mass appearing in Eq. (24) [through $p_2^+=(p_2^2+m_2^2)^{1/2}+p_z$], depending on whether the state is in the mixed symmetric or mixed antisymmetric part of the spin-flavor wave function.[26] We emphasize that the model is not entirely satisfactory as the Peierls-Yoccoz procedure does not produce an energy eigenstate, nor does it eliminate the center-of-mass momentum variable from the internal wave function of the nucleon. Furthermore, it has been pointed out that connected contributions to the twist-two piece of the scattering process also arise from four-quark intermediate states,[25] which are not considered in Eq. (24). As a consequence of these deficiencies the first moment of the spin-independent quark distributions (summed over flavors) is not guaranteed to be precisely three, nor is the Bjorken sum rule satisfied. We shall overcome these problems phenomenologically by simply adding in a term of the form $x^{-1/2}(1-x)^7$ to the quark distributions in order to mimic the four-quark term. We note that this does not affect g_1^p and g_1^n above $x=0.3$. Note however that the vanishing of the support above $x=1$ is guaranteed because of the δ function in Eq. (24) which ensures energy-momentum conservation explicitly. For a bag of radius $R=0.8$ fm the model predicts the following normalizations at the scale μ:

$$\begin{aligned}\int_0^1 u^{\uparrow}(x,\mu^2)dx&=1.53\ ,\\ \int_0^1 u^{\downarrow}(x,\mu^2)dx&=0.47\ ,\\ \int_0^1 d^{\uparrow}(x,\mu^2)dx&=0.37\ ,\\ \int_0^1 d^{\downarrow}(x,\mu^2)dx&=0.63\end{aligned} \tag{25}$$

giving the value of g_A corresponding to the Peierls-Yoccoz projected MIT bag model

$$g_A=\int_0^1\Delta q_3(x,\mu^2)dx=1.32 \tag{26}$$

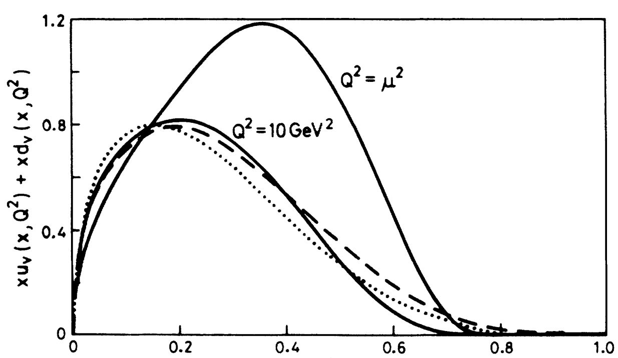

FIG. 1. $xu_v(x,Q^2)+xd_v(x,Q^2)$ at the model scale $Q^2=\mu^2$ and at $Q^2=10$ GeV^2 (solid lines). The dashed and dotted lines correspond to the Duke-Owens and Martin-Roberts-Stirling parametrizations of $xu_v(x,Q^2=10\text{ GeV}^2)+xd_v(x,Q^2=10\text{ GeV}^2)$, respectively (Figs. I.1b and I.2b of Ref. 23).

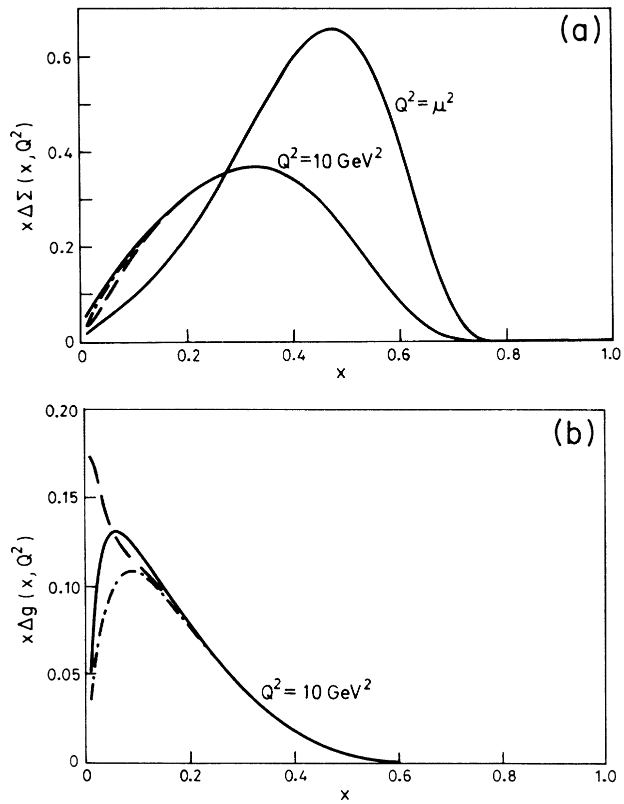

FIG. 2. The quark singlet (a) and polarized glue (b) distributions at $Q^2=\mu^2$ and $Q^2=10$ GeV^2 [note that $\Delta g(x,\mu^2)=0$]. The solid and dashed lines correspond to the addition of terms proportional to z^{-16} and z^{-10}, respectively, to ensure the convergence of the first moment as described in the text. The dotted-dashed line does not have this term added in.

and, for the first moments of the nucleon structure functions

$$\int_0^1 g_1^{p(n)}(x,\mu^2)dx = \int_0^1 [\pm\tfrac{1}{12}\Delta q_3(x,\mu^2)+\tfrac{1}{36}\Delta_8(x,\mu^2) + \tfrac{1}{9}\Delta\Sigma(x,\mu^2)]dx = 0.22(0.00)\ . \tag{27}$$

These are the same values as those expected with wave functions that are completely SU(6) symmetric. Clearly this is only an approximation to reality. For example, we have shown previously that a pion cloud reduces the integral of $g_1^p(x,Q^2)$ by about 15%.[27] It should be noted however that the distributions themselves are different

TABLE I. Moments of quark nonsinglet, singlet, and glue distributions. All moments are in percent of expected ones. The singlet moments correspond to those obtained from Eq. (23) with the first moment readjusted by a term z^{-10} (see text). The numbers in parentheses do not have this additional term.

Moment	$\Delta q_{NS}(n,Q^2)$	$\Delta\Sigma(n,Q^2)$	$\Delta g(n,Q^2)$
1	0.93	1.00 (0.99)	1.00 (0.40)
2	0.99	0.99 (1.00)	1.00 (0.84)
3	1.00	1.00 (1.00)	0.99 (0.95)
4	1.00	1.00 (1.00)	0.99 (0.98)

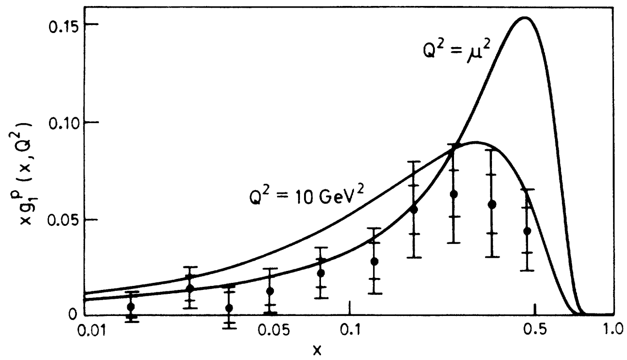

FIG. 3. The proton spin structure function $g_1^p(x,Q^2)$ at both the model scale $Q^2=\mu^2$ and $Q^2=10$ GeV^2. The data were taken from Ref. 10.

from the SU(6)-symmetric ones, in particular the neutron structure function is not zero everywhere. For the purpose of this paper we shall assume that

$$R=0.8\ \text{fm},\quad m_2^{\text{scalar(vector)}}=750\ (950)\ \text{MeV}$$

and $M=938$ MeV. The difference between the two diquark masses is fixed because of the N-Δ mass difference[26] while their absolute value has been treated as an adjustable parameter.

We now turn to the QCD evolution of the above model. In Fig. 1 we show the evolved function $xq_v(x,Q^2)$ and compare it with parametrized data at 10 GeV^2.[23] As r decreases, i.e., as Q^2/μ^2 increases, the momentum carried by the initial valence quarks is decreased and generates the rise of the quark and gluon sea contributions at low x. A value of $r\approx0.3$ (corresponding to $\mu\approx0.5$ GeV and $\alpha_s/2\pi=0.13$ if $\Lambda_{QCD}=0.2$ GeV) gives a reasonable approximation to the data. We emphasize that using a different low-energy model, in particular one including a pion cloud, will change this value of r (in fact, because the pions tend to carry some of the proton's momentum, the distributions will be peaked at smaller x and so the fitted value of r should increase).

Before we turn to the predictions this enables us to make for the spin-dependent structure functions we need to make a comment about the effect of the anomaly.

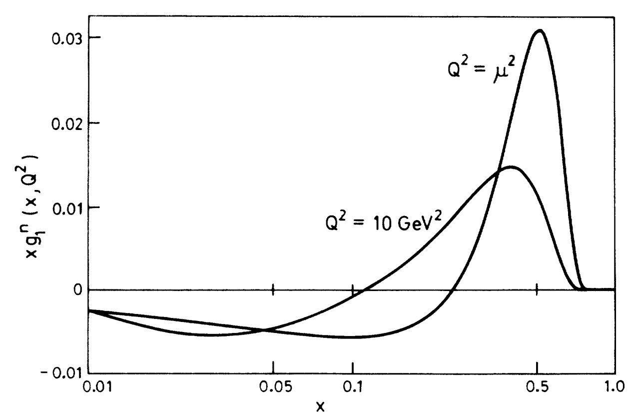

FIG. 4. The predicted neutron spin structure function $g_1^n(x,Q^2)$ at both the model scale $Q^2=\mu^2$ and $Q^2=10$ GeV^2.

Given the scale μ obtained above, one finds that the first moment of the glue distribution at $Q^2=10\ \mathrm{GeV}^2$ is about 0.8. It has been proposed that the anomaly produces an additional contribution to the first moment of the singlet of the form[12,13]

$$-\frac{f}{9}\frac{\alpha(Q^2)}{2\pi}\Delta g \quad (n=1,Q^2)\ .$$

With the above value of Δg $(n=1,Q^2)$ it only gives a contribution of order 0.01 to the sum rule. Furthermore, it has been shown[12] to depend on the regularization. We shall therefore neglect it.

In Fig. 2 we show the evolved quark singlet and glue distributions at $Q^2=10\ \mathrm{GeV}^2$, as well as the variation expected at low x due to the arbitrariness of the term inserted to reproduce the first moment. The region $x\geq 0.1$ is essentially unaffected by this term and hence reliable. This is also evident from the moments of the distributions, shown in Table I.

In Fig. 3 we show $xg_1^p(x,Q^2=10\ \mathrm{GeV}^2)$ corresponding to the quark distribution in Fig. 2, as well as the EMC data for $xg_1^p(x,Q^2=10\ \mathrm{GeV}^2)$. It is immediately evident that although the shape of the prediction for $xg_1^p(x,Q^2)$ matches the data, the first moment is not reproduced for the ratio of coupling constants considered here. A reduction of this ratio, translating into a smaller scale μ^2, decreases the first moment (if one includes the proposed effect of the anomaly).

Finally we turn to the prediction for $xg_1^n(x,Q^2)$ (Fig. 4). The nonzero value of this structure function has two origins: the glue splitting breaks the SU(6) symmetry of the wave function and the singlet and nonsinglet parts of the structure function evolve differently, giving rise to a nonzero neutron structure function at all $\mu^2\neq Q^2$, irrespective of whether the original wave function was SU(6) symmetric.

IV. CONCLUSIONS

We have presented a useful scheme for implementing the QCD evolution of spin-dependent quark and glue distributions. The method is computationally fast because the details of the QCD evolution are independent of the distribution. It is therefore particularly useful in fitting model predictions to data. We have applied the procedure to a model of the nucleon that includes the one gluon exchange responsible for the N-Δ mass splitting in a phenomenological way. This, as well as the QCD evolution itself, induces a nonzero neutron structure function $g_1^n(x,Q^2)$.

ACKNOWLEDGMENTS

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APPENDIX

We list below the higher-order coefficients occurring in Eq. (23). The asymptotic expansions given here have been generated using the algebraic manipulations routines in Ref. 28. Terms up to order $(z\ln az)^{-9}$ may be obtained from the authors [the factor $(n+p)^{-p}$ in Eq. (11) is included in the coefficients listed here]:

$$c_{\mathrm{NS}}[0,0]=1,\quad c_{\mathrm{NS}}[1,0]=-\frac{8u}{9}\ ,$$

$$c_{\mathrm{NS}}[2,0]=-\frac{76u}{81}+\frac{32u^2}{81}\ ,$$

$$c_{\mathrm{NS}}[3,0]=-\frac{40u}{27}+\frac{608u^2}{729}\frac{-256u^3}{2187}\ ,$$

$$c_{\mathrm{NS}}[4,0]=-\frac{1078u}{405}+\frac{11\,528u^2}{6561}-\frac{2432u^3}{6561}+\frac{512u^4}{19\,683}\ ,$$

$$c_{\mathrm{NS}}[5,0]=-\frac{136u}{27}+\frac{41\,072u^2}{10\,935}-\frac{57\,664u^3}{59\,049}+\frac{19\,456u^4}{177\,147}-\frac{4096u^5}{885\,735}\ ,$$

$$c_{\mathrm{NS}}[6,0]=\frac{-16\,636u}{1701}+\frac{264\,808u^2}{32\,805}-\frac{19\,329\,568u^3}{7\,971\,615}+\frac{184\,576u^4}{531\,441}-\frac{38\,912u^5}{1\,594\,323}+\frac{16\,384u^6}{23\,914\,845}\ ,$$

$$c_{qq^+}[2,2]=\frac{9}{25},\quad c_{qq^+}[3,2]=\frac{27}{25}-\frac{18u}{25}\ ,$$

$$c_{qq^+}[3,3]=\frac{27}{25},\quad c_{qq^+}[4,2]=\frac{9}{5}-\frac{91u}{25}+\frac{18u^2}{25}\ ,$$

$$c_{qq^+}[4,3]=\frac{1581}{250}-\frac{258u}{125},\quad c_{qq^+}[4,4]=\frac{5103}{2500}\ ,$$

$$c_{qq^+}[5,2]=\frac{27}{25}-\frac{57u}{5}+\frac{128u^2}{25}-\frac{12u^3}{25}\ ,$$

$$c_{qq^+}[5,3]=\frac{5571}{250}-\frac{2064u}{125}+\frac{246u^2}{125}\ ,$$

$$c_{qq^+}[5,4]=\frac{11\,772}{625}-\frac{4563u}{1250},\quad c_{qq^+}[5,5]=\frac{3159}{1250}\ ,$$

$$c_{qq^+}[6,2]=-\frac{27}{5}-\frac{6709u}{250}+\frac{10\,009u^2}{450}-\frac{22u^3}{5}+\frac{6u^4}{25}\ ,$$

$$c_{qq^+}[6,3]=\frac{29\,199}{500}-\frac{59\,117u}{750}+\frac{7493u^2}{375}-\frac{156u^3}{125}\ ,$$

$$c_{qq^+}[6,4]=\frac{201\,111}{2000}-\frac{105\,591u}{2500}+\frac{4039u^2}{1250}\ ,$$

$$c_{qq^+}[6,5]=\frac{444\,447}{12\,500}-\frac{12\,177u}{3125}\ ,$$

$$c_{qq^=}[6,6]=\frac{41\,553}{50\,000}\ ,$$

$$c_{qq^-}[0,0]=1,\quad c_{qq^-}[1,0]=\frac{-8u}{9}\ ,$$

$$c_{qq^-}[2,0]=-\frac{76u}{81}+\frac{32u^2}{81}\ ,$$

$$c_{qq^-}[2,1]=\frac{-4u}{15},\quad c_{qq^-}[2,2]=-\frac{9}{25}\ ,$$

$$c_{qq^-}[3,0]=-\frac{40u}{27}+\frac{608u^2}{729}-\frac{256u^3}{2187}\ ,$$

$$c_{qq-}[3,1]=\frac{-4u}{5}+\frac{32u^2}{135}\ ,$$

$$c_{qq-}[3,2]=-\frac{27}{25}-\frac{2u}{25},\quad c_{qq-}[3,3]=-\frac{27}{25}\ ,$$

$$c_{qq-}[4,0]=-\frac{1078u}{405}+\frac{11\,528u^2}{6561}-\frac{2432u^3}{6561}+\frac{512u^4}{19\,683}\ ,$$

$$c_{qq-}[4,1]=\frac{-4u}{3}+\frac{1168u^2}{1215}-\frac{128u^3}{1215}\ ,$$

$$c_{qq-}[4,2]=-\frac{9}{5}-\frac{47u}{45}+\frac{56u^2}{225}\ ,$$

$$c_{qq-}[4,3]=-\frac{1581}{250}+\frac{69u}{125},\quad c_{qq-}[4,4]=-\frac{5103}{2500}\ ,$$

$$c_{qq-}[5,0]=-\frac{136u}{27}+\frac{41\,072u^2}{10\,935}-\frac{57\,664u^3}{59\,049}+\frac{19\,456u^4}{177\,147}-\frac{4096u^5}{885\,735}\ ,$$

$$c_{qq-}[5,1]=\frac{-4u}{5}+\frac{944u^2}{405}-\frac{5888u^3}{10\,935}+\frac{1024u^4}{32\,805}\ ,$$

$$c_{qq-}[5,2]=-\frac{27}{25}-\frac{383u}{75}+\frac{1312u^2}{675}-\frac{896u^3}{6075}\ ,$$

$$c_{qq-}[5,3]=-\frac{5571}{250}+\frac{64u}{25}+\frac{16u^2}{375}\ ,$$

$$c_{qq-}[5,4]=-\frac{11\,772}{625}+\frac{2223u}{1250},\quad c_{qq-}[5,5]=-\frac{3159}{1250}\ ,$$

$$c_{qq-}[6,0]=-\frac{16\,636u}{1701}+\frac{264\,808u^2}{32\,805}-\frac{19\,329\,568u^3}{7\,971\,615}+\frac{184\,576u^4}{531\,441}-\frac{38\,912u^5}{1\,594\,323}+\frac{16\,384u^6}{23\,914\,845}\ ,$$

$$c_{qq-}[6,1]=4u+\frac{23\,432u^2}{6075}-\frac{163\,616u^3}{98\,415}+\frac{18\,944u^4}{98\,415}-\frac{2048u^5}{295\,245}\ ,$$

$$c_{qq-}[6,2]=\frac{27}{5}-\frac{36\,949u}{2250}+\frac{51\,988u^2}{6075}-\frac{7424u^3}{6075}+\frac{2816u^4}{55\,675}\ ,$$

$$c_{qq-}[6,3]=-\frac{29\,199}{500}+\frac{19\,517u}{4500}+\frac{1744u^2}{1125}-\frac{448u^3}{3375}\ ,$$

$$c_{qq-}[6,4]=-\frac{201\,111}{2000}+\frac{39\,747u}{2500}-\frac{358u^2}{625}\ ,$$

$$c_{qq-}[6,5]=-\frac{444\,447}{12\,500}+\frac{41\,013u}{12\,500}\ ,$$

$$c_{qq-}[6,6]=-\frac{41\,553}{50\,000}\ ,$$

$$c_{qg+}[1,1]=-\frac{9}{10}\ ,$$

$$c_{qg+}[2,1]=\frac{9u}{5},\quad c_{qg+}[2,2]=-\frac{27}{20}\ ,$$

$$c_{qg+}[3,1]=\frac{9}{5}+\frac{37u}{10}-\frac{9u^2}{5}\ ,$$

$$c_{qg+}[3,2]=-\frac{771}{200}+\frac{123u}{50},\quad c_{qg+}[3,3]=-\frac{1377}{1000}\ ,$$

$$c_{qg+}[4,1]=\frac{27}{5}+\frac{24u}{5}-\frac{37u^2}{5}+\frac{6u^3}{5}\ ,$$

$$c_{qg+}[4,2]=-\frac{171}{25}+\frac{627u}{50}-\frac{111u^2}{50}\ ,$$

$$c_{qg+}[4,3]=-\frac{9621}{1000}+\frac{1017u}{500},\quad c_{qg+}[4,4]=-\frac{243}{2000}\ ,$$

$$c_{qg+}[5,1]=\frac{63}{5}-\frac{21u}{100}-\frac{749u^2}{36}+\frac{37u^3}{5}-\frac{3u^4}{5}\ ,$$

$$c_{qg+}[5,2]=-\frac{2097}{400}+\frac{4969u}{120}-\frac{983u^2}{60}+\frac{33u^3}{25}\ ,$$

$$c_{qg+}[5,3]=-\frac{156\,553}{4000}+\frac{20\,687u}{1000}-\frac{673u^2}{500}\ ,$$

$$c_{qg+}[5,4]=-\frac{178\,929}{20\,000}-\frac{5589u}{5000}\ ,$$

$$c_{qg+}[5,5]=\frac{349\,191}{100\,000}\ ,$$

$$c_{qg+}[6,1]=27-27u-\frac{6737u^2}{150}+\frac{533u^3}{18}-\frac{74u^4}{15}+\frac{6u^5}{25}\ ,$$

$$c_{qg+}[6,2]=\frac{1701}{100}+\frac{10\,373u}{100}-\frac{15\,069u^2}{200}+\frac{962u^3}{75}-\frac{29u^4}{50}\ ,$$

$$c_{qg+}[6,3]=-\frac{230\,871}{2000}+\frac{234\,531u}{2000}-\frac{4841u^2}{250}+\frac{23u^3}{50}\ ,$$

$$c_{qg+}[6,4]=-\frac{3\,131\,613}{40\,000}+\frac{14\,589u}{2500}+\frac{11\,673u^2}{5000}\ ,$$

$$c_{qg+}[6,5]=\frac{998\,163}{50\,000}-\frac{433\,431u}{50\,000}\ ,$$

$$c_{qg+}[6,6]=\frac{391\,473}{40\,000}\ ,$$

$$c_{qg-}[1,1]=\frac{9}{10}\ ,$$

$$c_{qg-}[2,1]=\frac{-4u}{5},\quad c_{qg-}[2,2]=\frac{27}{20}\ ,$$

$$c_{qg-}[3,1]=-\frac{9}{5}-\frac{38u}{45}+\frac{16u^2}{45}\ ,$$

$$c_{qg-}[3,2]=\frac{771}{200}-\frac{36u}{25},\quad c_{qg-}[3,3]=\frac{1377}{1000}\ ,$$

$$c_{qg-}[4,1]=-\frac{27}{5}+\frac{4u}{15}+\frac{304u^2}{405}-\frac{128u^3}{1215}\ ,$$

$$c_{qg-}[4,2]=\frac{171}{25}-\frac{406u}{75}+\frac{56u^2}{75}\ ,$$

$$c_{qg-}[4,3]=\frac{9621}{1000}-\frac{243u}{125},\quad c_{qg-}[4,4]=\frac{243}{2000}\ ,$$

$$c_{qg-}[5,1]=-\frac{63}{5}+\frac{307u}{75}+\frac{3172u^2}{3645}-\frac{1216u^3}{3645}+\frac{256u^4}{10\,935}\ ,$$

$$c_{qg-}[5,2]=\frac{2097}{400}-\frac{16\,763u}{1350}+\frac{2372u^2}{675}-\frac{512u^3}{2025}\,,$$

$$c_{qg-}[5,3]=\frac{156\,553}{4000}-\frac{703u}{50}+\frac{152u^2}{125}\,,$$

$$c_{qg-}[5,4]=\frac{178\,929}{20\,000}-\frac{918u}{625},\quad c_{qg-}[5,5]=-\frac{349\,191}{100\,000}\,,$$

$$c_{qg-}[6,1]=-27+\frac{72u}{5}-\frac{1544u^2}{6075}-\frac{4384u^3}{6561}+\frac{9728u^4}{98\,415}-\frac{2048u^5}{492\,075}\,,$$

$$c_{qg-}[6,2]=-\frac{1701}{100}-\frac{164u}{9}+\frac{4034u^2}{405}-\frac{26\,176u^3}{18\,225}+\frac{128u^4}{2025}\,,$$

$$c_{qg-}[6,3]=\frac{230\,871}{2000}-\frac{88\,909u}{1500}+\frac{2152u^2}{225}-\frac{64u^3}{135}\,,$$

$$c_{qg-}[6,4]=\frac{3\,131\,613}{40\,000}-\frac{25\,779u}{1250}+\frac{876u^2}{625}\,,$$

$$c_{qg-}[6,5]=-\frac{998\,163}{50\,000}+\frac{17\,739u}{12\,500}\,,$$

$$c_{qg-}[6,6]=-\frac{391\,473}{40\,000}\,,$$

$$c_{gq+}[1,1]=-\frac{2}{5}\,,$$

$$c_{gq+}[2,1]=-\frac{6}{5}+\frac{4u}{5},\quad c_{gq+}[2,2]=-\frac{3}{5}\,,$$

$$c_{gq+}[3,1]=-\frac{14}{5}+\frac{182u}{45}-\frac{4u^2}{5}\,,$$

$$c_{gq+}[3,2]=-\frac{527}{150}+\frac{82u}{75},\quad c_{gq+}[3,3]=-\frac{153}{250}\,,$$

$$c_{gq+}[4,1]=-6+\frac{214u}{15}-\frac{256u^2}{45}+\frac{8u^3}{15}\,,$$

$$c_{gq+}[4,2]=-\frac{679}{50}+\frac{664u}{75}-\frac{74u^2}{75}\,,$$

$$c_{gq+}[4,3]=-\frac{764}{125}+\frac{113u}{125},\quad c_{gq+}[4,4]=-\frac{27}{500}\,,$$

$$c_{gq+}[5,1]=-\frac{62}{5}+\frac{3203u}{75}-\frac{10\,657u^2}{405}+\frac{44u^3}{9}-\frac{4u^4}{15}\,,$$

$$c_{qg+}[5,2]=-\frac{4307}{100}+\frac{60\,701u}{1350}-\frac{6913u^2}{675}+\frac{44u^3}{75}\,,$$

$$c_{gq+}[5,3]=-\frac{321\,577}{9000}+\frac{26\,789u}{2250}-\frac{673u^2}{1125}\,,$$

$$c_{gq+}[5,4]=-\frac{20\,691}{5000}-\frac{621u}{1250},\quad c_{gq+}[5,5]=\frac{38\,799}{25\,000}\,,$$

$$c_{gq+}[6,1]=-\frac{126}{5}+\frac{2903u}{25}-\frac{22\,253u^2}{225}+\frac{2254u^3}{81}-\frac{404u^4}{135}+\frac{8u^5}{75}\,,$$

$$c_{gq+}[6,2]=-\frac{2433}{20}+\frac{27\,149u}{150}-\frac{5779u^2}{90}+\frac{5036u^3}{675}-\frac{58u^4}{225}\,,$$

$$c_{gq+}[6,3]=-\frac{158\,497}{1000}+\frac{26\,351u}{300}-\frac{11\,701u^2}{1125}+\frac{46u^3}{225}\,,$$

$$c_{gq+}[6,4]=-\frac{472\,103}{10\,000}+\frac{1379u}{1250}+\frac{1297u^2}{1250}\,,$$

$$c_{gq+}[6,5]=\frac{338\,211}{25\,000}-\frac{48\,159u}{12\,500},\quad c_{gq+}[6,6]=\frac{43\,497}{10\,000}\,,$$

$$c_{gq-}[1,1]=\frac{2}{5}\,,$$

$$c_{gq-}[2,1]=\frac{6}{5}-\frac{16u}{45},\quad c_{gq-}[2,2]=\frac{3}{5}\,,$$

$$c_{gq-}[3,1]=\frac{14}{5}-\frac{584u}{405}+\frac{64u^2}{405}\,,$$

$$c_{gq-}[3,2]=\frac{527}{150}-\frac{16u}{25},\quad c_{gq-}[3,3]=\frac{153}{250}\,,$$

$$c_{gq-}[4,1]=6-\frac{568u}{135}+\frac{2944u^2}{3645}-\frac{512u^3}{10\,935}\,,$$

$$c_{gq-}[4,2]=\frac{679}{50}-\frac{584u}{135}+\frac{224u^2}{675}\,,$$

$$c_{gq-}[4,3]=\frac{764}{125}-\frac{108u}{125},\quad c_{gq-}[4,4]=\frac{27}{500}\,,$$

$$c_{gq-}[5,1]=\frac{62}{5}-\frac{7292u}{675}+\frac{92\,176u^2}{32\,805}-\frac{9472u^3}{32\,805}+\frac{1024u^4}{98\,415}\,,$$

$$c_{gq-}[5,2]=\frac{4307}{100}-\frac{112\,366u}{6075}+\frac{15\,536u^2}{6075}-\frac{2048u^3}{18\,225}\,,$$

$$c_{gq-}[5,3]=\frac{321\,577}{9000}-\frac{9946u}{1125}+\frac{608u^2}{1125}\,,$$

$$c_{gq-}[5,4]=\frac{20\,691}{5000}-\frac{408u}{625},\quad c_{gq-}[5,5]=-\frac{38\,799}{25\,000}\,,$$

$$c_{gq-}[6,1]=\frac{126}{5}-\frac{5852u}{225}+\frac{151\,568u^2}{18\,225}-\frac{343\,424u^3}{295\,245}+\frac{13\,312u^4}{177\,147}-\frac{8192u^5}{4\,428\,675}\,,$$

$$c_{gq-}[6,2]=\frac{2433}{20}-\frac{42\,922u}{675}+\frac{220\,504u^2}{18\,225}-\frac{6400u^3}{6561}+\frac{512u^4}{18\,225}\,,$$

$$c_{gq-}[6,3]=\frac{158\,497}{1000}-\frac{178\,423u}{3375}+\frac{59\,456u^2}{10\,125}-\frac{256u^3}{1215}\,,$$

$$c_{gq-}[6,4]=\frac{472\,103}{10\,000}-\frac{20\,858u}{1875}+\frac{1168u^2}{1875}\,,$$

$$c_{gq-}[6,5]=-\frac{338\,211}{25\,000}+\frac{1971u}{3125}\,,$$

$$c_{gq-}[6,6]=-\frac{43\,497}{10\,000}\,,$$

$$c_{gg^+}[0,0]=1,\quad c_{gg^+}[1,0]=-2u\ ,$$

$$c_{gg^+}[2,0]=-\frac{37u}{9}+2u^2\ ,$$

$$c_{gg^+}[2,1]=\frac{4u}{15},\quad c_{gg^+}[2,2]=-\frac{9}{25}\ ,$$

$$c_{gg^+}[3,0]=-\frac{28u}{3}+\frac{74u^2}{9}-\frac{4u^3}{3}\ ,$$

$$c_{gg^+}[3,1]=\frac{4u}{5}-\frac{8u^2}{15}\ ,$$

$$c_{gg^+}[3,2]=-\frac{27}{25}+\frac{28u}{25},\quad c_{gg^+}[3,3]=-\frac{27}{25}\ ,$$

$$c_{gg^+}[4,0]=-\frac{1799u}{90}+\frac{4393u^2}{162}-\frac{74u^3}{9}+\frac{2u^4}{3}\ ,$$

$$c_{gg^+}[4,1]=\frac{4u}{3}-\frac{364u^2}{135}+\frac{8u^3}{15}\ ,$$

$$c_{gg^+}[4,2]=-\frac{9}{5}+\frac{1346u}{225}-\frac{334u^2}{225}\ ,$$

$$c_{gg^+}[4,3]=-\frac{1581}{250}+\frac{321u}{125},\quad c_{gg^+}[4,4]=-\frac{5103}{2500}\ ,$$

$$c_{gg^+}[5,0]=-\frac{124u}{3}+\frac{10\,577u^2}{135}-\frac{2881u^3}{81}+\frac{148u^4}{27}-\frac{4u^5}{15}\ ,$$

$$c_{gg^+}[5,1]=\frac{4u}{5}-\frac{76u^2}{9}+\frac{512u^3}{135}-\frac{16u^4}{45}\ ,$$

$$c_{gg^+}[5,2]=-\frac{27}{25}+\frac{1474u}{75}-\frac{2528u^2}{225}+\frac{272u^3}{225}\ ,$$

$$c_{gg^+}[5,3]=-\frac{5571}{250}+\frac{7936u}{375}-\frac{1076u^2}{375}\ ,$$

$$c_{gg^+}[5,4]=-\frac{11\,772}{625}+\frac{2574u}{625},\quad c_{gg^+}[5,5]=-\frac{3159}{1250}\ ,$$

$$c_{gg^+}[6,0]=-\frac{15\,877u}{189}+\frac{168\,803u^2}{810}-\frac{2\,805\,899u^3}{21\,870}+\frac{2377u^4}{81}-\frac{74u^5}{27}+\frac{4u^6}{45}\ ,$$

$$c_{gg^+}[6,1]=-4u-\frac{13\,418u^2}{675}+\frac{20\,018u^3}{1215}-\frac{88u^4}{27}+\frac{8u^5}{45}\ ,$$

$$c_{gg^+}[6,2]=\frac{27}{5}+\frac{54\,523u}{1125}-\frac{41\,663u^2}{810}+\frac{23\,896u^3}{2025}-\frac{158u^4}{225}\ ,$$

$$c_{gg^+}[6,3]=-\frac{29\,199}{500}+\frac{1\,399\,291u}{13\,500}-\frac{102\,634u^2}{3375}+\frac{20\,714u^3}{10\,125}\ ,$$

$$c_{gg^+}[6,4]=-\frac{201\,111}{2000}+\frac{30\,513u}{625}-\frac{4941u^2}{1250}\ ,$$

$$c_{gg^+}[6,5]=-\frac{444\,447}{12\,500}+\frac{50\,247u}{12\,500}\ ,$$

$$c_{gg^+}[6,6]=-\frac{41\,553}{50\,000}\ ,$$

$$c_{gg^-}[2,2]=\frac{9}{25}\ ,$$

$$c_{gg^-}[3,2]=\frac{27}{25}-\frac{8u}{25}\ ,$$

$$c_{gg^-}[3,3]=\frac{27}{25}\ ,$$

$$c_{gg^-}[4,2]=\frac{9}{5}-\frac{292u}{225}+\frac{32u^2}{225}\ ,$$

$$c_{gg^-}[4,3]=\frac{1581}{250}-\frac{132u}{125}\ ,$$

$$c_{gg^-}[4,4]=\frac{5103}{2500}\ ,$$

$$c_{gg^-}[5,2]=\frac{27}{25}-\frac{236u}{75}+\frac{1472u^2}{2025}-\frac{256u^3}{6075}\ ,$$

$$c_{gg^-}[5,3]=\frac{5571}{250}-\frac{2704u}{375}+\frac{64u^2}{125}\ ,$$

$$c_{gg^-}[5,4]=\frac{11\,772}{625}-\frac{1404u}{625}\ ,$$

$$c_{gg^-}[5,5]=\frac{3159}{1250}\ ,$$

$$c_{gg^-}[6,2]=-\frac{27}{5}-\frac{5858u}{1125}+\frac{40\,904u^2}{18\,225}-\frac{4736u^3}{18\,225}+\frac{512u^4}{54\,675}\ ,$$

$$c_{gg^-}[6,3]=\frac{29\,199}{500}-\frac{98\,434u}{3375}+\frac{13\,504u^2}{3375}-\frac{1664u^3}{10\,125}\ ,$$

$$c_{gg^-}[6,4]=\frac{201\,111}{2000}-\frac{14\,052u}{625}+\frac{752u^2}{625}\ ,$$

$$c_{gg^-}[6,5]=\frac{444\,447}{12\,500}-\frac{10\,638u}{3125}\ ,$$

$$c_{gg^-}[6,6]=\frac{41\,553}{50\,000}\ .$$

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